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# 1 Building Mathematical Models

## 1.1 a

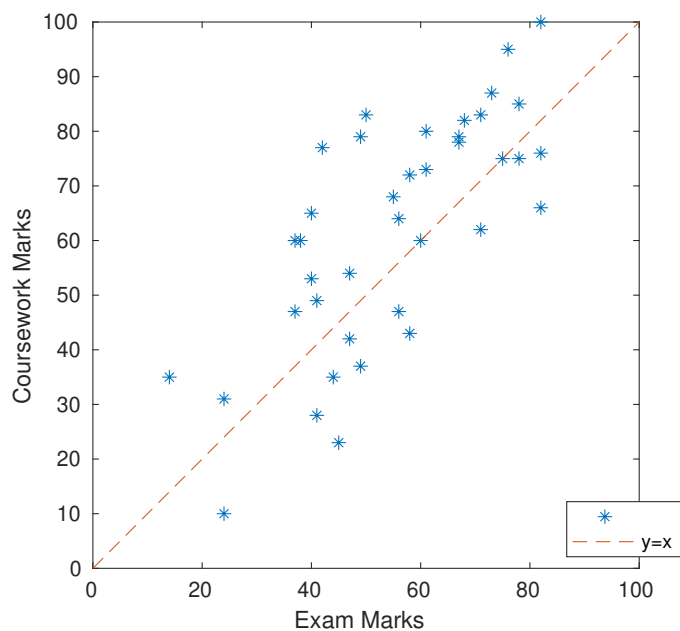


Figure 1: Coursework marks plotted against exam marks. Note the orange dashed line of symmetry

In Figure 1 the line of symmetry,  $x = y$ , was plotted. Because the data are reasonably spread around the line of symmetry, there is evidence for some proportionality between the two data sets.

Listing 1: Topic 1. Question a. Note all code per topic belongs to same file

```
1 figure,plot(examMarks,courseworkMarks, '*');
2 xlabel('Exam Marks'); ylabel('Coursework Marks')
3 axis square; axis ([0 100 0 100])
4 hold on; plot(0:100,0:100, '--'); hold off;
5 legend('', 'y=x', 'Location', 'best');
6 print('eps/topic1_a.eps', '-depsc')
```

## 1.2 b

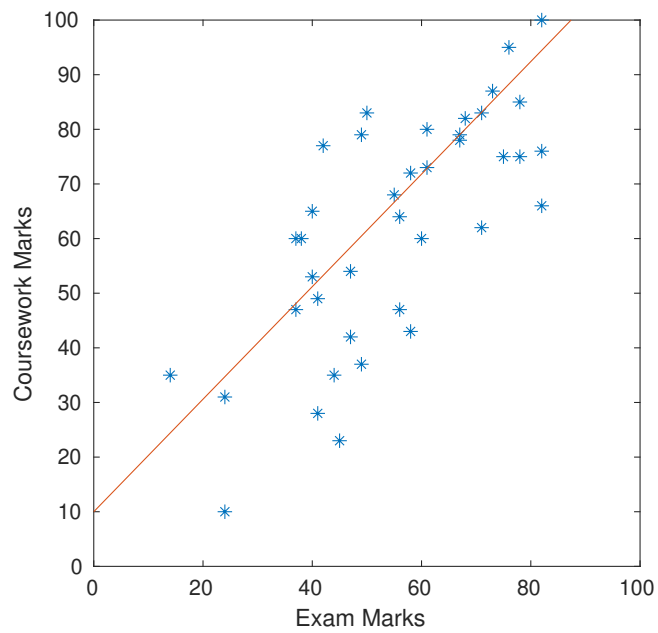


Figure 2: Coursework marks plotted against exam marks. Line of best fit estimated by eye

Given a line  $y = ax + b$ , the parameters were estimated as per below. This is represented in Figure 2.

$$\begin{aligned} a &= \frac{90 - 10}{78 - 0} = 1.03 \\ b &= 10 \end{aligned} \tag{1}$$

