HW4

November 12, 2025

Durrett: 2.4.3, 2.5.1, 2.5.3, 2.5.8, 3.2.1, 3.2.4, 3.2.6, 3.2.11, 3.2.13

Exercise 1 Let X_n, $n \geq 1$, be arbitrary r.v.'s on $(\Omega, \mathcal{F}, \mathsf{P})$ such that $\sum_{n=1}^{\infty} \pm X_n$ convergence P-a.s. for all choices of ± 1's. The goal is to show that $\sum_{n=1}^{\infty} X_n^2 < \infty$, a.s.

1. Let ξ_n be i.i.d. r.v.s on $(\Theta, \mathcal{G}, \mu)$ with $\mu(\xi_n = \pm 1) = \frac{1}{2}$. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathsf{P}}) = (\Omega \times \Theta, \mathcal{F} \otimes \mathcal{G}, \mathsf{P} \times \mu)$ be the product space. Using Fubini's theorem, show that

$$\tilde{\mathsf{P}}\left(\left\{(\omega, \theta) : \sum_{n=1}^{\infty} \xi_n(\theta) X_n(\omega) \text{ converges}\right\}\right) = 1,$$

   and hence for P-a.e. ω, $\sum_{n=1}^{\infty} \xi_n(\theta) X_n(\omega)$ converges for μ-a.e. θ.

2. Using Kolmogorov's one-series theorem on $(\Theta, \mathcal{G}, \mu)$ to conclude that for those ω in part 1,

$$\sum_{n=1}^{\infty} |X_n(\omega)|^2 = 2 \sum_{n=1}^{\infty} \operatorname*{Var}_{\theta}(\xi_n X_n)^2 := 2 \sum_{n=1}^{\infty} \int |\xi_n(\theta) X_n(\omega)|^2 \mu(d\theta) < \infty.$$