From System T to Continuation-Passing Style

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1 Syntax and typing definitions

$x,\ y$	variables		
i	integer literals		
$\tau,\ \sigma$	$::=$		types
	$\mid$ $\mathbb{Z}$		
	$\mid$ $\texttt{void}$		
	$\mid$ $\tau_1 \to \tau_2$		
	$\mid$ (τ)	S	
e	$::=$		annotated terms
	$\mid$ u^τ		
$u,\ v$	$::=$		raw terms
	$\mid$ x		
	$\mid$ i		
	$\mid$ $\lambda x : \tau.e$	bind x in e	
	$\mid$ $e_1\ e_2$		
	$\mid$ $e_1\ p\ e_2$		
	$\mid$ $\mathsf{if0}(e_1, e_2, e_3)$		
	$\mid$ $\mathsf{let}\ x = v\ \mathsf{in}\ u$	bind x in u	
	$\mid$ $\mathsf{halt}\ [\tau] e$		
	$\mid$ (u)	S	
p	$::=$		primitives
	$\mid$ $+$		
	$\mid$ $-$		
Γ	$::=$		contexts
	$\mid$		
	$\mid$ $\Gamma, x : \tau$		

$\boxed{\Gamma \vdash_{\mathrm{T}} e : \tau}$ annotated typing

$$\frac{\Gamma \vdash_{\mathrm{T}} u : \tau}{\Gamma \vdash_{\mathrm{T}} u^\tau : \tau} \quad \textsc{T\_ant\_ann}$$

$\boxed{\Gamma \vdash_{\mathrm{T}} u : \tau}$ typing

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash_{\mathrm{T}} x : \tau} \quad \textsc{T\_term\_var}$$

$$\frac{}{\Gamma \vdash_{\mathrm{T}} i : \mathbb{Z}} \quad \textsc{T\_term\_int}$$

$$\frac{\Gamma, x_1 : \tau_1 \vdash_{\mathrm{T}} e : \tau_2}{\Gamma \vdash_{\mathrm{T}} \lambda x_1 : \tau_1.e : \tau_1 \to \tau_2} \quad \textsc{T\_term\_lam}$$

$$\frac{\Gamma \vdash_{\mathrm{T}} e_1 : \tau_1 \to \tau_2 \quad \Gamma \vdash_{\mathrm{T}} e_2 : \tau_1}{\Gamma \vdash_{\mathrm{T}} e_1\, e_2 : \tau_2} \quad \text{T\_TERM\_APP}$$

$$\frac{\Gamma \vdash_{\mathrm{T}} e_1 : \mathbb{Z} \quad \Gamma \vdash_{\mathrm{T}} e_2 : \mathbb{Z}}{\Gamma \vdash_{\mathrm{T}} e_1\, p\, e_2 : \mathbb{Z}} \quad \text{T\_TERM\_PRIM}$$

$$\frac{\Gamma \vdash_{\mathrm{T}} e_1 : \mathbb{Z} \quad \Gamma \vdash_{\mathrm{T}} e_2 : \tau \quad \Gamma \vdash_{\mathrm{T}} e_3 : \tau}{\Gamma \vdash_{\mathrm{T}} \mathsf{if0}(e_1, e_2, e_3) : \tau} \quad \text{T\_TERM\_IF0}$$

$\boxed{\Gamma \vdash_{\mathrm{K}} e : \tau}$ annotated typing

$$\frac{\Gamma \vdash_{\mathrm{K}} u : \tau}{\Gamma \vdash_{\mathrm{K}} u^{\tau} : \tau} \quad \text{K\_ANT\_ANN}$$

