Lecture Note for Honor PDE

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1 Introduction

1.1 Derivation of PDEs

Many partial differential equations originate from physical models. Understanding these models provides valuable insight into the intuition underlying the equations. In this section, we demonstrate the derivation of several common PDEs from fundamental physical principles.

1.1.1 Transport equation

Let $u(t,x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be the unknown function. The variable t is the time coordinate, and x is the space coordinate. The variable u can be the density of something, the velocity field, etc.

For illustration, suppose that we are modeling the traffic flow and $u(t,x)$ is the density of cars at (t,x). Let $a < b$. We first have the *conservation of mass* equation

$$\frac{d}{dt}\Big(\int_a^b u(t,x)\,dx\Big) = J(t,a) - J(t,b). \tag{1.1}$$

Here, the LHS is the rate of change of the total number of cars, and $J(t,x)$ is the *flux* at (t,x): the number of cars moving from the left of x to the right of x in unit time.

Assume that u and J is smooth enough, so that we can differentiation and interchange the order of differentiation and integration. Taking the t-derivative in (1.1) yields

$$\int_a^b \partial_t u(t,x)\,dx = J(t,a) - J(t,b) = -\int_a^b \partial_x J(t,x)\,dx.$$

Then

$$\int_a^b \big[\partial_t u(t,x) + \partial_x J(t,x)\big]\,dx = 0.$$

Since a and b are arbitrary, and the integrand is a continuous function, we must have the relation

$$\partial_t u(t,x) + \partial_x J(t,x) = 0. \tag{1.2}$$

This is the differential form of (1.1).

Next, we need to relate J to u to eliminate the unknown J in order to close the equation for u. Since u is the density, by the physical meaning of flux we have

$$J(t,x) = u \cdot V(t,x),$$

where $V(t,x)$ is the velocity field. It remains to determine how the velocity depends on the density; this may differ from one model from another. Here are some examples.

- $V(t,x) = \text{const}$. Then (1.2) reduces to

$$\partial_t u + c\partial_x u = 0.$$

One can check that the general solution is given by $u(t,x) = \phi(x - ct)$, that is, the initial density profile $\phi(\cdot)$ moves with constant speed c.

- $V(t,x) = 1 - u$. This is a more realistic model for the traffic jam: the velocity is decreasing as the density increases, and at maximum density $u = 1$ the traffic flow completely stops. The resulting equation is

$$\partial_t u + \partial_x\big(u(1-u)\big) = 0 = \partial_t u + \partial_x u - 2u \cdot \partial_x u = 0.$$

  Although this equation seems simple, it is a nonlinear PDE and exhibits nontrivial behaviors such as formation of shocks.

We have the general form of the transport equation

$$\partial_t u + \partial_x\big(uV(u)\big) = 0, \tag{1.3}$$

where $V : \mathbb{R} \to \mathbb{R}$ is a function that depends on the model.

We can further generalize (1.3) to dimension $d > 1$. We first guess the form of the equation by matching the dimension, and then we will derive it rigourously using the conservation of mass.

Since u is the density, it is a multi-variate function

$$u(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}.$$

Since V gives the velocity, so V and $J = u \cdot V$ must be vector functions:

$$V(x) : \mathbb{R}^d \to \mathbb{R}^d, \quad J(t,x) = u \cdot V : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^d.$$

Looking the LHS of (1.3), $\partial_t u$ takes value in $\mathbb{R}$, so the differential operator must turn $J(t,x)$ into a function that maps $\mathbb{R}^d$ to $\mathbb{R}$. The only such operator is the divergence operator $\nabla\cdot$ acting on a vector function $f = (f_1, \dots, f_d)$

$$\nabla \cdot f = \nabla \cdot (f_1, \dots, f_d) \coloneqq \sum_{i=1}^{d} \frac{\partial}{\partial x_i} f_i.$$

Hence, we obtain a reasonable guess of the generalization of (1.3) in an arbitrary dimension $d > 1$:

$$\partial_t u + \nabla \cdot \big(uV(u)\big) = 0, \tag{1.4}$$

where $u : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ is the unknown function and $V : \mathbb{R} \to \mathbb{R}^d$ is a given function depending on the model.

