

Lecture Note for Honor PDE

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1 Introduction

1.1 Derivation of PDEs

Many PDEs come from physical models. Knowing these models help us understand the intuition behind the PDEs. In this section we illustrate how to derive some common PDEs from basic physical laws.

1.1.1 Transport equation

Let $u(t, x) : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ be the unknown function. The variable t is the time coordinate, and x is the space coordinate. The variable u can be the density of something, the velocity field, etc.

For illustration, suppose that we are modeling the traffic flow and $u(t, x)$ is the density of cars at (t, x) . Let $a < b$. We first have the *conservation of mass* equation

$$\frac{d}{dt} \left(\int_a^b u(t, x) dx \right) = J(t, a) - J(t, b). \quad (1.1)$$

Here, the LHS is the rate of change of the total number of cars, and $J(t, x)$ is the *flux* at (t, x) : the number of cars moving from the left of x to the right of x in unit time.

Assume that u and J is smooth enough, so that we can differentiate and interchange the order of differentiation and integration. Taking the t -derivative in (1.1) yields

$$\int_a^b \partial_t u(t, x) dx = J(t, a) - J(t, b) = - \int_a^b \partial_x J(t, x) dx.$$

Then

$$\int_a^b [\partial_t u(t, x) + \partial_x J(t, x)] dx = 0.$$

Since a and b are arbitrary, and the integrand is a continuous function, we must have the relation

$$\partial_t u(t, x) + \partial_x J(t, x) = 0. \quad (1.2)$$

This is the differential form of (1.1).

Next, we need to relate J to u to eliminate the unknown J in order to close the equation for u . Since u is the density, by the physical meaning of flux we have

$$J(t, x) = u \cdot V(t, x),$$

where $V(t, x)$ is the velocity field. It remains to determine how the velocity depends on the density; this may differ from one model from another. Here are some examples.

- $V(t, x) = \text{const.}$ Then (1.2) reduces to

$$\partial_t u + c \partial_x u = 0.$$

One can check that the general solution is given by $u(t, x) = \phi(x - ct)$, that is, the initial density profile $\phi(\cdot)$ moves with constant speed c .

- $V(t, x) = 1 - u$. This is a more realistic model for the traffic jam: the velocity is decreasing as the density increases, and at maximum density $u = 1$ the traffic flow completely stops. The resulting equation is

$$\partial_t u + \partial_x (u(1 - u)) = 0 = \partial_t u + \partial_x u - 2u \cdot \partial_x u = 0.$$

Although this equation seems simple, it is a nonlinear PDE and exhibits nontrivial behaviors such as formation of shocks.

We have the general form of the transport equation

$$\partial_t u + \partial_x (uV(u)) = 0, \tag{1.3}$$

where $V : \mathbb{R} \rightarrow \mathbb{R}$ is a function that depends on the model.

We can further generalize (1.3) to dimension $d > 1$. We first guess the form of the equation by matching the dimension, and then we will derive it rigourously using the conservation of mass.

Since u is the density, it is a multi-variate function

$$u(t, x) : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}.$$

Since V gives the velocity, so V and $J = u \cdot V$ must be vector functions:

$$V(x) : \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad J(t, x) = u \cdot V : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}^d.$$

Looking the LHS of (1.3), $\partial_t u$ takes value in \mathbb{R} , so the differential operator must turn $J(t, x)$ into a function that maps \mathbb{R}^d to \mathbb{R} . The only such operator is the divergence operator $\nabla \cdot$ acting on a vector function $f = (f_1, \dots, f_d)$

$$\nabla \cdot f = \nabla \cdot (f_1, \dots, f_d) := \sum_{i=1}^d \frac{\partial}{\partial x_i} f_i.$$

Hence, we obtain a reasonable guess of the generalization of (1.3) in an arbitrary dimension $d > 1$:

$$\partial_t u + \nabla \cdot (uV(u)) = 0, \tag{1.4}$$

where $u : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$ is the unknown function and $V : \mathbb{R} \rightarrow \mathbb{R}^d$ is a given function depending on the model.