$\boxed{\Gamma \vdash_{\mathrm{K}} u : \tau}$ typing

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash_{\mathrm{K}} x : \tau} \quad \text{K\_TERM\_VAR}$$

$$\frac{}{\Gamma \vdash_{\mathrm{K}} i : \mathbb{Z}} \quad \text{K\_TERM\_INT}$$

$$\frac{\Gamma, x : \tau_1 \vdash_{\mathrm{K}} e : \tau_2}{\Gamma \vdash_{\mathrm{K}} \lambda x : \tau_1 . e : \tau_1 \to \tau_2} \quad \text{K\_TERM\_LAM}$$

$$\frac{\Gamma \vdash_{\mathrm{K}} v : \tau \quad \Gamma, x : \tau \vdash_{\mathrm{K}} u : \mathtt{void}}{\Gamma \vdash_{\mathrm{K}} \mathsf{let}\, x = v \,\mathsf{in}\, u : \mathtt{void}} \quad \text{K\_TERM\_LET}$$

$$\frac{\Gamma \vdash_{\mathrm{K}} e_1 : \mathbb{Z} \quad \Gamma \vdash_{\mathrm{K}} e_2 : \mathbb{Z}}{\Gamma \vdash_{\mathrm{K}} e_1\, p\, e_2 : \mathbb{Z}} \quad \text{K\_TERM\_PRIM}$$

$$\frac{\Gamma \vdash_{\mathrm{K}} e' : \tau_1 \to \tau_2 \quad \Gamma \vdash_{\mathrm{K}} e : \tau}{\Gamma \vdash_{\mathrm{K}} e'\, e : \tau_2} \quad \text{K\_TERM\_APP}$$

$$\frac{\Gamma \vdash_{\mathrm{K}} e : \mathbb{Z} \quad \Gamma \vdash_{\mathrm{K}} e_1 : \mathtt{void} \quad \Gamma \vdash_{\mathrm{K}} e_2 : \mathtt{void}}{\Gamma \vdash_{\mathrm{K}} \mathsf{if0}(e, e_1, e_2) : \mathtt{void}} \quad \text{K\_TERM\_IF0}$$

$$\frac{\Gamma \vdash_{\mathrm{K}} e : \tau}{\Gamma \vdash_{\mathrm{K}} \mathsf{halt}\,[\tau] e : \mathtt{void}} \quad \text{K\_TERM\_HALT}$$

2 Translation

2.1 Type translation

$$\begin{aligned}
\mathcal{K}[\![\mathbb{Z}]\!] &\triangleq \mathbb{Z} \\
\mathcal{K}[\![\tau_1 \to \tau_2]\!] &\triangleq \mathcal{K}[\![\tau_1]\!] \to \mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!] \to \mathtt{void} \\
\mathcal{K}_{\mathtt{cont}}[\![\tau]\!] &\triangleq \mathcal{K}[\![\tau]\!] \to \mathtt{void}
\end{aligned}$$

