Formulation of the pull-out problem using index notation

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1. Background

The Python package NumPy provides very useful tools for scientific computing, such as the multidimensional array and summation tool over spatial dimensions. Using these tools, mathematic problems formulated with index notation can be efficiently and consistently implemented. We use the pull-out problem as a demonstrative example. The pull-out test is a common procedure to determine the property of the bond interface in composites. In pull-out tests, the reinforcements are pulled out from the matrix as shown in Fig. 1. Let us consider the following simple case for some insight into the mathematic background of the pull-out problems. For thin specimens made of cementitious composites, the shear deformation in matrix can generally be neglected, thus the problem can be simplified to a one dimensional problem as shown in Fig. 2. The reinforcement and the matrix are coupled through the bond interface in which the shear stress τ is characterized as a function of the slip s.

1.1. Constitutive laws

The constitutive laws of the reinforcement and the matrix are given as

$$\sigma_{\mathrm{f}} = D_{\mathrm{f}}(\varepsilon_{\mathrm{f}}) \tag{1}$$

$$\sigma_{\mathrm{m}} = D_{\mathrm{m}}(\varepsilon_{\mathrm{m}}), \tag{2}$$

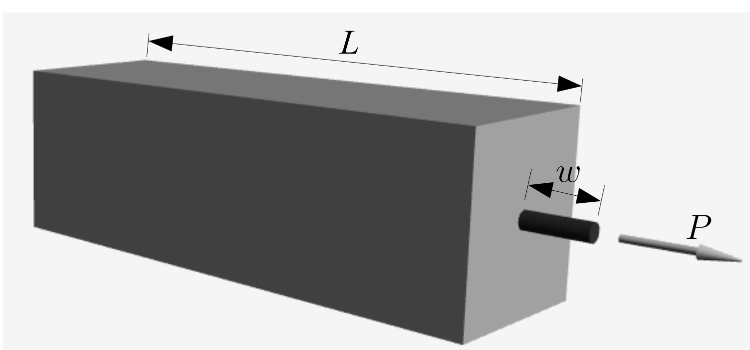

Figure 1: Schematic of typical pull-out tests

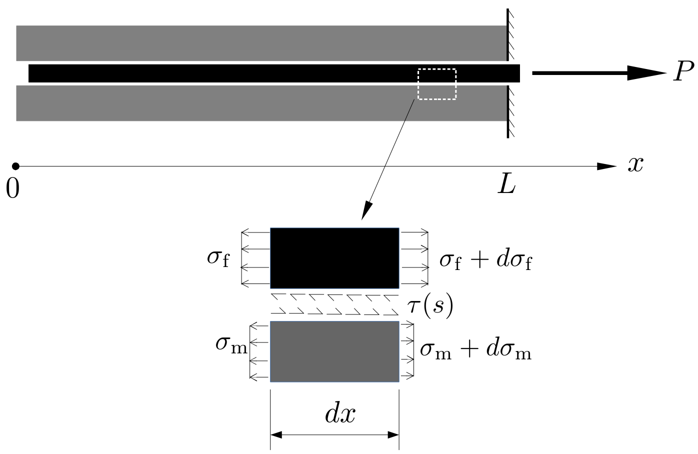

Figure 2: A simplified mechanical model of the pull-out problem

where σ_f and σ_m are the reinforcement stress and the matrix stress, ε_f and ε_m are the reinforcement strain and the matrix strain, respectively. And the constitutive law of the bond interface reads

$$\tau = \tau(s). \tag{3}$$

1.2. Kinematic

The strains in the matrix and reinforcement are given as

$$\varepsilon_\mathrm{f} = u_\mathrm{f,x} \tag{4}$$

$$\varepsilon_\mathrm{m} = u_\mathrm{m,x}, \tag{5}$$

where the index $(.)_{,x}$ denotes the derivative with respect to the spatial coordinate x, u_f and u_m are the reinforcement and matrix displacements, respectively. The relative slip in the bond interface can be expressed as

$$s = u_\mathrm{f} - u_\mathrm{m}. \tag{6}$$

1.3. Equilibrium

The equilibrium of an infinitesimal segment of the reinforcement leads to

$$A_\mathrm{f}\sigma_{\mathrm{f},x} - p\tau(s) = 0, \tag{7}$$

where A_f and p are the cross-sectional area and the perimeter of the reinforcement, receptively. Similarly, the equilibrium of the matrix can be expressed as

$$A_\mathrm{m}\sigma_{\mathrm{m},x} + p\tau(s) = 0, \tag{8}$$

in which A_m is the cross-sectional area of the matrix.

