

HW2

October 30, 2025

Durrett: 2.2.2, 2.2.4, 2.2.7, 2.3.8, 2.3.9, 2.4.2

In the next two problems we use λ to denote the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Exercise 1 Let $X_n, n \geq 1$, be r.v.s on $(\Omega, \mathcal{F}, \mathbf{P})$ with $\mathbf{E}|X_n| < \infty$. Let \mathcal{A} be all subsets of $\mathbb{N} = \{1, 2, \dots\}$, and μ be the counting measure on $(\mathbb{N}, \mathcal{A})$, that is,

$$\mu(A) = \text{number of elements in } A, \quad A \in \mathcal{A}.$$

On the product space $(\Omega \times \mathbb{N}, \mathcal{F} \otimes \mathcal{A}, \mathbf{P} \times \mu)$, show that,

1. the map $\mathbf{X}(\omega, n) = X_n(\omega)$ is $(\mathcal{F} \otimes \mathcal{A})$ -measurable;
2. if $\sum_{n=1}^{\infty} \mathbf{E}|X_n| < \infty$, then

$$\int_{\Omega \times \mathbb{N}} \mathbf{X}(\omega, n) (\mathbf{P} \times \mu)(d\omega dn) = \int_{\Omega} \sum_{n=1}^{\infty} X_n(\omega) \mathbf{P}(d\omega) = \sum_{n=1}^{\infty} \int_{\Omega} X_n(\omega) \mathbf{P}(d\omega).$$

Exercise 2 Let $X \geq 0$ be a r.v. on $(\Omega, \mathcal{F}, \mathbf{P})$. On the product space $(\Omega \times \mathbb{R}, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}), \mathbf{P} \times \lambda)$, show that,

1. $\{(\omega, y) : 0 \leq y \leq X(\omega)\} \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R})$;
Hint: the map $(x, y) \mapsto x - y$ is measurable;
2. the following equality holds:

$$\int_{\Omega} X(\omega) \mathbf{P}(d\omega) = (\mathbf{P} \times \lambda)(\{(\omega, y) : 0 \leq y \leq X(\omega)\}) = \int_0^{\infty} \mathbf{P}(X \geq y) \lambda(dy).$$

Conclude that

$$\mathbf{E}X \leq \sum_{n=0}^{\infty} \mathbf{P}(X \geq n).$$

Exercise 3 Use Fubini's Theorem to show that if $X \geq 0$ and $p > 0$, then

$$\mathbf{E}X^p = \int_0^{\infty} py^{p-1} \mathbf{P}(X > y) dy.$$

Hint: $x^p = \int_0^x py^{p-1} dy, x \geq 0$.

Exercise 4 Let $(X_n)_{n \geq 1}$ be independent. Show that

$$\sup_n X_n < \infty, \text{ a.s.} \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} \mathbb{P}(X_n > A) < \infty \text{ for some } A > 0.$$

Hint: for the “ \Leftarrow ” direction use the first Borel–Cantelli; for the “ \Rightarrow ” direction, use proof by contradiction and show $\sup_n X_n \geq A$ a.s. for every A using the second Borel–Cantelli when the condition does not hold.

Exercise 5 Recall from the “St. Petersburg game” we have the i.i.d. r.v.’s $(X_n)_{n \geq 1}$ with distribution

$$\mathbb{P}(X_1 = 2^j) = 2^{-j}, \quad j \geq 1.$$

1. Show that for every $M > 0$, $\sum_{n=1}^{\infty} \mathbb{P}(X_n \geq Mn \log_2 n) = \infty$.
2. Show that for every $M > 0$, $\mathbb{P}(\{\omega : X_n(\omega) \geq Mn \log_2 n, \text{ i.o.}\}) = 1$.
3. Show that for every $M > 0$, $\limsup_{n \rightarrow \infty} \frac{X_n}{n \log_2 n} \geq M$ a.s.
4. Show that $\limsup_{n \rightarrow \infty} \frac{S_n}{n \log_2 n} = \limsup_{n \rightarrow \infty} \frac{X_n}{n \log_2 n} = \infty$ a.s.