### 1.3 c

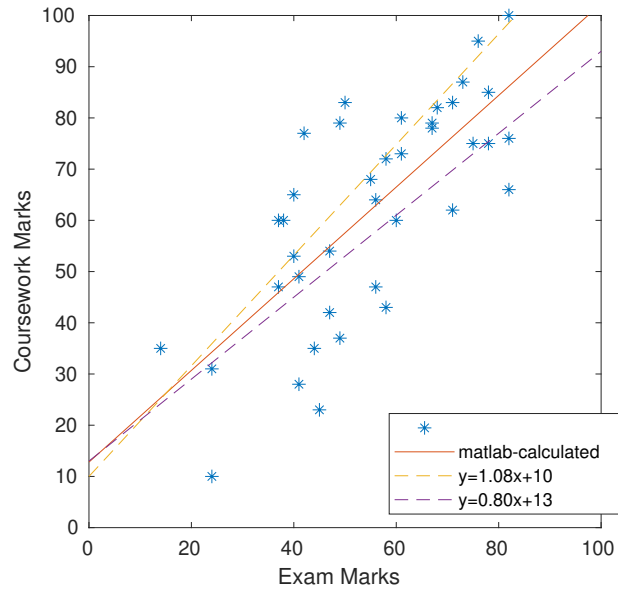


Figure 3: Coursework marks plotted against exam marks.

By comparing the maximum and minimum fits, we find the uncertainties on slope ( $\Delta a$ ) and y-intercept ( $\Delta b$ ) to be

$$Y_{\max} = 1.08x + 10; \quad Y_{\min} = 0.80x + 13$$

$$\therefore \Delta a = \frac{1.08 - 0.80}{2} = 0.14; \quad \Delta b = 1.5 \quad (2)$$

Listing 2: Topic 1. Question c. Note all code per topic belongs to same file

```

1 figure,plot(examMarks,courseworkMarks, '*');
2 xlabel('Exam Marks'); ylabel('Coursework Marks')
3 hold on;
4 axis square; axis ([0 100 0 100]);
5 p=polyfit(examMarks, courseworkMarks, 1); %Generate polynomial of best fit
6 xfit=0:100; yfit=polyval(p,xfit); %Line of best fit
7 plot(xfit,yfit)
8 plot(i,10+i.*1.08, '—'); %max slope
9 plot(i,13+i.*0.80, '—'); %min slope
10 legend('', 'matlab-calculated', 'y=1.08x+10', 'y=0.80x+13', 'Location', 'best')
11 hold off;

```

## 1.4 d

The general trend is similar, but Matlab calculates a higher y-intercept (12.7990) and a lower slope (0.8950). This suggests that I overvalue the density of points towards the extremity of the graph, while the software appropriately looks at all points equally.

## 1.5 e

$$\begin{aligned}f(x) &= 0.8950x + 12.7990 \\f(60) &= 0.8950 \cdot 60 + 12.7990 = 66.499\end{aligned}\tag{3}$$

Predicted grade would be 66.

## 1.6 f

We could present a more quantitative marker of reliability by measuring the correlation coefficient, but it can also be done qualitatively. There is a general correlation between exam marks and coursework marks, but the scattering suggests this will be imperfect (have an R-coefficient less than 1). From the data we have students with coursework marks around 60% getting exam marks ranging from approximately 40% to 80%. This is a really large spread that covers failing and getting first-class, so while the correlation is there, and it is accurate, it is not precise enough.

## 2 Employ assumptions to simplify systems

### 2.1 a

Length =  $L$ , Mass =  $M$ , Time =  $T$

- $[x] = L$
- $[m] = M$
- $[\frac{d^2x}{dt^2}] = \frac{L}{T^2}$
- $[k] = \frac{M}{T^2}$

### 2.2 b

$$\begin{aligned} A &\approx 10^{-3}; k \approx 10^3; m = 1; t = 10^{-3} \\ x(10^{-3}) &\approx 10^{-3} \cdot \cos\left(\cancel{10^{-3} \cdot \sqrt{10^3}}^1\right) \approx 10^{-3} \end{aligned} \tag{4}$$

## 3 Matrices and vectors

### 3.1 a

Listing 3: Topic 3. Question a. Note all code per topic belongs to same file

```
1 t=0:0.4:40
2 p=[20*sin(t); 20*cos(t); 10-(t./4).^2]
```