Next, we give a rigourous derivation using the conservation of mass. The key tool is the *Divergence Theorem*.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^d$ be a domain with piecewise \mathcal{C}^1 -boundary. Let $F : \bar{\Omega} \rightarrow \mathbb{R}^d$ be \mathcal{C}^1 . Then*

$$\int_{\partial\Omega} F \cdot \vec{n} dS = \int_{\Omega} \nabla \cdot F dx, \tag{1.5}$$

where dS denotes the surface element on $\partial\Omega$, and \vec{n} is the outer unit normal vector on $\partial\Omega$.

(1.5) is also referred to as the *Stokes formula*, or simply integration by parts, since from right to left a differential operator is removed.

Now let Ω be an arbitrary \mathcal{C}^1 -domain in \mathbb{R}^d . The total mass in Ω is given by $\int_{\Omega} u(t, x) dx$. By the conservation of mass, the rate of change of the total mass is a consequence of the flux of mass across of the boundary. Thus we have

$$\frac{d}{dt} \int_{\Omega} u(t, x) dx = - \int_{\Omega} J \cdot \vec{n} dS.$$

To double check the RHS: if the direction of the flux is tangent to the boundary at some point, that is $J \cdot \vec{n} = 0$, then there is no mass escaping from this point, justifying the form of the integrand. Also, if the flux J is point outwards and has the same direction as \vec{n} , this will result in a decrease of the mass, and hence the minus sign on the RHS.

Assuming u is smooth enough so that the order of differentiation and integration can be exchanged on the LHS, and using [Theorem 1.1](#) on the RHS, we obtain

$$\int_{\Omega} (\partial_t u + \nabla \cdot J) dx = 0.$$

Since this holds for an arbitrary \mathcal{C}^1 -domain, we have pointwise

$$\partial_t u + \nabla \cdot J = 0. \tag{1.6}$$

Plugging in $J = uV(u)$ we obtain [\(1.4\)](#).

For the last step we use the following simple result.

Lemma 1.2 • If $f \in \mathcal{C}(\mathbb{R}^d)$ and $\int_{\Omega} f(x) dx = 0$ for any rectangle $\Omega \subset \mathbb{R}^d$, then $f \equiv 0$.

• If $f \in L^1_{\text{loc}}(\mathbb{R})$ and $\int_{\Omega} f(x) dx = 0$ for any rectangle $\Omega \subset \mathbb{R}^d$, then $f = 0$ almost everywhere.

As we will see, the transport equation may develop singularity no matter how smooth the initial condition is, so [\(1.4\)](#) may not hold for every point, but it is at least safe to say that it holds almost everywhere.

1.2 heat equation

In [\(1.6\)](#), we can also interpret the function u as the temperature and J as the heat flow; then [\(1.6\)](#) is a consequence of another conservation law — *the conservation of energy*, as verified by Joule's experiment. To close the equation, we need to relate J to u . Fourier's law states that the heat flow is negatively proportional to the gradient of the temperature field, that is,

$$J = -c \nabla u,$$

where the constant c is the *heat conductivity*. Here, the gradient operator ∇ is defined by

$$(\nabla f)(x_1, \dots, x_d) = (\partial_{x_1} f(x_1, \dots, x_d), \dots, \partial_{x_d} f(x_1, \dots, x_d)).$$

Combined with [\(1.6\)](#), we obtain

$$\partial_t u = -\nabla \cdot (-c \nabla u) = \sum_{i=1}^d \partial_{x_i x_i} u =: c \Delta u. \tag{1.7}$$

The operator Δ is called the Laplacian operator. (1.7) is called the *heat equation*. Usually we set $c = 1$.

The heat equation also models the phenomenon of *diffusion*. Let u be the concentration of some substance in the fluid (which is essentially like the density). The particles of this substance will move under external force, but even without any external force, the principle of diffusion says that the particles will move from higher concentration to lower concentration. More specifically, *Fick's law* states that the J is again negatively proportional to the gradient of u , and hence the diffusion is modeled by the heat equation as well.