Next, we give a rigourous derivation using the conservation of mass. The key tool is the *Divergence Theorem*.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^d$ be a domain with piecewise $\mathcal{C}^1$-boundary. Let $F : \bar{\Omega} \to \mathbb{R}^d$ be $\mathcal{C}^1$. Then*

$$\int_{\partial\Omega} F \cdot \vec{n}\, dS = \int_{\Omega} \nabla \cdot F\, dx, \tag{1.5}$$

where dS denotes the surface element on $\partial\Omega$, and $\vec{n}$ is the outer unit normal vector on $\partial\Omega$.

(1.5) is also refered to as the *Stokes formula*, or simply integration by parts, since from right to left a differential operator is removed.

Now let Ω be an arbitrary $\mathcal{C}^1$-domain in $\mathbb{R}^d$. The total mass in Ω is given by $\int_\Omega u(t,x)\,dx$. By the conservation of mass, the rate of change of the total mass is a conseqeunce of the flux of mass across of the boundary. Thus we have

$$\frac{d}{dt}\int_\Omega u(t,x)\,dx = -\int_\Omega J\cdot\vec{n}\,dS.$$

To double check the RHS: if the direction of the flux is tangent to the boundary at some point, that is $J\cdot\vec{n}=0$, then there is no mass escaping from this point, justifying the form of the integrand. Also, if the flux J is point outwards and has the same direction as $\vec{n}$, this will result in a decrease of the mass, and hence the minus sign on the RHS.

Assuming u is smooth enough so that the order of differentiation and integration can be exchanged on the LHS, and using Theorem 1.1 on the RHS, we obtain

$$\int_\Omega (\partial_t u + \nabla\cdot J)\,dx = 0.$$

Since this holds for an arbitrary $\mathcal{C}^1$-domain, we have pointwise

$$\partial_t u + \nabla\cdot J = 0. \tag{1.6}$$

Plugging in $J = uV(u)$ we obtain (1.4).

For the last step we use the following simple result.

Lemma 1.2

- *If $f\in\mathcal{C}(\mathbb{R}^d)$ and $\int_\Omega f(x)\,dx = 0$ for any rectangle $\Omega\subset\mathbb{R}^d$, then $f\equiv 0$.*
- *If $f\in L^1_{\mathrm{loc}}(\mathbb{R})$ and $\int_\Omega f(x)\,dx = 0$ for any rectangle $\Omega\subset\mathbb{R}^d$, then $f=0$ almost everywhere.*

As we will see, the transport equation may develop singularity no matter how smooth the initial condition is, so (1.4) may not hold for every point, but it is at least safe to say that it holds almost everywhere.

1.1.2 Heat equation

In (1.6), we may interpret u as the temperature and J as the heat flux; then (1.6) follows from *the conservation of energy*, as confirmed by Joule's experiment. To close the equation, we need to relate J to u. Fourier's law states that the heat flux is proportional to the negative gradient of the temperature field, expressed as

$$J = -c\nabla u,$$

where the constant c denotes the *thermal conductivity*. Here, the gradient operator ∇ is defined by

$$(\nabla f)(x_1,\dots,x_d) = \big(\partial_{x_1} f(x_1,\dots,x_d),\dots,\partial_{x_d} f(x_1,\dots,x_d)\big).$$

Combined with (1.6), we obtain

$$\partial_t u = -\nabla\cdot(-c\nabla u) = \sum_{i=1}^d \partial_{x_i x_i} u =: c\Delta u. \tag{1.7}$$

The operator Δ is called the Laplacian operator. (1.7) is called the *heat equation*. Usually we set $c=1$.

The heat equation also models the phenomenon of *diffusion*. Let u represent the concentration of a substance within the fluid, analogous to the density. Particles of this substance may move under external forces, but even in the absence of such external forces, diffusion causes particles to move from regions of higher concentration to lower concentration. Specifically, *Fick's law* states that the flux J is proportional to the negative gradient of u, and hence the diffusion is modeled by the heat equation as well.

The heat equation is a second-order PDE since it involves second partial derivatives. It is classified as a *parabolic equation* since the time derivative is only first order, analogous to the *parabola equation* $t = x^2$.