2.2 Program translation

$$
\begin{array}{lll}
\mathcal{K}_{\mathsf{prog}}[\![u^\tau]\!] & \triangleq & \mathcal{K}_{\mathsf{exp}}[\![u^\tau]\!](\lambda x : \mathcal{K}[\![\tau]\!].\mathsf{halt}[\mathcal{K}[\![\tau]\!]]x^{\mathcal{K}[\![\tau]\!]})^{\mathcal{K}_{\mathsf{cont}}[\![\tau]\!]} \\
\mathcal{K}_{\mathsf{exp}}[\![y^\tau]\!]k & \triangleq & k(y^{\mathcal{K}[\![\tau]\!]}) \\
\mathcal{K}_{\mathsf{exp}}[\![i^\tau]\!]k & \triangleq & k(i^{\mathcal{K}[\![\tau]\!]}) \\
\mathcal{K}_{\mathsf{exp}}[\![(\lambda x_1 : \tau_1.u_2^{\tau_2})^\tau]\!]k & \triangleq & k((\lambda x : \mathcal{K}[\![\tau_1]\!].\lambda c : \mathcal{K}_{\mathsf{cont}}[\![\tau_2]\!]. \\
 & & \quad \mathcal{K}_{\mathsf{exp}}[\![u_2^{t_2}]\!]c^{\mathcal{K}_{\mathsf{cont}}[\![\tau_2]\!]})^{\mathcal{K}[\![\tau]\!]}) \\
\mathcal{K}_{\mathsf{exp}}[\![(u_1^{\tau_1}u_2^{\tau_2})^\tau]\!]k & \triangleq & \mathcal{K}_{\mathsf{exp}}[\![u_1^{\tau_1}]\!](\lambda x_1 : \mathcal{K}[\![\tau_1]\!]. \\
 & & \quad \mathcal{K}_{\mathsf{exp}}[\![u_2^{\tau_2}]\!](\lambda x_2 : \mathcal{K}[\![\tau_2]\!]. \\
 & & \quad\quad x_1^{\mathcal{K}[\![\tau_1]\!]}x_2^{\mathcal{K}[\![\tau_2]\!]}k)^{\mathcal{K}_{\mathsf{cont}}[\![\tau_2]\!]})^{\mathcal{K}_{\mathsf{cont}}[\![\tau_1]\!]} \\
\mathcal{K}_{\mathsf{exp}}[\![(e_1\ p\ e_2)^\tau]\!]k & \triangleq & \mathcal{K}_{\mathsf{exp}}[\![e_1]\!](\lambda x_1 : \mathbb{Z}. \\
 & & \quad \mathcal{K}_{\mathsf{exp}}[\![e_2]\!](\lambda x_2 : \mathbb{Z}. \\
 & & \quad\quad \mathsf{let}\ y = x_1\ p\ x_2\ \mathsf{in}\ k(y^{\mathbb{Z}}))^{\mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!]})^{\mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!]} \\
\mathcal{K}_{\mathsf{exp}}[\![\mathsf{if0}(e_1, e_2, e_3)^\tau]\!]k & \triangleq & \mathcal{K}_{\mathsf{exp}}[\![e_1]\!](\lambda x : \mathbb{Z}. \\
 & & \quad \mathsf{if0}(x^{\mathbb{Z}}, \mathcal{K}_{\mathsf{exp}}[\![e_2]\!]k, \mathcal{K}_{\mathsf{exp}}[\![e_3]\!]k))^{\mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!]}
\end{array}
$$

3 Type correctness

3.1 Terms

Lemma 1. $\Gamma \vdash_{\mathrm{T}} e : \tau \implies \mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} k : \mathcal{K}_{\mathsf{cont}}[\![\tau]\!] \implies \mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{exp}}[\![e]\!]k : \mathtt{void}$

Proof. T_ANT_ANN is the only typing derivation of the hypothesis. Therefore,

$$e = u^\tau \tag{1}$$

$$\Gamma \vdash_{\mathrm{T}} u^\tau : \tau \tag{2}$$

We must show

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} k : \mathcal{K}_{\mathsf{cont}}[\![\tau]\!] \implies \mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{exp}}[\![u^\tau]\!]k : \mathtt{void} \tag{3}$$

By induction on a derivation of typing judgement (2):

1. If the last rule in the derivation is T_TERM_VAR, then

$$u = x \tag{4}$$

$$\Gamma(x) = \tau \tag{5}$$

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} k : \mathcal{K}[\![\tau]\!] \to \mathtt{void} \tag{6}$$

We must show

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{exp}}[\![x^\tau]\!]k : \mathtt{void} \tag{7}$$

i.e.

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} k(x^{\mathcal{K}[\![\tau]\!]}) : \mathtt{void} \tag{8}$$

By K_TERM_APP and typing judgement (6), this follows from

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} x^{\mathcal{K}[\![\tau]\!]} : \mathcal{K}[\![\tau]\!] \tag{9}$$

By K_ANT_ANN, this follows from

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} x : \mathcal{K}[\![\tau]\!] \tag{10}$$

By K_TERM_VAR, this follows from

$$\mathcal{K} \circ \Gamma(x) = \mathcal{K}[\![\tau]\!] \tag{11}$$

which is immediate from typing judgement (5).

2. If the last rule in the derivation is T_TERM_INT

   We have

$$u = i \tag{12}$$

$$\tau = \mathbb{Z} \tag{13}$$

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} k : \mathbb{Z} \to \mathtt{void} \tag{14}$$

   We must show

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \mathcal{K}_{\mathtt{exp}}[\![i^{\mathbb{Z}}]\!]k : \mathtt{void} \tag{15}$$

   i.e.

