Mathematics Extended Essay

How many primes are there between 1 and n?

word count: 3500

Contents

1 Introduction

From Euclid proving there were infinitely many prime numbers circa 300 BC, to formulas that were thought to guarantee prime numbers, to the prime number theorem proven in 1896, prime numbers have been an intriguing field that has seen many of its breakthroughs through what was in the forefront of mathematics. The incredibly long history of the topic, as well as my personal curiosity led me to the following research question: **How many primes are there between** 1 **and any integer** n**?**.

Surprisingly, various fields of mathematics are required in order to tackle the question – from number theory to calculus to complex analysis. The research will consequently be divided into two discreet sections: prime finding functions and counting functions. The former consists of more traditional methods based in number theory that require each prime between 1 and n to be found and counted. We discuss trial divisions and its importance; the sieve of Erastothenes; and finally, build a preliminary distribution of primes for somewhat large values of n. The second section, prime counting functions, is largely based on Derbyshire's book "Prime Obsession" discussing the work of German mathematician Bernhard Riemann, and in it we will discuss some of the body of work which led to the prime number theorem. It focuses on presenting the largest developments in the field, and the move away from the "simple" number theory. Finally, it presents a function, and shows an example calculation, that, based on the (possible) proof of the Riemann Hypothesis, can reliably answer how many primes there are between 1 and n.

2 Prime Finding

2.1 Trial and error

Prime numbers are the building blocks of integers, and through the development of mathematics have had fairly diverse approaches to its definition, however with algebra it can simply be stated that a prime is a natural number larger than 1 that has exactly two positive divisors: 1 and itself. It thus leads to one of the most basic, but important theorems of number theory (Joyce, 1996, Book IX, Proposition 14):

Theorem 1 ***Fundamental Theorem of Arithmetic****. Any positive integer larger than 1 can be written as a unique product of prime numbers.*

A simple example would be the composite number 12, which is described by the unique product of the numbers 3 and two times 2, that is $3 \cdot 2^2$. A pre-algebraic description for primes is somewhat more complicated in its definition, though with some visual aid we can utilise it to establish our first attempt at counting primes (Horsley, 1772, p.327):

Definition 1 *A prime number is such a one as to have no integral divisor but unity*[1].

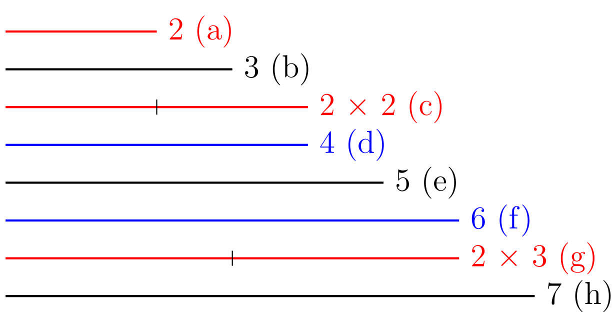


Figure 1: A pre-algebraic representation of primes through strings of increasing length

Figure 1 shows the attempts, through trial and error, to establish which numbers have whole ("integral") factors and which don't, and while it may be incredibly verbose, it is

[1] Horsley's english modernised for easier comprehension.

worth discussing step-by-step. Line a is the first, and can only be divided by 1 ("unity"), and is therefore, **prime**. Line b cannot be divided into whole multiples of line a, therefore b is also prime. line d can be divided into two times line a, (as shown with line c), therefore d is **composite**, and so on. Now if we want to establish which numbers are prime from 1 to 10, we will have to do trial divisions for every single number (or line) by every other prime predecessor: a) is 5 divisible by 2? No; b) Is 5 divisible by 3? No – then 5 must be prime, and so on with 6, 7, etc.

This allows us to introduce the concept of prime counting function, $\pi(n)$, and through trial divisions up to 10, affirm that $\pi(10) = 4$.

Definition 2 *Denoted by $\pi(n)$, the prime counting function tells us the exact number of prime numbers less than or equal to some number n, or in other words, the **distribution of primes** less than or equal to n.*

Although trial and error seems to include many redundancies, many prime finding algorithms boil down to it. A great example is The Great Internet Mersenne Prime Search – GIMPS, a distributed computing effort to find Mersenne primes (see Definition 3). The process consists of, among others, a series of optimised trial divisions and the creation of a modified sieve of Eratosthenes, which will be discussed in detail further in the paper.