1.1.3 Wave equation

The wave equation models the wave phenomena in elastic media. Let Ω be a domain representing an elastic object, such as a string, a rod, or membrane. For simplicity, we take $\Omega = (a, b)$ as an example. The unknown function $u(t, x) : \mathbb{R} \times \Omega \to \mathbb{R}$ under consideration is the displacement of the object from its equilibrium position. By Newton's second law, we have

$$\partial_{tt} u(t, x) = F(t, x),$$

where $F(t, x)$ is the force acting at position x. To determine this force, we invoke *Hooke's law*, which states that the elastic force is negatively proportional to the displacement

$$F = -k\Delta L.$$

We imagine there are two small springs on the intervals $(x - \Delta x, x)$ and $(x, x + \Delta x)$. The net force at (t, x) results from the combination of the elastic forces from these springs. Applying the Hooke's law gives

$$F(t, x) \approx F_1 + F_2 = -k\Big(u(t, x) - u(t, x - \Delta x)\Big) - k\big(u(t, x) - u(t, x + \Delta x)\big) \approx k(\Delta x)^2 \partial_{xx} u(t, x).$$

Combining all these and assuming $k(\Delta x)^2 \to c$ as $\Delta x \to 0$, we obtain the *wave equation*

$$\partial_{tt} u = c\partial_{xx} u.$$

The wave equation in dimensions $d > 1$ can be derived analogously or postulated as

$$\partial_{tt} u = c\Delta u.$$

This equation is classified as the *hyperbolic equation* since both the time and space derivatives are of second order and have the opposite signs, which resembles the *hyperbola equation* $t^2 = x^2$.

1.1.4 Laplace equation

Consider the heat equation in a domain Ω, with boundary condition $u\big|_{\partial\Omega} = \varphi$ and initial condition $u\big|_{t=0} = u_0$. From a physical perspective, if the temperature is fixed at the boundary, eventually the temperature field will reach an equilibrium state, that is, there is $u_* : \Omega \to \mathbb{R}$ such that $u(t, x) \to u_*(x)$ as $t \to \infty$, where u_* may or may not depend on u_0. Since $v(t, x) = u_*(x)$ also satisfies the heat equation as it is the equilibrium, we obtain

$$\Delta u_* = 0, \quad u_*\big|_{\partial\Omega} = \varphi. \tag{1.8}$$

This is known as the *Laplace equation*. It is classified as an *elliptic equation* since all second derivatives have the same sign, resembling the ellipse equation $ax^2 + by^2 = 1$.

We now derive the Laplace equation using the *calculus of variation*, a powerful tool to obtain PDEs. We consider the following minimization problem:

$$\inf_{u\big|_{\partial\Omega}=\varphi} \int_\Omega |\nabla u|^2(x)\,dx =: \inf_{u\big|_{\partial\Omega}=\varphi} I[u]. \tag{1.9}$$

The square bracket $[\cdot]$ stresses that I is a *functional*, that is, a "function" of functions. Assume that u_* achieves the minimum of (1.9), that is,

$$I[u_*] = \min_{u\big|_{\partial\Omega}=\varphi} I[u].$$

Intuitively, $I[u]$ is the L^2-norm of the heat flow corresponding to the temperature field u, and if the L^2-norm is minimized, the temperature field is at the equilibrium state.

Assuming u_* is the minimum function, let us derive conditions that u_* must satisfy. Let $v \in \mathcal{C}_0^\infty(\Omega)$ be arbitrary. We introduce perturbation of u_* as

$$u_\varepsilon = u_* + \varepsilon v, \quad \varepsilon \in \mathbb{R}$$

Since v vanishes at $\partial\Omega$, the function u_ε satisfies the boundary condition. Let $f(\varepsilon) = I[u_\varepsilon]$. Since f achieves minimum at $\varepsilon = 0$, we must have $f'(0) = 0$ provided that the derivative exist. We do not know if f is actually differentiable, but assuming that all functions are nice, this is indeed the case and we have:

$$0 = f'(0) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_\Omega |\nabla u_* + \varepsilon\nabla v|^2\,dx = 2\int_\Omega \nabla u_* \cdot \nabla v\,dx = 0. \tag{1.10}$$

To proceed, we use the following useful integration-by-part formula.