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} k(i^{\mathcal{K}[\![\mathbb{Z}]\!]}) : \mathtt{void} \tag{16}$$

   By K_TERM_APP and typing judgement (14), this follows from

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} i^{\mathcal{K}[\![\mathbb{Z}]\!]} : \mathbb{Z} \tag{17}$$

   By K_ANT_ANN, this follows from

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} i : \mathbb{Z} \tag{18}$$

   which is immediate from K_TERM_INT.

3. If the last rule in the derivation is T_TERM_LAM, then

$$u = \lambda x : \tau_1.e_2 \tag{19}$$

$$\tau = \tau_1 \to \tau_2 \tag{20}$$

$$\Gamma, x : \tau_1 \vdash_{\mathrm{T}} e_2 : \tau_2 \tag{21}$$

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} k : \mathcal{K}[\![\tau_1 \to \tau_2]\!] \to \mathtt{void} \tag{22}$$

   T_ANT_ANN is the only derivation of typing judgement (21). Therefore,

$$e_2 = u_2^{\tau_2} \tag{23}$$

$$\Gamma, x : \tau_1 \vdash_{\mathrm{T}} u_2 : \tau_2 \tag{24}$$

   We must show

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \mathcal{K}_{\mathtt{exp}}[\![\lambda x : \tau_1.u_2^{\tau_2}]\!]k : \mathtt{void} \tag{25}$$

   i.e.

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} k((\lambda x : \mathcal{K}[\![\tau_1]\!].\lambda c : \mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!].\mathcal{K}_{\mathtt{exp}}[\![u_2^{\tau_2}]\!]c^{\mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!]})^{\mathcal{K}[\![\tau_1 \to \tau_2]\!]}) : \mathtt{void} \tag{26}$$

   By K_TERM_APP and typing judgement (22), this follows from

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} (\lambda x : \mathcal{K}[\![\tau_1]\!].\lambda c : \mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!].\mathcal{K}_{\mathtt{exp}}[\![u_2^{\tau_2}]\!]c^{\mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!]})^{\mathcal{K}[\![\tau_1 \to \tau_2]\!]} : \mathcal{K}[\![\tau_1 \to \tau_2]\!] \tag{27}$$

   By K_ANT_ANN, this follows from

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \lambda x : \mathcal{K}[\![\tau_1]\!].\lambda c : \mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!].\mathcal{K}_{\mathtt{exp}}[\![u_2^{\tau_2}]\!]c^{\mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!]} : \mathcal{K}[\![\tau_1]\!] \to \mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!] \to \mathtt{void} \tag{28}$$

   By K_TERM_LAM, this follows from

$$\mathcal{K} \circ \Gamma, x : \mathcal{K}[\![\tau_1]\!] \vdash_{\mathrm{K}} \lambda c : \mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!].\mathcal{K}_{\mathtt{exp}}[\![u_2^{\tau_2}]\!]c^{\mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!]} : \mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!] \to \mathtt{void} \tag{29}$$

   which follows

$$\mathcal{K} \circ \Gamma, x : \mathcal{K}[\![\tau_1]\!], c : \mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!] \vdash_{\mathrm{K}} \mathcal{K}_{\mathtt{exp}}[\![u_2^{\tau_2}]\!]c^{\mathcal{K}_{\mathtt{cont}}[\![\tau_2]\!]} : \mathtt{void} \tag{30}$$

   which is immediate from K_ANT_ANN, K_TERM_VAR and the induction hypothesis.

4. If the last rule in the derivation is T_TERM_APP, then

$$u = e_1 e_2 \tag{31}$$

$$\Gamma \vdash_{\mathrm{T}} e_1 : \tau_1 \to \tau \tag{32}$$

$$\Gamma \vdash_{\mathrm{T}} e_2 : \tau_1 \tag{33}$$

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} k : \mathcal{K}_{\texttt{cont}}[\![\tau]\!] \tag{34}$$

T_ANT_ANN is the only derivation of typing judgements (32)(33). Therefore,

$$e_1 = u_1^{\tau_1 \to \tau} \tag{35}$$

$$e_2 = u_2^{\tau_1} \tag{36}$$

$$\Gamma \vdash_{\mathrm{T}} u_1 : \tau_1 \to \tau \tag{37}$$

$$\Gamma \vdash_{\mathrm{T}} u_2 : \tau_1 \tag{38}$$

We must show

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \mathcal{K}_{\texttt{exp}}[\![(u_1^{\tau_1 \to \tau} u_2^{\tau_1})^{\tau}]\!]k : \texttt{void} \tag{39}$$

i.e.