2. Weak formulation and finite element discretization of the pull-out problem using index notation

2.1. Mappings

In order to use the index notation we replace the subscripts $(.)_{\mathrm{m}}$ and $(.)_{\mathrm{f}}$ with the numbers 1 and 2. In the present case of the pull-out problem, index 1 represents the matrix m and index 2 the fibers f, thus the equilibrium can be rewritten as

$$\begin{aligned} \sigma(u_1)_{1,x} - \tau(u_2 - u_1) = 0 \\ \tau(u_2 - u_1) + \sigma(u_2)_{2,x} = 0 \end{aligned} \tag{9}$$

$$u_1(\sigma(u_1)_{1,x} - \tau(u_2 - u_1)) + u_2(\tau(u_2 - u_1) + \sigma(u_2)_{2,x}) = 0$$

$$\begin{aligned} u_1(\sigma(u_1)_{1,x}) + u_2(\sigma(u_2)_{2,x}) + \\ u_1(-\tau(u_2 - u_1)) + u_2(\tau(u_2 - u_1)) = 0 \end{aligned} \tag{10}$$

2.2. Weak formulation of the multi-layer problem

In order to provide a more general model of the pull-out problem defined by the differential Eqs. (7) and (8) we construct a weak form of the boundary value problem within the domain $\Omega := [0, L]$ shown in Fig. 2 . The corresponding essential and natural boundary conditions are specified as

$$u_c \quad = \quad \bar{u}_c(\theta) \text{ on } \Gamma_{u_c} \quad \text{and} \quad \sigma_c A_c \quad = \quad \bar{t}_c(\theta) \text{ on } \Gamma_{t_c} \tag{11}$$

Denoting the integration of the product of the terms u, v over V as $(u, v)_V$, the weak formulation can be expressed as

$$(\delta u_c, A_c\sigma_{c,x} + (-1)^c\, p_c\tau)_\Omega + (\delta u_c, u_c - \bar{u}_c(\theta))_{\Gamma_{u_c}} + (\delta u_c, -A_c\sigma_c + \bar{t}_c(\theta))_{\Gamma_{t_c}} = 0 \tag{12}$$

Since $\delta u_c = 0$ on Γ_{u_c}, the second and fifth terms in Eq. (12) vanish. Using integration by parts, the orders of the the stress derivatives $\sigma_{c,x}$ can be reduced as follows

$$(\delta u_c, A_c\sigma_{c,x})_\Omega = -\left(\delta u_{c,x}, A_c\sigma_c\right)_\Omega + (\delta u_c, A_c\sigma_c)_\Gamma \tag{13}$$

Finally, by substituting Eq. (13) for both matrix and fibers into Eq. (12), the following variational formulation of the pull-out problem is obtained

$$(\delta u_{c,x}, A_c\sigma_c)_\Omega + (\delta u_c\,(-1)^c\,, p_c\tau)_\Omega - (\delta u_c, \bar{t}_c(\theta))_{\Gamma_{t_c}} = 0. \tag{14}$$

Note that the term $\delta u_c(-1)^c = \delta(u_2 - u_1) = \delta s$ renders the virtual slip between the components.

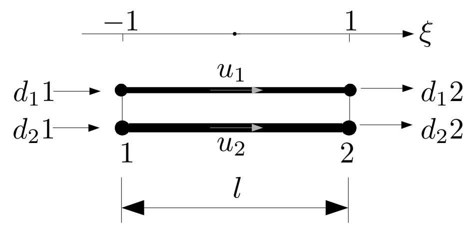

Figure 3: The two node element with bond interface

2.3. Finite element discretization

The finite element used here is shown in Fig. reffig:element. The element has two nodes and two independent displacement fields u_{f} and u_{m}. The displacement field of a material component $c = 1 \ldots N_c$ for N_c number of components is approximated using the shape functions

$$\begin{aligned} u_c &= N_I \; d_{cI} & u_{ce} &= N_i d_{cI(e,i)} \\ \delta u_c &= N_I \; \delta d_{cI} & \delta u_{ce} &= N_i \delta d_{cI(e,i)} \end{aligned} \tag{15}$$

with index $I = 1 \ldots N_d$ representing a displacement degree of freedom. The shape functions are given as

$$N_1 = \frac{1}{2}(1 - \xi), \tag{16}$$

$$N_2 = \frac{1}{2}(1 + \xi). \tag{17}$$

The strain field in each material component is then expressed as

$$u_{c,x} = B_I \; d_{cI} \tag{18}$$

$$\delta u_{c,x} = B_I \; \delta d_{cI} \tag{19}$$

where $B_I = N_{I,x} = N_{I,\xi} J(\xi)^{-1}$. As a consequence, the slip field is approximated as