Lemma 1.3 *Let Ω be a $\mathcal{C}^1$-domain and $u, v \in \mathcal{C}^1(\bar\Omega) \cap \mathcal{C}^2(\Omega)$. Then*

$$\int_\Omega \nabla u \cdot \nabla v\,dx = -\int_\Omega u\Delta v\,dx + \int_{\partial\Omega} u\frac{\partial v}{\partial n}\,dS, \tag{1.11}$$

$$\int_\Omega \nabla u \cdot \nabla v\,dx = -\int_\Omega v\Delta u\,dx + \int_{\partial\Omega} v\frac{\partial u}{\partial n}\,dS, \tag{1.12}$$

$$\int_\Omega u\Delta v - v\Delta u\,dx = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\,dS. \tag{1.13}$$

Proof: The last identity follows from taking difference of the first two. Since the roles of u and v are symmetric, it suffices to prove (1.11). Indeed, consider the vector function $F = u\nabla v : \Omega \to \mathbb{R}^d$. Then $\nabla\cdot F = \nabla u\cdot\nabla v + u\Delta v$ and $F\cdot\vec{n} = u\frac{\partial v}{\partial n}$. Applying Theorem 1.1 to F yields the desired conclusion. □

Using Lemma 1.3, we can continue with (1.10) to obtain

$$0 = \int_\Omega \nabla u_* \cdot \nabla v\,dx = -\int_\Omega v\Delta u_*\,dx, \quad \forall v \in \mathcal{C}_0^\infty(\Omega). \tag{1.14}$$

There is no boundary term after integration by parts since v vanishes at the boundary. Since (1.14) holds for arbitrary $v \in \mathcal{C}_0^\infty(\Omega)$, a variant of Lemma 1.2 implies that $\Delta u_* = 0$ pointwise assuming its continuity. Hence we derive the Laplace equation again.

As another example, the variational problem

$$\inf_{u\big|_{\partial\Omega}=\varphi} \int_\Omega \sqrt{1 + |\nabla u|^2}\,dx$$

gives rise the *minimal surface equation*. This will be left as an exercise.

1.1.5 *Viscous Burgers equation and fluid equation

Let us consider the velocity field $u(t,x)$ of particles moving on $\mathbb{R}$. By Newton's law, we have

$$\text{acceleration of particles at } (t_0,x_0) = \text{friction} + \text{external force}. \tag{1.15}$$

First, let us express the acceleration field from the velocity field. The naive guess $\partial_t u$ is wrong, since the particles at position x are not the same for different t. To get the correct form of the acceleration, we must follow a fixed particle. Let $x(t)$ be the trajectory of the particle passing (t_0,x_0) (that is, $x(t_0)=x_0$). Then by definition

$$\dot{x}(t) = u\big(t,x(t)\big).$$

Hence,

$$\ddot{x}(t) = \frac{d}{dt}u\big(t,x(t)\big) = \partial_t u + \dot{x}\cdot\partial_x u = \partial_t u + u\cdot\partial_x u. \tag{1.16}$$

This gives the LHS of (1.15).

For the RHS, first, the friction force is modeled by $\partial_{xx}u$. To understand why second derivative appears, it suffices to note that if $\partial_{xx}u = 0$, then u is linear and there is no friction in the sheer transform. Last, the external force is modeled by an arbitrary function $f(t,x)$. Combining all these, we obtain the full *viscous Burgers equation*:

$$\partial_t u + u\partial_x u = \partial_{xx}u + f(t,x).$$

Its multi-dimensional analogue is

$$\partial_t u + u\cdot\nabla u = \Delta u + f(t,x), \quad u:\mathbb{R}_+\times\mathbb{R}^d\to\mathbb{R}.$$

The Burgers equation is a mixture of the "transport term" $u\partial_x u$ and the "diffusion term" $\partial_{xx}u$.

The Burgers equation is a toy model for fluid dynamics. Here we also mention the celebrated Navier-Stokes equation, and by now we can understand the physical meaning of all the terms in the equation. Assuming the fluid is incompressible (meaning the density is constant), the Navier-Stokes equation reduced to an equation of the velocity field: $u(t,x):\mathbb{R}_+\times\mathbb{R}^d$, $d=2,3$,

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla p = \Delta u + f, \\ \nabla\cdot u = 0. \end{cases}$$

Here, we recognize the material derivative term $\partial_t u + u\cdot\nabla u$, which is the acceleration field. All the other are forcing terms: the pressure term ∇p, the friction Δu, and the external force f. The divergence-free constraint comes from conservation of mass:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho u) = 0 \implies \nabla\cdot u = 0.$$

Although the pressure p is also unknown and first equation seems under-determined, the divergence-free constraint can in fact eliminate the pressure term in the first equation.