The heat equation is a second-order PDE since it involves second partial derivatives. It is classified as *parabolic equation* since the time derivative is only first order, analogous to the *parabola equation* $t = x^2$.

1.3 wave equation

The wave equation models the phenomenon of waves in elastic mechanics. Let Ω be a domain, say, representing an elastic object like a string, a stick, a membrane or so on. For simplicity we take $\Omega = (a, b)$. The unknown function $u(t, x) : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ under consideration is the displacement of the object from its equilibrium position. By Newton's second law, we have

$$\partial_{tt}u(t, x) = F(t, x),$$

where $F(t, x)$ is the force exerted upon the location x . To calculate the force at (t, x) , we recall the Hooke's law for a spring, which says the elastic force is negatively proportional to the displacement from the equilibrium position:

$$F = -k\Delta L.$$

We can imagine there are two small springs on $(x - \Delta x, x)$ and $(x, x + \Delta x)$. The force at (t, x) is a combination of the elastic force from these two small springs. Using the Hooke's law we have

$$F(t, x) \approx F_1 + F_2 = -k(u(t, x) - u(t, x - \Delta x)) - k(u(t, x) - u(t, x + \Delta x)) \approx k(\Delta x)^2 \partial_{xx}u(t, x).$$

Combining all these and assuming $k(\Delta x)^2 \rightarrow c$ as $\Delta x \rightarrow 0$, we obtain the *wave equation*

$$\partial_{tt}u = c\partial_{xx}u.$$

The wave equation in dimension $d > 1$ can be derived in a similar way, or be guessed as

$$\partial_{tt}u = c\Delta u.$$

It is classified as the *hyperbolic equation* since both the time and space derivatives are of second order and have the opposite signs, which resembles the *hyperbola equation* $t^2 = x^2$.

1.4 Laplace equation

Consider the heat equation in a domain Ω , with boundary condition $u|_{\partial\Omega} = \varphi$ and initial condition $u|_{t=0} = u_0$. Physically, if the temperature is fixed at the boundary, eventually the temperature field will reach equilibrium, that is, there is $u_* : \Omega \rightarrow \mathbb{R}$ such that $u(t, x) \rightarrow u_*(x)$ as $t \rightarrow \infty$, where u_∞ may or may not depend on u_0 . Since $v(t, x) = u_*(x)$ also satisfies the heat equation as it is the equilibrium, we obtain

$$\Delta u_* = 0, \quad u_*|_{\partial\Omega} = \varphi. \quad (1.8)$$

This is the *Laplace equation* for the obvious reason. It is classified as the *elliptic equation* since all second derivatives have the same sign, resembling the ellipse equation $ax^2 + by^2 = 1$.

Next we will derive the Laplace equation from *calculus of variation*, which is a powerful tool to derive PDEs. We consider the following optimization problem

$$\inf_{u|_{\partial\Omega}=\varphi} \int_{\Omega} |\nabla u|^2(x) dx =: \inf_{u|_{\partial\Omega}=\varphi} I[u]. \quad (1.9)$$

The square bracket $[\cdot]$ stresses that I is a *functional*, that is, a function of functions. Assume that u_* achieves the minimum of (1.9), that is,

$$I[u_*] = \min_{u|_{\partial\Omega}=\varphi} I[u].$$

Intuitively, $I[u]$ is the L^2 -norm of the heat flow corresponding to the temperature field u , and if the L^2 -norm is minimized, the temperature is at the equilibrium.

Assuming u_* is the minimum function, let us derive conditions that u_* must satisfy. Let $v \in C_0^\infty(\Omega)$ be arbitrary. We introduce perturbation of u_* as

$$u_\varepsilon = u_* + \varepsilon v, \quad \varepsilon \in \mathbb{R}$$

Since v vanishes at $\partial\Omega$, the function u_ε satisfies the boundary condition. Let $f(\varepsilon) = I[u_\varepsilon]$. Since f achieves minimum at $\varepsilon = 0$, we must have $f'(0) = 0$. We do not know if f is actually differentiable, but assuming that all functions are nice, we can compute formally:

$$0 = f'(0) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \int_{\Omega} |\nabla u_* + \varepsilon \nabla v|^2 dx = 2 \int_{\Omega} \nabla u_* \cdot \nabla v dx = 0. \quad (1.10)$$

To proceed, we use the following useful integration-by-part formula.