$$s = u_2 - u_1 = u_c \, (-1)^c = (-1)^c \, N_I d_{cI}. \tag{20}$$

Eq. (14) can be rewritten as

$$\delta d_{cI} \, (B_I, A_c \sigma_c)_\Omega + \delta d_{cI} \, ((-1)^c \, N_I, p_c \tau)_\Omega - \delta d_{cI} \, (N_I, \bar{t}_c(\theta))_{\Gamma_{t_c}} = 0, \tag{21}$$

Since δd_{cI} is arbitrary, and $(N_I, \bar{t}_c)_{\Gamma_{t_c}}$ can be simplified to nodal loads as $\bar{t}_{cI}$ Eq. (21) can be reduced to

$$R_{cI}(\theta) = (B_I, A_c \sigma_c)_\Omega + ((-1)^c \, N_I, p_c \tau)_\Omega - \bar{t}_{cI}(\theta) = 0. \tag{22}$$

2.4. Iterative solution algorithm

In case of nonlinear material behavior assumed either for the matrix, reinforcement or bond, Eq. (22) must be prepared for iterative solution strategies by means of linearization i.e. by Taylor expansion neglecting quadratic and higher order terms. The expansion up to the linear term reads

$$R^{(\theta)}_{cI}(d^{(k)}_{dJ}) \approx R^{(\theta)}_{cI}(d^{(k-1)}_{dJ}) + \left. \frac{\partial R_{cI}}{\partial d_{dJ}} \right|_{d^{(k-1)}_{dJ}} \Delta d^{(k)}_{dJ}. \tag{23}$$

Assuming nonlinear material behavior of the components c, the derivative of the residual R_{cI} with respect to d_{cI} reads

$$\frac{\partial \sigma_c}{\partial d_{dJ}} = \frac{\partial \sigma_c}{\partial \varepsilon_d}\frac{\partial \varepsilon_d}{\partial d_{dJ}} = \frac{\partial \sigma_c}{\partial \varepsilon_d} B_J \tag{24}$$

Similarly, for nonlinear bond behavior, the derivative of shear flow with respect to

$$\frac{\partial \tau}{\partial d_{dJ}} = \frac{\partial \tau}{\partial s}\frac{\partial s}{\partial d_{dJ}} = \frac{\partial \tau}{\partial s} N_J \left(-1\right)^d \tag{25}$$

The derivative of the residual then reads

$$\left.\frac{\partial R_{cI}}{\partial d_{dJ}}\right|_{d_{dJ}^{k-1}} = K_{cIdJ}^{(k-1)} = A_c \int_\Omega B_I B_J \left.\frac{\partial \sigma_c}{\partial \varepsilon_d}\right|_{d_{dJ}^{k-1}} \mathrm{d}x \tag{26}$$

$$+ \left(-1\right)^{c+d} p_c \int_\Omega N_I N_J \left.\frac{\partial \tau}{\partial s}\right|_{d_{dJ}^{k-1}} \mathrm{d}x \tag{27}$$

Assuming a linear elastic constitutive law of the components the derivatives of σ_c with respect to ε_d read

$$\frac{\partial \sigma_c}{\partial \varepsilon_d} = \delta_{cd} E_c \tag{28}$$

and linear bond leads to

$$\frac{\partial \tau}{\partial s} = G. \tag{29}$$

Finally, substituting Eqs. (26) to (28) into Eq. (23), the following incremental form of equilibrium is obtained,

$$K_{cIdJ}^{(k-1)} \Delta d_{dJ}^{(k)} = R_{cI}^{(\theta)}(d_{dJ}^{(k-1)}), \tag{30}$$

in which the tangential stiffness matrix can be written as

$$K_{cIdJ}^{(k-1)} = \left[A_c B_{I(e,i)M(e,m)} B_{J(e,j)M(e,m)} \left.\frac{\partial \sigma_c}{\partial \varepsilon_d}\right|_{d_{dJ}^{k-1}} \right. \tag{31}$$

$$\left. + \left(-1\right)^{c+d} p N_{I(e,i)M(e,m)} N_{J(e,j)M(e,m)} \left.\frac{\partial \tau}{\partial s}\right|_{d_{dJ}^{k-1}} \right] w_{M(e,m)} \left|J\right|_{M(e,m)}. \tag{32}$$