1.1.6 *Maxwell equation

In this section we briefly look at the Maxwell's equation that models the electro-magnetic field. The unknowns are the electric field E and the magnetic field B, both are vector functions on $\mathbb{R}^3$. All of the four equations can be written down in the differential form and in the integral form.

- Gauss's law:
$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}, \quad \int_{\partial\Omega} E \cdot \vec{n}\, dS = \frac{1}{\varepsilon_0} \int_{\Omega} \rho\, dx.$$
Here, ρ is the electric charge density, ε_0 is a physical constant, and Ω is an arbitrary domain.

- Gauss's law for magnetism:
$$\nabla \cdot B = 0, \quad \int_{\partial\Omega} B \cdot \vec{n}\, dS.$$

- Faraday's equation (electric generated from a changing magnetic field):
$$\nabla \times E = -\frac{\partial B}{\partial t}, \quad \oint_{\partial\Sigma} E \cdot d\ell = -\int_{\Sigma} \frac{\partial B}{\partial t} \cdot dA,$$
where Σ is any surface.

- Ampère's circuital law (magnetic field generated by currents):
$$\nabla \times B = \mu_0(J + \varepsilon_0 \frac{\partial E}{\partial t}), \quad \oint_{\partial\Sigma} B \cdot d\ell = \int_{\Sigma} \mu_0\big(J + \varepsilon_0 \frac{\partial E}{\partial t}\big) \cdot dA,$$
where J is the current.

It is well-known that electro-magnetic field related to waves. To see this from the equation, we consider the vacuum case where $\rho = J \equiv 0$. Then we have

$$\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \Delta E = -\Delta E = -\frac{\partial}{\partial t}(\nabla \times B) = -\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial^2 t},$$

so E satisfies the wave equation, where the wave speed (i.e., the light speed) is $c = \sqrt{\mu_0 \varepsilon_0}$. A similar calculation yields a wave equation for B.

1.2 key questions in this course

This course will focus on four elementary partial differential equations (PDEs), which model fundamental physical phenomena and serve as foundational components for more complex PDEs:

- the transport equation: $\partial_t u + \partial_x V(u) = f$;
- the Laplace equation: $\Delta u = f$;
- the heat equation: $\partial_t u = \Delta u$;
- the wave equation: $\partial_{tt} u = \Delta u$.

One part of the course is devoted to how to write down solutions of the PDEs, using techniques like Fourier analysis, separation of variables and etc. A more important part is to develop *well-posedness* theory without an explicit form of the solution. The well-posedness theory is three-fold:

- existence of solution, including suitable conditions on the boundary and initial condition, regularity requirement;
- uniqueness of solution
- stability: how sensitive the solution is to initial and boundary data.

For a rigorous well-posedness theory we must be accurate about the solution space. A key concept is the *classical solution*, where all the derivatives appearing in the PDE are continuous function so that the PDEs make sense pointwise. When there are both time and space derivative, we use $\mathcal{C}^{\alpha,\beta}$ to indicate the space of functions that has α-th order continuous derivative in t and β-th order continuous derivative in space. For example, classical solutions of the first order transport equation live in $\mathcal{C}^{1,1}$, for the heat equation $\mathcal{C}^{1,2}$, and for the wave equation $\mathcal{C}^{2,2}$. We may also spend some time discussing how to define weak solutions, solutions that have a lower regularity requirement.

2 First-order transport equation

In this section we study the first-order transport equation:

$$\begin{cases} \partial_t u + b(t,x,u)\cdot \partial_x u = f(t,x,u), \\ u(0,x) = \phi(x), \end{cases} \qquad u(t,x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}. \tag{2.1}$$

2.1 Method of characteristics

2.1.1 Constant b

Suppose $b(t,x,u) = V$ is a constant and $f \equiv 0$. The equation becomes

$$\partial_t u + V\partial_x u = 0, \quad u(0,x) = \phi(x).$$

We introduce $U(t) = u(t, x_0 + Vt)$ where u is a solution and $x_0 \in \mathbb{R}$ is fixed. Then

$$\dot{U}(t) = \partial_t u(t, x_0 + Vt) + V\partial_x u(t, x_0 + Vt) = 0.$$

Hence,

$$U(t) \equiv U(0) = u(0, x_0) = \phi(x_0),$$

and we have

$$u(t,x) = \phi(x - Vt).$$

The curves $\eta(t) = x_0 + Vt$ are called *characteristics*. Intuitively, the initial data ϕ is propagating along these curves.