### 3.2 b

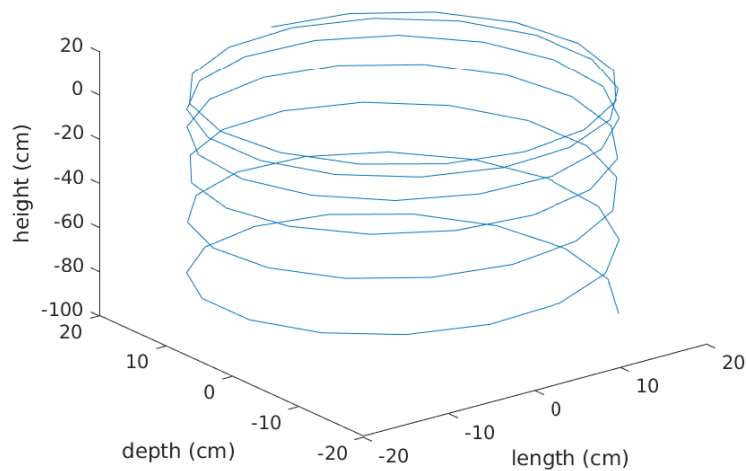


Figure 4: Particle position from  $t = 0$ s to  $t = 40$ s with 0.4 second intervals.

Listing 4: Code for Topic 3. Question b. Note all code per topic belongs to same file

```
1 plot3(p(1,:),p(2,:),p(3,:))
2 xlabel('length (cm)'),ylabel('depth (cm)'), zlabel('height (cm)')
```

### 3.3 c

Given that the speed is the modulus of velocity,

$$|\vec{v}(15)| = \left| \begin{bmatrix} 20 \cos(15) \\ -20 \sin(15) \\ -\frac{2}{16}15 \end{bmatrix} \right| = \left| \begin{bmatrix} -15.1938 \\ -13.0058 \\ -1.8750 \end{bmatrix} \right| = 20.0877 \text{ cms}^{-1} \quad (5)$$

### 3.4 d

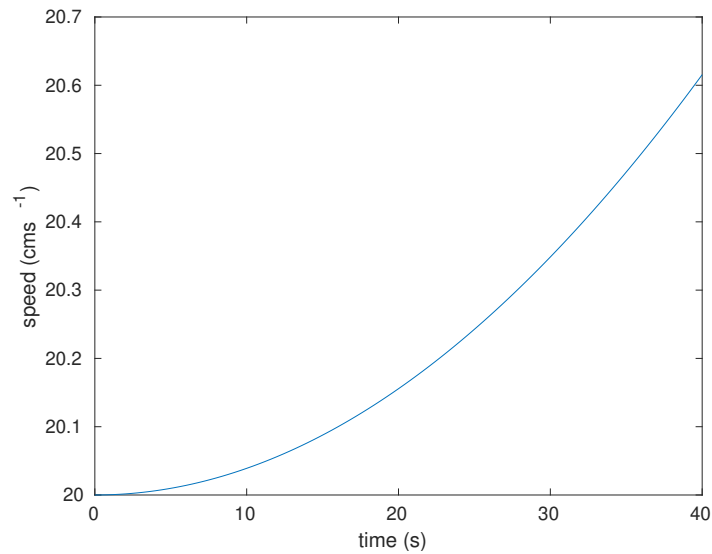


Figure 5: Particle speed varying with time.

Listing 5: Topic 3. Question d. Note all code per topic belongs to same file

```
1 v=[20*cos(t); -20*sin(t); -2.*t./16]
2 speed=sqrt(sum(v.^2))
3 plot(t,speed)
4 xlabel('time (s)'),ylabel('speed (cms-1)')
```



## 4 Complex numbers

### 4.1 a

Given that  $j^\alpha$  is a positive imaginary number, we can conclude that its argument is  $\frac{\pi}{2}$ , that is

$$\begin{aligned} (re^{jx})^\alpha &= r^\alpha (\cos(x\alpha) + j \sin(x\alpha)) \quad (\text{De Moivre's theorem}) \\ j^\alpha &= r^\alpha e^{jx\alpha} \implies x = \frac{\pi}{2}; r = 1 \\ \therefore j^\alpha &= e^{j\frac{\pi\alpha}{2}} = \cos\left(\frac{\alpha\pi}{2}\right) + j \sin\left(\frac{\alpha\pi}{2}\right) \end{aligned} \tag{6}$$