Lemma 1.3 *Let Ω be a C^1 -domain and $u, v \in C^1(\bar{\Omega}) \cap C^2(\Omega)$. Then*

$$\int_{\Omega} \nabla u \cdot \nabla v dx = - \int_{\Omega} u \Delta v dx + \int_{\partial\Omega} u \frac{\partial v}{\partial n} dS, \quad (1.11)$$

$$\int_{\Omega} \nabla u \cdot \nabla v dx = - \int_{\Omega} v \Delta u dx + \int_{\partial\Omega} v \frac{\partial u}{\partial n} dS, \quad (1.12)$$

$$\int_{\Omega} u \Delta v - v \Delta u dx = \int_{\partial\Omega} u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} dS. \quad (1.13)$$

Proof: The last identity follows from taking difference of the first two. Since the roles of u and v are symmetric, it suffices to prove (1.11). Indeed, consider the vector function $F = u \nabla v : \Omega \rightarrow \mathbb{R}^d$. Then $\nabla \cdot F = \nabla u \cdot \nabla v + u \Delta v$ and $F \cdot \vec{n} = u \frac{\partial v}{\partial n}$. Applying Theorem 1.1 to F yields the desired conclusion. \square

Using Lemma 1.3, we can continue with (1.10) to obtain

$$0 = \int_{\Omega} \nabla u_* \cdot \nabla v dx = - \int_{\Omega} v \Delta u_* dx, \quad \forall v \in C_0^\infty(\Omega). \quad (1.14)$$

There is no boundary term after integration by parts since v vanishes at the boundary. Since (1.14) holds for arbitrary $v \in C_0^\infty(\Omega)$, a variant of Lemma 1.2 implies that $\Delta u_* = 0$ pointwise assuming its continuity. Hence we derive the Laplace equation again.

As another example, the variational problem

$$\inf_{u|_{\partial\Omega}=\varphi} \int_{\Omega} \sqrt{1 + |\nabla u|^2} dx$$

gives rise the *minimal surface equation*. This will be left as an exercise.

1.5 *Viscous Burgers equation and fluid equation

Let us consider the velocity field $u(t, x)$ of particles moving on \mathbb{R} . By Newton's law, we have

$$\text{acceleration of particles at } (t_0, x_0) = \text{friction} + \text{external force}. \quad (1.15)$$

First, let us express the acceleration field from the velocity field. The naive guess $\partial_t u$ is wrong, since the particles at position x are not the same for different t . To get the correct form of the acceleration, we must follow a fixed particle. Let $x(t)$ be the trajectory of the particle passing (t_0, x_0) (that is, $x(t_0) = x_0$). Then by definition

$$\dot{x}(t) = u(t, x(t)).$$

Hence,

$$\ddot{x}(t) = \frac{d}{dt}u(t, x(t)) = \partial_t u + \dot{x} \cdot \partial_x u = \partial_t u + u \cdot \partial_x u. \quad (1.16)$$

This gives the LHS of (1.15).

For the RHS, first, the friction force is modeled by $\partial_{xx}u$. To understand why second derivative appears, it suffices to note that if $\partial_{xx}u = 0$, then u is linear and there is no friction in the shear transform. Last, the external force is modeled by an arbitrary function $f(t, x)$. Combining all these, we obtain the full *viscous Burgers equation*:

$$\partial_t u + u \partial_x u = \partial_{xx}u + f(t, x).$$

Its multi-dimensional analogue is

$$\partial_t u + u \cdot \nabla u = \Delta u + f(t, x), \quad u : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}.$$

The Burgers equation is a mixture of the “transport term” $u \partial_x u$ and the “diffusion term” $\partial_{xx}u$.