As a remark, if $\phi \in \mathcal{C}^1$, then $u = \phi(x - Vt) \in \mathcal{C}^{1,1}$ is a classical solution. But even if $\phi \notin \mathcal{C}^1$, this is still the only plausible solution to the PDE, despite being non-classical. From this example, we see that dealing with non-classical solutions is already inevitable even for very simple PDE,

2.2 A non-homogeneous example

We consider the following equation:

$$\begin{cases} \partial_t u + x\partial_x u = u + x, \\ u(0,x) = \phi(x). \end{cases} \tag{2.2}$$

We are seeking characteristics $\eta(t)$ so that

$$U(t) = u\big(t, \eta(t)\big)$$

solves a simple ODE. We clearly have

$$\dot{\eta}(t) = \eta \implies \eta(t) = C_1 e^t.$$

Plugging into U, we have

$$\dot{U}(t) = U(t) + C_1 e^t.$$

The general solution for this ODE is

$$U(t) = C_1 t e^t + C_2 e^t.$$

Finally, we need to determine the constants C_1 and C_2. We have

$$\eta(t) = C_1 e^t = x, \quad U(0) = C_2 = \phi(C_1) \implies C_1 = xe^{-t}, \quad C_2 = \phi(xe^{-t}).$$

Hence, the solution to the PDE is

$$u = xt + \phi(xe^{-t})e^t.$$

One can check by direct computation that it indeed solves the original PDE.

2.3 Genera linear case

We consider the general linear case

$$\begin{cases} \partial_t u + b(t,x)\partial_x u = f(t,x,u), \\ u(0,x) = \phi(x). \end{cases} \tag{2.3}$$

We state a well-posedness result.

Theorem 2.1 *Assume that $b \in \mathcal{C}^{0,1}$, $\phi \in \mathcal{C}^1$ and $f \in \mathcal{C}^{0,1,1}$. Then there exists a unique solution to (2.3).*

Proof: We consider the characteristic ODE

$$\dot{\eta}(t) = b\big(t, \eta(t)\big), \quad \eta(0) = x_0.$$

Since b is Lipschitz in x, by standard ODE theory, there is a unique solution for every initial condition x_0. Moreover, the solution map

$$\Phi_t : x_0 \mapsto \eta(t; x_0)$$

is a $\mathcal{C}^1$-diffeomorphism of $\mathbb{R}$, that is, both Φ_t and Φ_t^{-1} are in $\mathcal{C}^1$. Indeed, Φ_t' satisfies the ODE

$$\frac{d}{dt}\Big(\Phi_t'\Big) = \partial_x b\big(t, \Phi(t)\big)\Phi_t', \quad \Phi_0' = 1.$$

Let u_1 and u_2 be two $\mathcal{C}^{1,1}$-solutions of the PDE and let

$$w_i(t) = u_i\big(t, \eta(t)\big), \quad i = 1, 2. \tag{2.4}$$

Then w_i solves the ODE

$$\dot{w}_i(t) = f\big(t, \eta(t), w_i(t)\big), \quad w_i(0) = \phi\big(\eta(0)\big). \tag{2.5}$$

Since the above ODE has unique solution, we have $w_1 = w_2$. Hence the PDE has unique solution.