Definition 3 *Mersenne primes, denoted by M_p, are numbers with form $2^p - 1$, such that M_p and p are prime. For example, $M_3 = 2^3 - 1 = 7$ or $M_5 = 31$.*

The optimised trial divisions are possible due to a few properties of Mersenne primes – whose proofs go beyond the scope of this paper – which allow to reduce the number of operations that must be performed. Amongst others(Mersenne Research Inc., 2009):

Definition 4

1. *any factor q of $M_p = 2^p - 1$ must have form $2kp + 1$. That is, $\frac{M_p}{q} = \frac{2^p-1}{2kp+1} = N,\ N \in \mathbb{N}$.*

2. *$q = 1(\mod 8)$ or $7(\mod 8)$.*

3. q must be prime.

We must also introduce an important definition regarding the magnitude of factor q that will be used throughout this study. If we consider a natural number n such that $q_1 \cdot q_2 = n$, the largest both factors can simultaneously be is $q_1 = q_2 = q$, that is $q^2 = n$, leading to the following definition:

Definition 5 *The smallest factor of n must be equal to or less than $\sqrt{n}$.*

To exemplify the process, the primality of two possible Mersenne primes will be checked, $M_{11} = 2^{11} - 1 = 2047$ and $M_{17} = 2^{17} - 1 = 131071$. If we first consider M_{11}, as per Definition 5, $q \leq \sqrt{2047} \cong 45$, so the first step is to check factor $q = 2kp + 1$, replacing k for 1, 2, 3, etc. such that $q \leq 45$.

For k = 1, $q = 2 \cdot 11 \cdot 1 + 1 = 23$

For k = 2, $q = 45$

Followed by trial divisions of M_{11} by q such that the result is a natural number (as per Definition 4.1).

$$\frac{2047}{23} = 89$$

Therefore M_{11} is not prime, and we have found its factors: $M_{11} = 23 \cdot 89 = 2047$. Repeating the process for M_{17}, and given that for M_{17}, $q \leq \sqrt{131071} \cong 362$

For k = 1, $q = 2 \cdot 17 \cdot 1 + 1 = 35$

For k = 2, $q = 69$

For k = 3, $q = 103$

$q = 137, 171, 205, 239, 273, 307$ or 341

Definition 4.2 tells us that q = 1 or 7 (mod 8):

$$\begin{array}{lll} \cancel{35 \mod 8 = 3} & \cancel{171 \mod 8 = 3} & 273 \mod 8 = 1 \\ \cancel{69 \mod 8 = 5} & \cancel{205 \mod 8 = 5} & \cancel{307 \mod 8 = 3} \\ 103 \mod 8 = 7 & 239 \mod 8 = 7 & \cancel{341 \mod 8 = 5} \\ 137 \mod 8 = 1 & & \end{array}$$

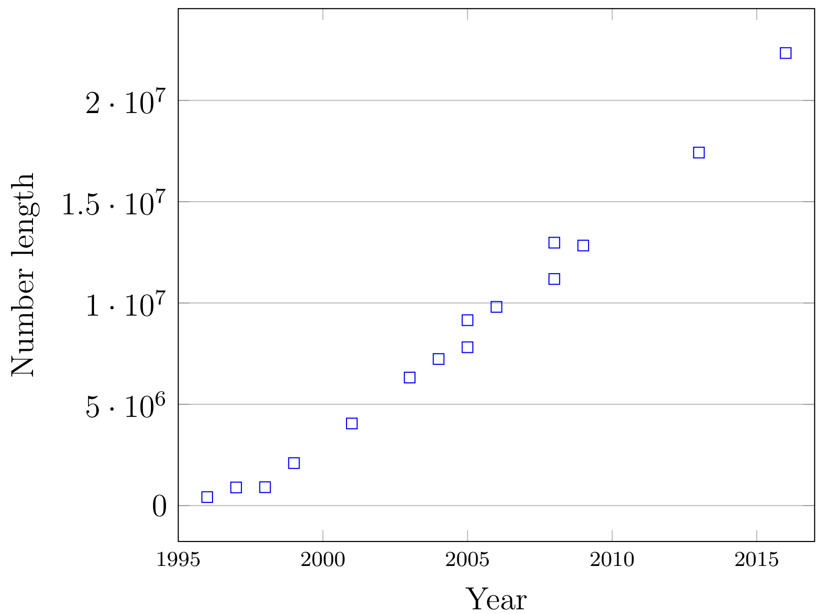
Finally, as per Definition 4.3 and indeed Theorem 1, 273 is excluded for being composite ($3 \cdot 7 \cdot 13 = 273$). Therefore only the following trial divisions must be performed in order to verify the primality of 131071:

$$\frac{131071}{103} = 1272 + \frac{55}{103} \quad \text{or} \quad \frac{131071}{137} = 956 + \frac{99}{137}$$
$$\text{or} \quad \frac{131071}{239} = 548 + \frac{99}{239} \quad \text{or} \quad \frac{131071}{273} = 480 + \frac{31}{273}$$

None of which resulted in natural numbers, and thus $M_{17} = 131071$ is prime. This means that with four trial divisions, we were able to with certainty establish that 131071 is prime! Indeed, these optimisations are so powerful that the current largest prime has over 22 million digits and was found through GIMPS utilising, amongst others, this method. Figure 2 gives a great insight into the incredible jump in size of primes found in the last circa 20 years since its inception – from around 420,000 digits in 1996 to 10,00,000 in 2006 to 22,000,000 in 2016.

It is clear that this efficiency is only applicable to a very narrow scope, and serves the purpose of checking the primality of a fairly specific set of numbers. There is, in fact, a method that contrasts greatly to performing trial divisions for each number, and it is called a sieve of Erastothenes.

Figure 2: Distribution of GIMPS-found Mersenne primes (Mersenne Research Inc., 2016).



2.2 Sieve of Erathosthenes

The sieve of Eratosthenes, attributed to ancient Greek mathematician Eratosthenes of Cyrene, is an incredibly powerful algorithm which allows us to quickly exclude composite numbers from a list (Horsley, 1772, pp.332-335) by "counting up". In lieu of a strictly mathematical definition, the process can be best described by the following pseudocode ("Sieve of Eratosthenes", n.d.):

Input: an intenger n, such that $n > 1$
Let A be an array of boolean[2] values, indexed by integers 2 to n, initially all set to true.
  for $i = 2, 3, 4, ...,$ not exceeding $\sqrt{n}$ **do**
    if $A[i]$ is **true then**
      for $j = i^2, i^2 + i, i^2 + 2i, i^2 + 3i, ...,$ not exceeding n **do**
        $A[j] := false$
      end
    end
end
Output: all i such that $A[i]$ is true.

Algorithm 2: Sieve of Eratosthenes ("Sieve of Eratosthenes", n.d.)

[2]Boolean is a data type that can only either be true or false.

In order to find all primes from 1 to 20 – that is, $\pi(20)$ – we begin by excluding all the multiples of the lowest prime, namely 2:

2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20

Followed by doing the same with the next lowest prime, 3:

2, 3, 5, 7, 9, 11, 13, 15, 17, 19

And repeating the process until all primes in the list have been found.

Thus, we have found the primes less than or equal to 20, namely $2, 3, 5, 7, 11, 13, 17, 19$ – or in other words, $\pi(20) = 7$. While this ancient technique is quite powerful compared to the basic trial and error method discussed earlier, we are still forced to find each individual prime. Nevertheless, because it's just a repetitive task, we can have a computer run the algorithm (Appendix 7.2.1). In doing so, we find all primes less than 10^8 before running into hardware limitations (Raw data in Appendix 7.1.1), and analyse our findings.