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \mathcal{K}_{\texttt{exp}}[\![{u_1}^{\tau_1 \to \tau}]\!](\lambda x_1 : \mathcal{K}[\![\tau_1 \to \tau]\!].\mathcal{K}_{\texttt{exp}}[\![u_2^{\tau_1}]\!](\lambda x_2 : \mathcal{K}[\![\tau_1]\!]. \\ x_1^{\mathcal{K}[\![\tau_1 \to \tau]\!]} x_2^{\mathcal{K}[\![\tau_1]\!]} k)^{\mathcal{K}_{\texttt{cont}}[\![\tau_1]\!]})^{\mathcal{K}_{\texttt{cont}}[\![\tau_1 \to \tau]\!]} : \texttt{void} \tag{40}$$

By the induction hypothesis, this follows from

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} (\lambda x_1 : \mathcal{K}[\![\tau_1 \to \tau]\!].\mathcal{K}_{\texttt{exp}}[\![u_2^{\tau_1}]\!](\lambda x_2 : \mathcal{K}[\![\tau_1]\!]. \\ x_1^{\mathcal{K}[\![\tau_1 \to \tau]\!]} x_2^{\mathcal{K}[\![\tau_1]\!]} k)^{\mathcal{K}_{\texttt{cont}}[\![\tau_1]\!]})^{\mathcal{K}_{\texttt{cont}}[\![\tau_1 \to \tau]\!]} : \mathcal{K}_{\texttt{cont}}[\![\tau_1 \to \tau]\!] \tag{41}$$

By K_ANT_ANN, this follows from

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \lambda u_1 : \mathcal{K}[\![\tau_1 \to \tau]\!].\mathcal{K}_{\texttt{exp}}[\![u_2^{\tau_1}]\!](\lambda x_2 : \mathcal{K}[\![\tau_1]\!].u_1^{\mathcal{K}[\![\tau_1 \to \tau]\!]} x_2^{\mathcal{K}[\![\tau_1]\!]} k)^{\mathcal{K}_{\texttt{cont}}[\![\tau_1]\!]} : \mathcal{K}[\![\tau_1 \to \tau]\!] \to \texttt{void} \tag{42}$$

By K_TERM_LAM, this follows from

$$\mathcal{K} \circ \Gamma, x_1 : \mathcal{K}[\![\tau_1 \to \tau]\!] \vdash_{\mathrm{K}} \mathcal{K}_{\texttt{exp}}[\![u_2^{\tau_1}]\!](\lambda x_2 : \mathcal{K}[\![\tau_1]\!].x_1^{\mathcal{K}[\![\tau_1 \to \tau]\!]} x_2^{\mathcal{K}[\![\tau_1]\!]} k)^{\mathcal{K}_{\texttt{cont}}[\![\tau_1]\!]} : \texttt{void} \tag{43}$$

By the induction hypothesis, We must show

$$\mathcal{K} \circ \Gamma, x_1 : \mathcal{K}[\![\tau_1 \to \tau]\!] \vdash_{\mathrm{K}} (\lambda x_2 : \mathcal{K}[\![\tau_1]\!].x_1^{\mathcal{K}[\![\tau_1 \to \tau]\!]} x_2^{\mathcal{K}[\![\tau_1]\!]} k)^{\mathcal{K}_{\texttt{cont}}[\![\tau_1]\!]} : \mathcal{K}_{\texttt{cont}}[\![\tau_1]\!] \tag{44}$$