For the existence of the solution, let $w(t; w_0)$ be the solution to the ODE (2.5) with initial condition w_0. Then one can check that

$$u(t,x) = w\Big(t; \phi\big(\Phi_t^{-1}(x)\big)\Big)$$

is a $\mathcal{C}^{1,1}$-function that solves the PDE. The detailed computation will be omitted. For concrete equations, the justification will be more straightforward. □

2.4 Nonlinear equation

In nonlinear transport equations, the function $b = b(t, x, u)$ also depends on u. In this case, we have to solve the ODEs of η and w together:

$$\begin{cases} \dot{\eta}(t) = b(t, \eta, w), \\ \dot{w}(t) = f(t, \eta, w). \end{cases}$$

2.4.1 Burgers equation

The Burgers equation is one of the simplest nonlinear PDEs. We start from the homogeneous equation ($f \equiv 0$).

$$\partial_t u + u\partial_x u = 0, \quad u(0, x) = \phi(x).$$

The characteristic ODE system is

$$\dot{\eta}(t) = w, \quad \dot{w}(t) = 0.$$

The second equation indicates that w is constant, implying that η is a linear function: $\eta(t) = x_0 + t\phi(x_0) = x$. Physically, this corresponds to particles moving at constant velocity due to the absence of external forcing f. The characteristics, representing particle trajectories, are therefore straight lines. To determine the velocity field at (t, x), one may identify the origin of the particle arriving at the point (t, x), and retrieve its velocity.

In nonlinear scenarios, however, characteristics may intersect, causing the correspondence $x \mapsto x_0$ to cease being one-to-one. If multiple characteristics pass through a point (t, x), it implies that particles carrying different velocity meet at (t, x), causing the velocity field at (t, x) is undetermined. On the other hand, one can check that a necessary and sufficient condition to avoid intersection is that ϕ is increasing, but in such case, certain points (t, x) may lack any passing characteristics, again leaving the velocity field undetermined.

Through two examples we will illustrate how to resolve these issues.

1. Rarefaction solution

   Suppose the initial condition is given by

   $$\phi(x) = \begin{cases} 0, & x \le 0, \\ 1, & x > 0. \end{cases}$$

   By looking at the characteristics, we have

   $$u(t, x) = \begin{cases} 0, & x \le 0 \\ 1, & x \ge t. \end{cases}$$

   There is no characteristic in the region $0 < x < t$, leaving the solution undetermined. Let us try to construct a reasonable solution. We notice that the initial condition ϕ is already discontinuous. We cannot expect our constructed solutions to be continuous, but we should make the discontinuous point as few as possible. One possible choice is

   $$u(t, x) = \begin{cases} 0, & x \le kt, \\ 1, & x > kt, \end{cases}$$

   where $k \in (0, 1)$. The solution is only discontinuous along the line $x = kt$.

Are these solutions reasonable? From the point of view of differentiability it seems yes: apart from the curve $x = kt$, the function is continuously differentiable and satisfies the PDE. But it turns out that these are *non-physical* solution.

To obtain a physical solution, we note that the root issue is that the initial condition is not $\mathcal{C}^1$. Nonsmooth function is merely a pure mathematical object; we should think of the discontinuous function ϕ as an idealization of another function that has an abrupt near 0:

$$\phi_\varepsilon(x) = \begin{cases} 0, & x \le 0, \\ x/\varepsilon, & 0 \le x \le \varepsilon, \\ 1, & x \ge \varepsilon, \end{cases}$$

where ε is so small that make ϕ_ε look like discontinuous. With the initial condition $\phi_\varepsilon(x)$ one can check that characteristics fill the whole space, as by letting $\varepsilon \to 0$, we obtain another solution to the original PDE

$$u(x) = \begin{cases} 0, & x \le 0, \\ x/t, & ,0 < x < t, \\ 1, & x \ge t. \end{cases}$$

This is the so-called *rarefaction solution.*

2. Shocks Now we assume the initial condition takes the form

$$\phi(x) = \begin{cases} 1, & x < 0, \\ 0, & x \ge 0. \end{cases}$$

Since $\phi(x)$ is not increasing, characteristics will intersect. It is not hard to see that for any fixed $k \in (0, 1)$, the following function

$$u(t, x) = \begin{cases} 1, & x < kt, \\ 0, & x \ge kt. \end{cases}$$

is a solution, with singularity only on the curve $x = kt$.

Again, not all k corresponds to physical solution. The previous trick of smoothing ϕ no longer help. To determine the correct value of k, we need to understand the effect of collision, which is not quite modeled by this equation.

We will not dive deep into the theory at this moment, but we will mention two things.