The Burgers equation is a toy model for fluid dynamics. Here we also mention the celebrated Navier-Stokes equation, and by now we can understand the physical meaning of all the terms in the equation. Assuming the fluid is incompressible (meaning the density is constant), the Navier-Stokes equation reduced to an equation of the velocity field: $u(t, x) : \mathbb{R}_+ \times \mathbb{R}^d, d = 2, 3$,

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \Delta u + f, \\ \nabla \cdot u = 0. \end{cases}$$

Here, we recognize the material derivative term $\partial_t u + u \cdot \nabla u$, which is the acceleration field. All the other are forcing terms: the pressure term ∇p , the friction Δu , and the external force f . The divergence-free constraint comes from conservation of mass:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \implies \nabla \cdot u = 0.$$

Although the pressure p is also unknown and first equation seems under-determined, the divergence-free constraint can in fact eliminate the pressure term in the first equation.

1.6 *Maxwell equation

In this section we briefly look at the Maxwell's equation that models the electro-magnetic field. The unknowns are the electric field E and the magnetic field B , both are vector functions on \mathbb{R}^3 . All of the four equations can be written down in the differential form and in the integral form.

- Gauss's law:

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}, \quad \int_{\partial\Omega} E \cdot \vec{n} dS = \frac{1}{\varepsilon_0} \int_{\Omega} \rho dx.$$

Here, ρ is the electric charge density, ε_0 is a physical constant, and Ω is an arbitrary domain.

- Gauss's law for magnetism:

$$\nabla \cdot B = 0, \quad \int_{\partial\Omega} B \cdot \vec{n} dS.$$

- Faraday's equation (electric generated from a changing magnetic field):

$$\nabla \times E = -\frac{\partial B}{\partial t}, \quad \oint_{\partial\Sigma} E \cdot d\ell = - \int_{\Sigma} \frac{\partial B}{\partial t} \cdot dA,$$

where Σ is any surface.

- Ampère's circuital law (magnetic field generated by currents):

$$\nabla \times B = \mu_0(J + \varepsilon_0 \frac{\partial E}{\partial t}), \quad \oint_{\partial\Sigma} B \cdot d\ell = \int_{\Sigma} \mu_0(J + \varepsilon_0 \frac{\partial E}{\partial t}) \cdot dA,$$

where J is the current.

It is well-known that electro-magnetic field related to waves. To see this from the equation, we consider the vacuum case where $\rho = J \equiv 0$. Then we have

$$\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \Delta E = -\Delta E = -\frac{\partial}{\partial t}(\nabla \times B) = -\mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2},$$

so E satisfies the wave equation, where the wave speed (i.e., the light speed) is $c = \sqrt{\mu_0 \varepsilon_0}$. A similar calculation yields a wave equation for B .

1.7 key questions in this course

In this course we will focus on four elementary PDEs, which model four basic physical phenomenon and are the building blocks of more complicated PDEs:

- the transport equation $\partial_t u + \partial_x V(u) = f$;
- the Laplace equation $\Delta u = f$;
- the heat equation $\partial_t u = \Delta u$;
- the wave equation $\partial_{tt} u = \Delta u$.

One part of the course is devoted to how to write down solutions of the PDEs, using techniques like Fourier analysis, separation of variables and etc. A more important part is to develop *well-posedness* theory without an explicit form of the solution. The well-posedness theory is three-fold:

- existence of solution, including suitable conditions on the boundary and initial condition, regularity requirement;
- uniqueness of solution
- stability: how sensitive the solution is to initial and boundary data.

For a rigorous well-posedness theory we must be accurate about the solution space. A key concept is the *classical solution*, where all the derivatives appearing in the PDE are continuous function so that the PDEs make sense pointwise. When there are both time and space derivative, we use $\mathcal{C}^{\alpha,\beta}$ to indicate the space of functions that has α -th order continuous derivative in t and β -th order continuous derivative in space. For example, classical solutions of the first order transport equation live in $\mathcal{C}^{1,1}$, for the heat equation $\mathcal{C}^{1,2}$, and for the wave equation $\mathcal{C}^{2,2}$. We may also spend some time discussing how to define weak solutions, solutions that have a lower regularity requirement.