By K_ANT_ANN, this follows from

$$\mathcal{K} \circ \Gamma, x_1 : \mathcal{K}[\![\tau_1 \to \tau]\!] \vdash_{\mathrm{K}} \lambda x_2 : \mathcal{K}[\![\tau_1]\!].x_1^{\mathcal{K}[\![\tau_1 \to \tau]\!]} x_2^{\mathcal{K}[\![\tau_1]\!]} k : \mathcal{K}[\![\tau_1]\!] \to \texttt{void} \tag{45}$$

By K_TERM_LAM, this follows from

$$\mathcal{K} \circ \Gamma, x_1 : \mathcal{K}[\![\tau_1 \to \tau]\!], x_2 : \mathcal{K}[\![\tau_1]\!] \vdash_{\mathrm{K}} x_1^{\mathcal{K}[\![\tau_1 \to \tau]\!]} x_2^{\mathcal{K}[\![\tau_1]\!]} k : \texttt{void} \tag{46}$$

which is immediate from K_TERM_APP, K_ANT_ANN, K_TERM_VAR and typing judgement (34).

5. If the last rule in the derivation is T_TERM_PRIM, then

$$u = e_1 \; p \; e_2 \tag{47}$$

$$\tau = \mathbb{Z} \tag{48}$$

$$\Gamma \vdash_{\mathrm{T}} e_1 : \mathbb{Z} \tag{49}$$

$$\Gamma \vdash_{\mathrm{T}} e_2 : \mathbb{Z} \tag{50}$$

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} k : \mathbb{Z} \to \texttt{void} \tag{51}$$

T_ANT_ANN is the only derivation of typing judgement (49)(50). Therefore,

$$e_1 = u_1^{\mathbb{Z}} \tag{52}$$

$$e_2 = u_2^{\mathbb{Z}} \tag{53}$$

$$\Gamma \vdash_{\mathrm{T}} u_1 : \mathbb{Z} \tag{54}$$

$$\Gamma \vdash_{\mathrm{T}} u_2 : \mathbb{Z} \tag{55}$$

We must show

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{exp}}[\![(u_1^{\mathbb{Z}}\ p\ u_2^{\mathbb{Z}})^{\mathbb{Z}}]\!]k : \mathtt{void} \tag{56}$$

i.e.

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{exp}}[\![u_1^{\mathbb{Z}}]\!](\lambda x_1 : \mathbb{Z}.\mathcal{K}_{\mathsf{exp}}[\![u_2^{\mathbb{Z}}]\!](\lambda x_2 : \mathbb{Z}.\mathsf{let}\ y = x_1\ p\ x_2\ \mathsf{in}\ k(y^{\mathbb{Z}}))^{\mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!]})^{\mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!]} : \mathtt{void} \tag{57}$$

By the induction hypothesis, this follows from

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} (\lambda x_1 : \mathbb{Z}.\mathcal{K}_{\mathsf{exp}}[\![u_2]\!](\lambda x_2 : \mathbb{Z}.\mathsf{let}\ y = x_1\ p\ x_2\ \mathsf{in}\ k(y^{\mathbb{Z}}))^{\mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!]})^{\mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!]} : \mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!] \tag{58}$$

By K_ANT_ANN, this follows from

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \lambda x_1 : \mathbb{Z}.\mathcal{K}_{\mathsf{exp}}[\![u_2]\!](\lambda x_2 : \mathbb{Z}.\mathsf{let}\ y = x_1\ p\ x_2\ \mathsf{in}\ k(y^{\mathbb{Z}}))^{\mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!]} : \mathbb{Z} \to \mathtt{void} \tag{59}$$

By K_TERM_LAM, this follows from

$$\mathcal{K} \circ \Gamma, x_1 : \mathbb{Z} \vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{exp}}[\![u_2]\!](\lambda x_2 : \mathbb{Z}.\mathsf{let}\ y = x_1\ p\ x_2\ \mathsf{in}\ k(y^{\mathbb{Z}}))^{\mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!]} : \mathtt{void} \tag{60}$$