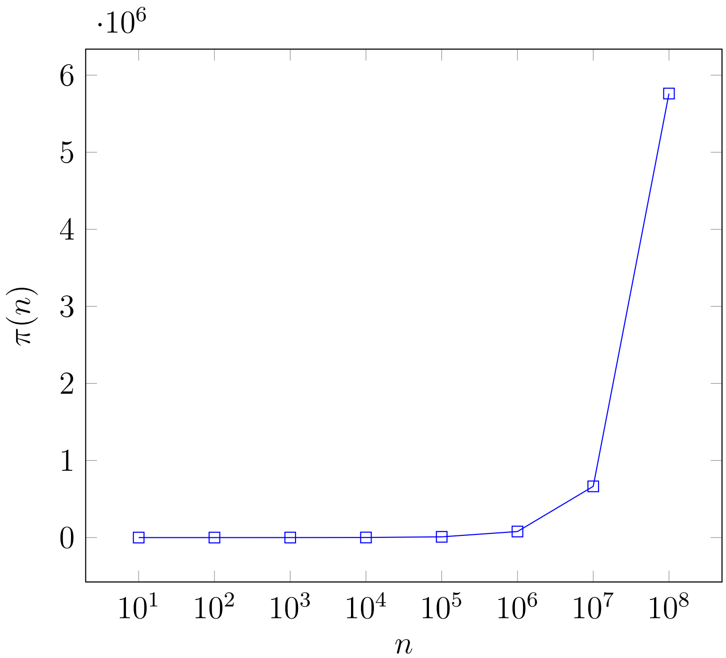
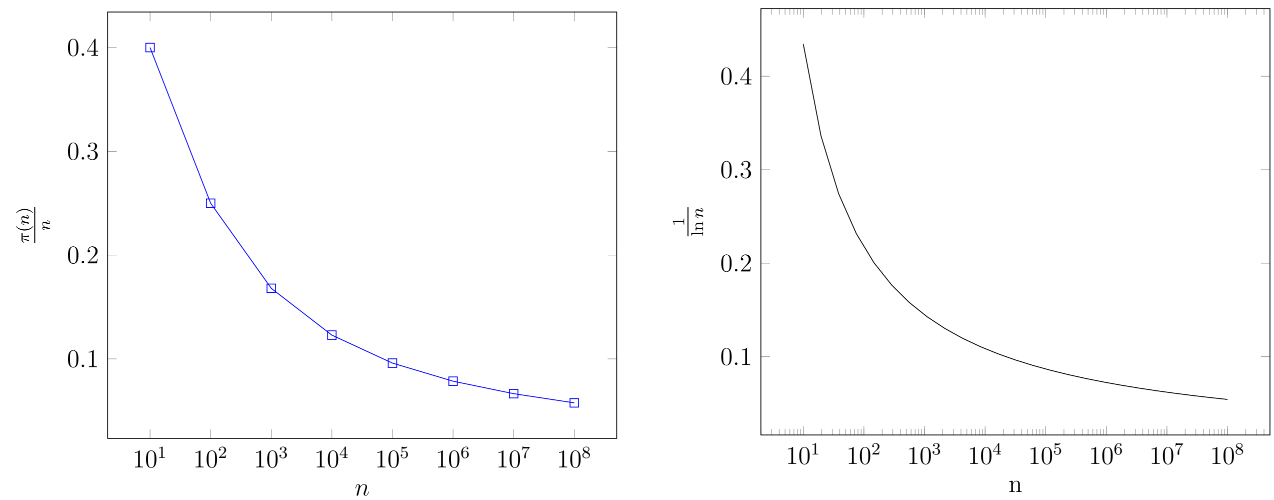


Figure 3: Distribution of primes

Figure 3 seems to suggest some kind of exponential growth, however because the later values are so much larger than the first, the almost flat line we observe in the beginning hinders our analysis a little. Instead of plotting $\pi(n)$, we plot the frequency of primes –

that is $\frac{\pi(n)}{n}$ (Seen in figure 4a). In looking for a "relative" of the exponential graph we hypothesised, we settle on $\frac{1}{\ln n}$, and present its graph in figure 4b for comparison.



(a) Distribution of primes found utilising a Sieve of Eratosthenes. (Appendix 7.2.1)

(b) $\frac{1}{\ln n}$ stipulated to follow the same tendency as $\frac{\pi(n)}{n}$

Figure 4: Comparison between the graphs for $\frac{\pi(n)}{n}$ and $\frac{1}{\ln n}$.

A comparison of the graphs suggests that their values converge for larger values of n. Thus we may present our first conjecture:

$$\begin{aligned} \frac{\pi(n)}{n} &\approx \frac{1}{\ln n} \quad \text{for large values of n} \\ \therefore \pi(n) &\approx \frac{n}{\ln n} \quad \text{for large values of n} \end{aligned} \tag{1}$$

Although the sieve itself could not be used to establish a significant distribution of primes for any n, equation 1 gives us the first stepping stone in our study for large values of n. Taking 10^{10}, for instance, yields us $\frac{10^{10}}{\ln 10^{10}} \approx 4.3 \cdot 10^8$, a really good estimate considering $\pi(10^{10}) \approx 4.5 \cdot 10^8$ (Caldwell, n.d.).