First, the correct way of smoothing the PDE is to introduce the viscous term:

$$\partial_t u + u \cdot \partial_x u = \varepsilon \partial_{xx} u.$$

As we will see, the appearance of $\varepsilon \partial_{xx} u$ will make possible the existence of classical solution. The added term $\partial_{xx} u$ represents the friction force, a term ignored when deriving the Burgers equation but correctly handles the intersection of characteristics, the collision. By letting $\varepsilon \to 0$, one may get the unique physical solution.

Second, the correct answer is $k = 1/2$. The interface $x = kt$ is called shocks, where particle of velocity of 0 and 1 meet and stick together. Essentially by conservation of momentum, the shock will travel at velocity $\frac{1}{2}(1 + 0) = 1/2$, which gives the physically correct value of k.

A comprehensive study of the first-order transport equation needs a good understanding of the second-order diffusion equation. This should be a good motivation for the next section.

3 Heat equation

The heat equation takes the form

$$\begin{cases} \partial_t u = \Delta u, & t > 0, x \in \Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases}$$

plus some boundary condition. We will take this opportunity to introduce three basic types of boundary condition in the PDE theory. In the context of the heat equation, all these boundary conditions have concrete physical meaning.

Dirichlet boundary condition

$$u\big|_{\partial\Omega} = \mu.$$

This means that the temperature at the boundary is fixed, like a thermal bath or in the ice water.

Neumann boundary condition

$$\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0.$$

This models the insulation, where there is no heat flux across the boundary.

Mixed (Robin) boundary condition

$$-k\frac{\partial u}{\partial n} = H\big(u(t,x) - \mu(t,x)\big), \quad x \in \Omega, t > 0.$$

Physically the parameter k and H should be positive: the LHS is the heat flux across the boundary, the RHS is difference of the internal temperature and the surrounding temperature. In the limit $k \downarrow 0$, this converges to the Dirichlet boundary condition, where the heat transfer is instant and the internal and external temperature is identical. In the limit $H \downarrow 0$, there is no heat flux at the boundary and this is the Neumann boundary condition. A more general way to write the mixed boundary condition is

$$\alpha u + \beta\frac{\partial u}{\partial n} = \mu,$$

where $\alpha, \beta \in \mathbb{R}$.

Take the Dirichlet boundary condition as an example, we will present the definition of a *classical solution.*

Definition 3.1 *Let $\Omega \subset \mathbb{R}^d$ be a domain with continuous boundary. A classical solution to the PDE*

$$\begin{cases} \partial_t u = \Delta u, & t > 0,\ x \in \Omega, \\ u(t,x) = \mu(t,x), & t \geq 0,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega \end{cases}$$

is a function $u \in \mathcal{C}([0,\infty) \times \bar{\Omega}) \cap \mathcal{C}^{1,2}((0,\infty) \times \Omega)$ that satisfies the equation and the boundary/initial condition.

For Neumann and mixed boundary conditions, the domain should have $\mathcal{C}^1$-boundary in order to define the normal derivative $\partial u/\partial n$.

In this section we will focus on the following aspects of the heat equations, each of which will lead to a set of tools to study the equation:

- linear equation,
- Fourier transform,
- smoothing effect of Δ,
- maximum principle/energy method.

3.1 Principle of superposition

Consider the heat equation

$$\begin{cases} \partial_t u = \Delta u + f, & t > 0,\ x \in \Omega, \\ u(t,x) = 0, & t > 0,\ x \in \partial\Omega, \\ u(0,x) = g(x), & x \in \Omega. \end{cases} \tag{3.1}$$

We have the *principle of superposition* due to the linearity of the equation.

Theorem 3.1 *If $u_i \in \mathcal{C}([0,\infty) \times \bar{\Omega}) \cap \mathcal{C}^{1,2}((0,\infty) \times \Omega)$, $i = 1, 2$, are classical solutions to (3.1) with data (f_i, g_i), then $\alpha u_1 + \beta u_2$ is a classical solution to (3.1) with data $\big(\alpha f_1 + \beta f_2, \alpha g_1 + \beta g_2\big)$.*

Proof: It follows from the linearity of the operators ∂_t and Δ:

$$\partial_t(\alpha u_1 + \beta u_2) = \alpha \partial_t u_1 + \beta \partial_t u_2,$$
$$\Delta(\alpha u_1 + \beta u_2) = \alpha \Delta u_1 + \beta \Delta u_2.$$

□

3.2 Energy method