Lecture Note for Honor PDE

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1 Introduction

1.1 Derivation of PDEs

Many PDEs come from physcical models. Knowing these models help us understand the intuition behind the PDEs. In this section we illustrate how to derive some common PDEs from basic physical laws.

1.1.1 Transport equation

Let $u(t,x) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be the unknown function. The variable t is the time coordinate, and x is the space coordinate. The variable u can be the density of something, the velocity field, etc.

For illustration, suppose that we are modeling the traffic flow and $u(t,x)$ is the density of cars at (t,x). Let $a < b$. We first have the *conservation of mass* equation

$$\frac{d}{dt}\Big(\int_a^b u(t,x)\,dx\Big) = J(t,a) - J(t,b). \tag{1.1}$$

Here, the LHS is the rate of change of the total number of cars, and $J(t,x)$ is the *flux* at (t,x): the number of cars moving from the left of x to the right of x in unit time.

Assume that u and J is smooth enough, so that we can differientiate and interchange the order of differietiation and integration. Taking the t-derivative in (1.1) yields

$$\int_a^b \partial_t u(t,x)\,dx = J(t,a) - J(t,b) = -\int_a^b \partial_x \partial_x J(t,x)\,dx.$$

Then

$$\int_a^b \big[\partial_t u(t,x) + \partial_x J(t,x)\big]\,dx = 0.$$

Since a and b are arbitrary, and the integrand is a continuous function, we must have the relation

$$\partial_t u(t,x) + \partial_x J(t,x) = 0. \tag{1.2}$$

This is the differential form of (1.1).

Next, we need to relate J to u to elimate the unknown J in order to close the equation for u. Since u is the density, by the physical meaning of flux we have

$$J(t,x) = u \cdot V(t,x),$$

where $V(t,x)$ is the velocity field. It remains to determine how the velocity depends on the density; this may differ from one model from another. Here are some examples.

- $V(t,x) = \text{const}$. Then (1.2) reduces to

$$\partial_t u + c\partial_x u = 0.$$

One can check that the general solution is given by $u(t,x) = \phi(x - ct)$, that is, the initial density profile $\phi(\cdot)$ moves with constant speed c.

- $V(t,x) = 1 - u$. This is a more realistic model for the traffic jam: the velocity is decreasing as the density increases, and at maximum density $u = 1$ the traffic flow completely stops. The resulting equation is

  $$\partial_t u + \partial_x \big(u(1-u)\big) = 0 = \partial_t u + \partial_x u - 2u \cdot \partial_x u = 0.$$

  Although this equation seems simple, it is a nonlinear PDE and exhibits nontrivial behaviors such as formation of shocks.