Correspondingly, we are faced with two questions:

1. Figure 4a describes the behaviour up to 10^8. Can this behaviour be expected to hold as n becomes infinitely large?

2. Does the frequency of prime numbers eventually reach zero?

The former is in a way the very point of this study, whilst the latter requires a bit more information than a simple yes or no. As will be discussed further in this study, models for the distribution of primes are based on asymptotic analysis, which assume infinitely large values of n. This assumption is made possible by Euclid's Theorem, which proves the following:

Theorem 2 *There are infinitely many prime numbers.*

Its proof follows an incredibly elegant logic (Clawson, 1996, pp. 147, 148).

Proof 1 *(i) Suppose there is a finite number of primes* p_1, p_2,..., p_n*;*

(ii) Let $p \,|\, p \in \{p_1, p_2, ..., p_n\}$*;*

(iii) Let P *be a multiple of any combination of these primes plus one, that is,* $P = p_1 p_2 ... p_n + 1$*. As such,* P *is either prime or composite.*

(iii) If P *cannot be divided by* p*, then as per theorem 1, it must itself be prime, and therefore (i) is incomplete.*

(iv) If P *is composite, it must therefore be divisible by* p*, which is impossible since* $P \equiv 1 \pmod{p}$*;*

The knowledge amassed so far allows us to move away from the idea of finding each individual prime and move towards prime counting functions.

3 Prime Counting

3.1 Prime Number Theorem

In the late 18th and early 19th centuries both Gauss and Legendre independently conjectured a similar relationship to the one which we arrived and described in equation 1 (Shanks, 1979, pp. 15-17). This body of work, plus the advents we will be discussing in this paper form what later, when proven, will be called prime number theorem.

Theorem 3 *the* ***Prime Number Theorem*** *describes the asymptotic distribution of prime numbers.*

In other words, the prime number theorem is the name for the most accurate equation for $\pi(n)$ possible. As it has presently been proven and the name is often used by mathematicians, we will utilise it, however, it is important to acknowledge that in the early 19th century it had not yet been proven. In order to formalise our findings (and Gauss' work), we will be calling this function **Ln(n)**. Our stipulation from (1) of needing large values of n remains true, though we utilise the standard asymptotic notation ($\sim$), as seen below.

$$\pi(n) \sim Ln(n) := \frac{n}{\ln n} \tag{2}$$

There were other attempts at improving the accuracy and describing the magnitude of $\pi(n)$, including some by famous mathematicians, such as Legendre and Chebyshev (Derbyshire, 2003, pp. 55,124-125). However, the most interesting and important one is heavily used in Riemann's unproven work (Derbyshire, 2003, p.116) – describing $\pi(n)$ as the area under the curve of the $\frac{1}{\ln n}$ graph from 0 to n. This second major step in prime number theorem will henceforth be called **li(n)**, and a calculus description of it will be used[3]:

[3]Though originally opting for "the European notation", denoted by $Li(n) = \int_2^n \frac{1}{\ln x} dx$, the "American notation", $li(n)$, will ultimately be utilised. A discussion of why will be in the Conclusion section as a part of further considerations.

$$\pi(n) \sim li(n) := \int_0^n \frac{1}{\ln x} dx \tag{3}$$

By comparing the magnitude of the error between the respective functions and known values of $\pi(n)$ (Sloane and Plouffe, 2016, Wilson v, 2000a,b) we have the following table:

n	$\pi(n)$	$\lvert Ln(n) - \pi(n)\rvert$	Percentage error	$\lvert li(n) - \pi(n)\rvert$	Percentage error
10^3	168	23	13.69%	10	5.95%
10^6	78498	6116	7.79%	130	0.17%
10^9	50847534	2592592	5.10%	1701	0.0033%
10^{12}	37607912018	1416705193	3.77%	38263	0.0001%

Table 1: Comparison between prime counting functions (Sloane and Plouffe, 2016, Wilson v, 2000a,b)

Table 1 very clearly shows the trend of accuracy improvement for larger values of n that was discussed. In fact, the difference in accuracy between the functions is so large that by $n = 10^{24}$ the margin of error for $Ln(n)$ is $\frac{1}{10^2}$% – compared to $li(n)$'s $\frac{1}{10^{10}}$%.