### 4.2 b

$$\begin{aligned} Z &= R_\infty + \frac{R_0 - R_\infty}{1 + (j\omega\tau)^\alpha} = R_\infty + \frac{R_0 - R_\infty}{1 + (\omega\tau)^\alpha \cdot j^\alpha} \\ &= R_\infty + \frac{R_0 - R_\infty}{1 + \theta^\alpha (c + js)} = R_\infty + \frac{R_0 - R_\infty}{1 + \theta^\alpha c + \theta^\alpha js} \\ &= R_\infty + \frac{R_0 - R_\infty}{1 + \theta^\alpha c + \theta^\alpha js} \cdot \frac{1 + c\theta^\alpha - js\theta^\alpha}{1 + c\theta^\alpha - js\theta^\alpha} \\ &= R_\infty + \frac{(R_0 - R_\infty)(1 + c\theta^\alpha - js\theta^\alpha)}{1 + c\theta^\alpha + \cancel{js\theta^\alpha} + c\theta^\alpha + c^2\theta^{2\alpha} + \cancel{cjs\theta^{2\alpha}} - \cancel{js\theta^\alpha} - \cancel{jsc\theta^{2\alpha}} - (js)^2\theta^{2\alpha}} \tag{7} \\ &= R_\infty + \frac{(R_0 - R_\infty)(1 + c\theta^\alpha - js\theta^\alpha)}{1 + 2c\theta^\alpha + \theta^{2\alpha}(c^2 - j^2s^2)} \\ &= R_\infty + \frac{(R_0 - R_\infty)(1 + c\theta^\alpha - js\theta^\alpha)}{1 + 2c\theta^\alpha + \theta^{2\alpha}(\cancel{c^2 + s^2})^1} \quad (\text{from } \cos^2(x) + \sin^2(x) = 1) \\ \therefore Z &= R_\infty + \frac{(R_0 - R_\infty)(1 + c\theta^\alpha - js\theta^\alpha)}{1 + 2c\theta^\alpha + \theta^{2\alpha}} \quad \text{as required} \end{aligned}$$

### 4.3 c

Given that  $\Delta = R_0 - R_\infty$ , and  $q = 1 + 2c\theta^\alpha + \theta^{2\alpha}$ ,

$$Z = R_\infty + \Delta \frac{1 + c\theta^\alpha - js\theta^\alpha}{q}$$

$$Z = R_\infty + \frac{\Delta + \Delta c\theta^\alpha}{q} - j \frac{s\Delta\theta^\alpha}{q}$$

$$\Rightarrow |Z| = \sqrt{\left(R_\infty + \frac{\Delta + \Delta c\theta^\alpha}{q}\right)^2 + \left(\frac{s\Delta\theta^\alpha}{q}\right)^2}$$

$$|Z| = \sqrt{R_\infty^2 + \frac{2R_\infty\Delta + 2R_\infty\Delta c\theta^\alpha}{q} + \frac{\Delta^2 + 2\Delta^2 c\theta^\alpha + \Delta^2 c^2\theta^{2\alpha}}{q^2} + \frac{(s\Delta\theta^\alpha)^2}{q^2}}$$

$$[\text{note that } \theta^{2\alpha}\Delta^2(c^2 + s^2) = \theta^{2\alpha}\Delta^2]$$

$$|Z| = \sqrt{R_\infty^2 + \frac{2R_\infty\Delta + 2R_\infty\Delta c\theta^\alpha}{q} + \cancel{\Delta^2\left(\frac{1 + 2c\theta^\alpha + \theta^{2\alpha}}{q^2}\right)}^{\frac{1}{q}}}$$

$$|Z| = \sqrt{\frac{qR_\infty^2 + 2R_\infty\Delta + 2R_\infty\Delta c\theta^\alpha + \Delta^2}{q}}$$

$$|Z| = \sqrt{\frac{(1 + 2c\theta^\alpha + \theta^{2\alpha})R_\infty^2 + 2R_\infty(R_0 - R_\infty) + 2R_\infty(R_0 - R_\infty)c\theta^\alpha + (R_0 - R_\infty)^2}{1 + 2c\theta^\alpha + \theta^{2\alpha}}}$$