By the induction hypothesis, this follows from

$$\mathcal{K} \circ \Gamma, x_1 : \mathbb{Z} \vdash_{\mathrm{K}} (\lambda u_2 : \mathbb{Z}.\mathsf{let}\ y = x_1\ p\ u_2\ \mathsf{in}\ k(y^{\mathbb{Z}}))^{\mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!]} : \mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!] \tag{61}$$

By K_ANT_ANN, this follows from

$$\mathcal{K} \circ \Gamma, x_1 : \mathbb{Z} \vdash_{\mathrm{K}} \lambda x_2 : \mathbb{Z}.\mathsf{let}\ y = x_1\ p\ x_2\ \mathsf{in}\ k(y^{\mathbb{Z}}) : \mathbb{Z} \to \mathtt{void} \tag{62}$$

By K_TERM_LAM, this follows from

$$\mathcal{K} \circ \Gamma, x_1 : \mathbb{Z}, x_2 : \mathbb{Z} \vdash_{\mathrm{K}} \mathsf{let}\ y = x_1\ p\ x_2\ \mathsf{in}\ k(y^{\mathbb{Z}}) : \mathtt{void} \tag{63}$$

By K_ANT_LET, K_ANT_PRIM and K_ANT_VAR, We must show

$$\mathcal{K} \circ \Gamma, x_1 : \mathbb{Z}, x_2 : \mathbb{Z}, y : \mathbb{Z} \vdash_{\mathrm{K}} k(y^{\mathbb{Z}}) : \mathtt{void} \tag{64}$$

By K_TERM_APP and typing judgement (51), this follows from

$$\mathcal{K} \circ \Gamma, x_1 : \mathbb{Z}, x_2 : \mathbb{Z}, y : \mathbb{Z} \vdash_{\mathrm{K}} y^{\mathbb{Z}} : \mathbb{Z} \tag{65}$$

which is immediate from K_ANT_ANN and K_ANT_VAR.

6. If the last rule in the derivation is T_TERM_IF0, then

$$u = \mathsf{if0}(e_1, e_2, e_3) \tag{66}$$

$$\Gamma \vdash_{\mathrm{T}} e_1 : \mathbb{Z} \tag{67}$$

$$\Gamma \vdash_{\mathrm{T}} e_2 : \tau \tag{68}$$

$$\Gamma \vdash_{\mathrm{T}} e_3 : \tau \tag{69}$$

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} k : \mathcal{K}_{\mathsf{cont}}[\![\tau]\!] \tag{70}$$

T_ANT_ANN is only constructor of typing judgements (67)(68)(69). Therefore,

$$e_1 = u_1^{\mathbb{Z}} \tag{71}$$

$$e_2 = u_2^{\tau} \tag{72}$$

$$e_3 = u_3^\tau \tag{73}$$

$$\Gamma \vdash_{\mathrm{T}} u_1 : \mathbb{Z} \tag{74}$$

$$\Gamma \vdash_{\mathrm{T}} u_2 : \tau \tag{75}$$

$$\Gamma \vdash_{\mathrm{T}} u_3 : \tau \tag{76}$$

We must show

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{exp}}[\![\mathsf{if0}(u_1^{\mathbb{Z}}, u_2^\tau, u_3^\tau)]\!]k : \mathtt{void} \tag{77}$$

i.e.

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{exp}}[\![u_1^{\mathbb{Z}}]\!](\lambda x : \mathbb{Z}.\mathsf{if0}(x^{\mathbb{Z}}, \mathcal{K}_{\mathsf{exp}}[\![u_2^\tau]\!]k, \mathcal{K}_{\mathsf{exp}}[\![u_3^\tau]\!]k))^{\mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!]} : \mathtt{void} \tag{78}$$

By the induction hypothesis, We must show

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} (\lambda x : \mathbb{Z}.\mathsf{if0}(x^{\mathbb{Z}}, \mathcal{K}_{\mathsf{exp}}[\![u_2^\tau]\!]k, \mathcal{K}_{\mathsf{exp}}[\![u_3^\tau]\!]k))^{\mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!]} : \mathcal{K}_{\mathsf{cont}}[\![\mathbb{Z}]\!] \tag{79}$$