At this stage, we can quite successfully describe the number of primes between 1 and n for large values of n through the **principal term** $li(n)$. Accordingly, the final step in describing a satisfactorily accurate function is to introduce an equation that qualifies the difference between $Li(n)$ and $\pi(n)$, which we will call the **remainder term**, $\Delta(n)$, formalised below. In order to do so, we discuss the Zeta function and the Riemann Hypothesis.

$$\pi(n) = li(n) \pm \Delta(n) \tag{4}$$

4 Remainder Term

4.1 The Zeta Function

In 1859, in "On the number of primes less than a given magnitude" – his only contribution to number theory (Derbyshire, 2003), Riemann extended Euler's work in the Zeta Function and conjectured the now Millennium Prize worthy Riemann hypothesis. His work in analytic number theory was central to the quantifying of the remainder term, and de la Vallé Poussin and Hadamard's proof to the prime number theorem. Indeed, there are many interesting aspects to the Zeta function going far beyond the scope of its link to the distribution of prime numbers, including unexpected results for some of its arguments (Derbyshire, 2003, p.153).

The function has its origins as a specific case of a Dirichlet series – namely any series of the form:

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

Euler was famously quite interested in infinite series, and among his work there is a specific case of a Dirichlet series for which $a_n = 1$, and $s \in \mathbb{R}$ which he called the zeta function. Furthermore, Euler proved an extremely important link between the function and prime numbers (Derbyshire, 2003, p.135).

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \frac{1}{1 - \frac{1}{p^s}} \quad \text{for } s > 1, p \text{ prime} \tag{5}$$

One of the important portions of Riemann's work is the expansion of the domain of $\zeta(s)$ to the complex plane, specifically $\mathbb{C} \setminus \{1\}$. The equations that resulted from his analytic continuation can be seen in Appendix 7.3, equations 11 and 12. The full details of Riemann's work are far too complex and go beyond what is reasonable in this study, so we must take as *fait accompli* that the so-called non-trivial zeros of the function define the magnitude of the remainder term, $\Delta(n)$, (Grime, 2014).

There are infinitely many so-called trivial zeros[4], thus, we must specify the concept of non-trivial zeros, ρ. Furthermore, it is important to note that from this point on, the knowledge that is discussed is based on the veracity of the still unproven Riemann Hypothesis.

Definition 6 *ρ refers to the non-trivial zeros of $\zeta(s)$, that is:*

$\rho \subset \{s \in \mathbb{C} \setminus \{1\} \mid \zeta(s) = 0 \ \cap s \neq -2, -4, -6, \cdots\}$

Definition 7 ***Riemann Hypothesis*** *states that "All non-trivial zeros of the zeta function, $\zeta(s)$, have $\Re(s) = \frac{1}{2}$."*

An important aspect of the Riemann Hypothesis which will be utilised later in the study is that if ρ is a zero to the zeta function, then so is its conjugate, $\bar{\rho}$. This will be emphasised later in the study.

4.2 The Weighted Prime Counting Function

The last piece of the puzzle is the weighted prime counting function $J(n)$ which Riemann expressed in relation to the zeta function. The mathematics to reach the expression for $\pi(n)$ in terms of $J(n)$ are incredible, though far too complex to be reasonably discussed. In short what will be presented is $\pi(n)$ expressed in terms of $J(n)$, and $J(n)$ expressed in terms of $\zeta(s)$, and therefore $\pi(n)$ expressed in terms of $\zeta(s)$ (Derbyshire, 2003, pp.249, 302, 328).

$$\pi(n) = \sum_{k} \frac{\mu(k)}{k} J(\sqrt[k]{n}) \tag{6}$$

such that:

$\mu(k) = 0$ if k has a square factor

$\mu(k) = -1$ if k is prime, or the product of an odd number of different primes

$\mu(k) = 1$ for $k = 1$, or if k is the product of an even number of different primes