2 First-order transport equation

In this section we study the first-order transport equation:

$$\begin{cases} \partial_t u + b(t, x, u) \cdot \partial_x u = f(t, x, u), \\ u(0, x) = \phi(x), \end{cases} \quad u(t, x) : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}. \quad (2.1)$$

2.1 Method of characteristics

2.1.1 Constant b

Suppose $b(t, x, u) = V$ is a constant and $f \equiv 0$. The equation becomes

$$\partial_t u + V \partial_x u = 0, \quad u(0, x) = \phi(x).$$

We introduce $U(t) = u(t, x_0 + Vt)$ where u is a solution and $x_0 \in \mathbb{R}$ is fixed. Then

$$\dot{U}(t) = \partial_t u(t, x_0 + Vt) + V \partial_x u(t, x_0 + Vt) = 0.$$

Hence,

$$U(t) \equiv U(0) = u(0, x_0) = \phi(x_0),$$

and we have

$$u(t, x) = \phi(x - Vt).$$

The curves $\eta(t) = x_0 + Vt$ are called *characteristics*. Intuitively, the initial data ϕ is propagating along these curves.

As a remark, if $\phi \in \mathcal{C}^1$, then $u = \phi(x - Vt) \in \mathcal{C}^{1,1}$ is a classical solution. But even $\phi \notin \mathcal{C}^1$, this is still the only plausible solution to the PDE. Indeed, even for this simplest PDE, dealing with non-classical solutions is already inevitable.

2.2 A non-homogeneous example

We consider the following equation:

$$\begin{cases} \partial_t u + x \partial_x u = u + x, \\ u(0, x) = \phi(x). \end{cases} \quad (2.2)$$

We are seeking characteristics $\eta(t)$ so that

$$U(t) = u(t, \eta(t))$$

solves a simple ODE. We clearly have

$$\dot{\eta}(t) = \eta \implies \eta(t) = C_1 e^t.$$

Plugging into U , we have

$$\dot{U}(t) = U(t) + C_1 e^t.$$

The general solution for this ODE is

$$U(t) = C_1 t e^t + C_2 e^t.$$

Finally, we need to determine the constants C_1 and C_2 . We have

$$\eta(t) = C_1 e^t = x, \quad U(0) = C_2 = \phi(C_1) \implies C_1 = x e^{-t}, \quad C_2 = \phi(x e^{-t}).$$

Hence, the solution to the PDE is

$$u = xt + \phi(x e^{-t}) e^t.$$

One can check by direct computation that it indeed solves the original PDE.

2.3 General linear case

We consider the general linear case

$$\begin{cases} \partial_t u + b(t, x) \partial_x u = f(t, x, u), \\ u(0, x) = \phi(x). \end{cases} \quad (2.3)$$

We state a well-posedness result.

Theorem 2.1 *Assume that $b \in \mathcal{C}^{0,1}$, $\phi \in \mathcal{C}^1$ and $f \in \mathcal{C}^{0,1,1}$. Then there exists a unique solution to (2.3).*

Proof: We consider the characteristic ODE

$$\dot{\eta}(t) = b(t, \eta(t)), \quad \eta(0) = x_0.$$

Since b is Lipschitz in x , by standard ODE theory, there is a unique solution for every initial condition x_0 . Moreover, the solution map

$$\Phi_t : x_0 \mapsto \eta(t; x_0)$$

is a \mathcal{C}^1 -diffeomorphism of \mathbb{R} , that is, both Φ_t and Φ_t^{-1} are in \mathcal{C}^1 . For example, Φ'_t satisfies the ODE

$$\frac{d}{dt}(\Phi'_t) = \partial_x b(t, \Phi(t)) \Phi'_t, \quad \Phi'_0 = 1.$$

□