$$|Z| = \sqrt{\frac{\cancel{R_\infty^2} + \cancel{2c\theta^\alpha R_\infty^2} + R_\infty^2\theta^{2\alpha} + \cancel{2R_\infty R_0} - \cancel{2R_\infty^2} + 2R_\infty R_0 c\theta^\alpha - \cancel{2R_\infty^2 c\theta^\alpha} + R_0^2 + \cancel{R_\infty^2} - \cancel{2R_0 R_\infty}}{1 + 2c\theta^\alpha + \theta^{2\alpha}}}$$

$$\therefore |Z| = \sqrt{\frac{R_\infty^2\theta^{2\alpha} + 2cR_\infty R_0\theta^\alpha + R_0^2}{1 + 2c\theta^\alpha + \theta^{2\alpha}}} \quad \text{as required}$$

(8)

### 4.4 d

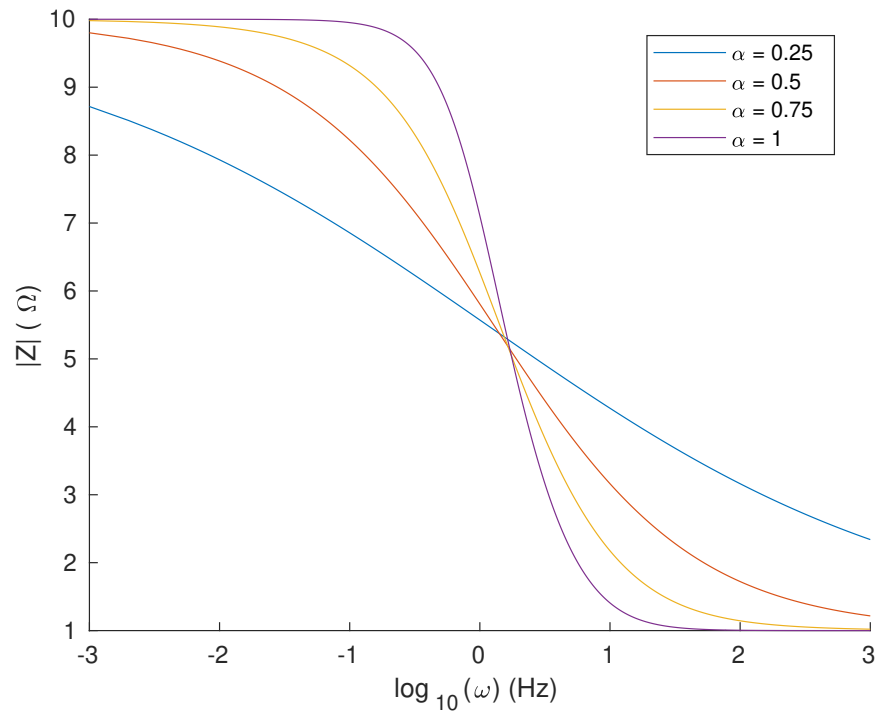


Figure 6: Magnitude of impedance plotted against angular frequency

Listing 6: Code for Topic 4. Question d.

```

1 R_infty = 1
2 R_0 = 10
3 tau = 1
4
5 omega=0.001:0.001:10^3
6 theta = omega*tau
7 figure, hold on;
8 for alpha = [0.25, 0.5, 0.75, 1]
9     c=cos(alpha*pi/2)
10     modZ = sqrt( ((R_infty)^2.*theta.^(2*alpha) + 2*c*R_0*R_infty.*theta.^alpha + (R_0)^2)./
11                 (1+2*c*theta.^alpha + theta.^(2.*alpha)) )
12     plot(log10(omega), modZ);
13 end
14 xlabel('log_{10}(\omega) (Hz)'); ylabel('|Z| (\omega)')
15 legend('\alpha = 0.25', '\alpha = 0.5', '\alpha = 0.75', '\alpha = 1', 'Location','best')
16 print('eps/topic4_d.eps', '-depsc')

```

## 4.5 e

We see variations of logistic curves, which is somewhat surprising, given that the equation for  $|Z|$  is quite different from the standard  $f(x)$ , such that

$$f(x) = \frac{L}{1 + e^{-k(x-x_0)}} \quad (9)$$