By K_ANT_ANN, this follows from

$$\mathcal{K} \circ \Gamma \vdash_{\mathrm{K}} \lambda x : \mathbb{Z}.\mathsf{if0}(x^{\mathbb{Z}}, \mathcal{K}_{\mathsf{exp}}[\![u_2^\tau]\!]k, \mathcal{K}_{\mathsf{exp}}[\![u_3^\tau]\!]k) : \mathbb{Z} \to \mathtt{void} \tag{80}$$

By K_TERM_LAM, this follows from

$$\mathcal{K} \circ \Gamma, x : \mathbb{Z} \vdash_{\mathrm{K}} \mathsf{if0}(x^{\mathbb{Z}}, \mathcal{K}_{\mathsf{exp}}[\![u_2^\tau]\!]k, \mathcal{K}_{\mathsf{exp}}[\![u_3^\tau]\!]k) : \mathtt{void} \tag{81}$$

By K_TERM_IF0, this follows from

$$\mathcal{K} \circ \Gamma, x : \mathbb{Z} \vdash_{\mathrm{K}} x^{\mathbb{Z}} : \mathbb{Z} \tag{82}$$

$$\mathcal{K} \circ \Gamma, x : \mathbb{Z} \vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{exp}}[\![u_2^\tau]\!]k : \mathtt{void} \tag{83}$$

$$\mathcal{K} \circ \Gamma, x : \mathbb{Z} \vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{exp}}[\![u_3^\tau]\!]k : \mathtt{void} \tag{84}$$

in which (82) is immediate from K_ANT_ANN and K_TERM_VAR. (83)(84) come from the induction hypotheses, K_ANT_ANN and typing judgements (70)(75)(76).

□

3.2 Programs

Theorem 1. $\vdash_{\mathrm{T}} e : \tau \implies \vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{prog}}[\![e]\!] : \mathtt{void}$

Proof. T_ANT_ANN is the only typing derivation for the hypothesis. Therefore,

$$e = u^\tau \tag{85}$$

$$\vdash_{\mathrm{T}} u^\tau : \tau \tag{86}$$

We must show

$$\vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{prog}}[\![u^\tau]\!] : \mathtt{void} \tag{87}$$

i.e.

$$\vdash_{\mathrm{K}} \mathcal{K}_{\mathsf{exp}}[\![u^\tau]\!](\lambda x : \mathcal{K}[\![\tau]\!].\mathsf{halt}[\mathcal{K}[\![\tau]\!]]x^{\mathcal{K}[\![\tau]\!]})^{\mathcal{K}_{\mathsf{cont}}[\![\tau]\!]} : \mathtt{void} \tag{88}$$

By Lemma 1, this follows from

$$\vdash_{\mathrm{K}} (\lambda x : \mathcal{K}[\![\tau]\!].\mathsf{halt}[\mathcal{K}[\![\tau]\!]]x^{\mathcal{K}[\![\tau]\!]})^{\mathcal{K}_{\mathsf{cont}}[\![\tau]\!]} : \mathcal{K}_{\mathsf{cont}}[\![\tau]\!] \tag{89}$$

By K_ANT_ANN, this follows from

$$\vdash_{\mathrm{K}} \lambda x : \mathcal{K}[\![\tau]\!].\mathsf{halt}[\mathcal{K}[\![\tau]\!]]x^{\mathcal{K}[\![\tau]\!]} : \mathcal{K}[\![\tau]\!] \to \mathtt{void} \tag{90}$$

By K_TERM_LAM, this follows from

$$x : \mathcal{K}[\![\tau]\!] \vdash_{\mathrm{K}} \mathsf{halt}[\mathcal{K}[\![\tau]\!]]x^{\mathcal{K}[\![\tau]\!]} : \mathtt{void} \tag{91}$$

By K_TERM_HALT, this follows from

$$x : \mathcal{K}[\![\tau]\!] \vdash_{\mathrm{K}} x^{\mathcal{K}[\![\tau]\!]} : \mathcal{K}[\![\tau]\!] \tag{92}$$

which is immediate from K_ANT_ANN and K_TERM_VAR. □