[4]That is, values of s such that $\zeta(s) = 0$. This is easy to see from Appendix 7.3(12), from $\sin(\frac{\pi s}{2})$ being 0 for every even negative value of the argument (-2, -4, -6, $\cdots$).

and

$$J(n) = Li(n) - \sum_{\rho} Li(n^{\rho}) - \ln 2 + \int_{n}^{\infty} \frac{1}{t(t^2-1)\ln t} dt$$

$$\begin{aligned}
\therefore \pi(n) = &\left(Li(n) - \sum_{\rho} Li(n^{\rho}) - \ln 2 + \int_{n}^{\infty} \frac{1}{t(t^2-1)\ln t} dt \right) \\
&- \frac{1}{2}\left(Li(\sqrt{n}) - \sum_{\rho} Li(n^{\frac{\rho}{2}}) - \ln 2 + \int_{\sqrt{n}}^{\infty} \frac{1}{t(t^2-1)\ln t} dt \right) \\
&- \frac{1}{3}\left(Li(\sqrt[3]{n}) - \sum_{\rho} Li(n^{\frac{\rho}{3}}) - \ln 2 + \int_{\sqrt[3]{n}}^{\infty} \frac{1}{t(t^2-1)\ln t} dt \right) \\
&- \frac{1}{5}\left(Li(\sqrt[5]{n}) - \sum_{\rho} Li(n^{\frac{\rho}{5}}) - \ln 2 + \int_{\sqrt[5]{n}}^{\infty} \frac{1}{t(t^2-1)\ln t} dt \right) \\
&+ \frac{1}{6}\left(Li(\sqrt[6]{n}) - \sum_{\rho} Li(n^{\frac{\rho}{6}}) - \ln 2 + \int_{\sqrt[6]{n}}^{\infty} \frac{1}{t(t^2-1)\ln t} dt \right) \\
&- \cdots
\end{aligned} \tag{7}$$

One characteristic of $J(n)$ we will utilise is that $J(n) = 0$ for $n < 2$. This[5]means that if we look at Equation 6: for $k > 6$, $\sqrt[k]{100} < 2$, so $J(\sqrt[k]{100}) = 0$. Therefore, the calculations will be done up to $k = 6$

With these equations we actually have the solution to our question of how many primes there are between 1 and n. However, it is quite clear the equation is incredibly complex, and as such, it is worth breaking it into its constituent parts and discussing them individually, using the example of $\pi(100)$ to better understand it.

In section 3.1 we called $li(n)$ the principal term, and the rest to be the remainder term. The convention still stands, though now we subdivide the remainder term and consider the

[5]Appendix 7.3 (13) presents $J(n)$ with respect to $\pi(n)$, which shows each successive step when $n < 2$ results in $\pi(n) = 0$

sum of each of these sub-terms, that is:

Principal term: $li(n)$

Secondary term: $-\dfrac{1}{2}li(n^{\frac{1}{2}}) - \dfrac{1}{3}li(n^{\frac{1}{3}}) - \dfrac{1}{5}li(n^{\frac{1}{5}}) + \cdots$

Periodic term: $-\sum_{\rho} li(n^{\rho}) + \dfrac{1}{2}\sum_{\rho} li(n^{\frac{\rho}{2}}) + \dfrac{1}{3}\sum_{\rho} li(n^{\frac{\rho}{3}}) + \cdots$

Log term: $-\ln 2 + \dfrac{1}{2}\ln 2 + \dfrac{1}{3}\ln 2 + \cdots$

Integral term: $\displaystyle\int_{n}^{\infty} \frac{1}{t(t^2-1)\ln t}dt - \frac{1}{2}\int_{\frac{\sqrt{n}}{2}}^{\infty} \frac{1}{t(t^2-1)\ln t}dt - \frac{1}{3}\int_{\frac{\sqrt[3]{n}}{3}}^{\infty} \frac{1}{t(t^2-1)\ln t}dt - \cdots$

Principal term

$$li(100) = \int_{0}^{100} \frac{1}{\ln x}dx = 30.1261416$$

Secondary term

$$\begin{aligned}
-\frac{1}{2}li(100^{\frac{1}{2}}) &= -3.08279975 \\
-\frac{1}{3}li(100^{\frac{1}{3}}) &= -1.13555052 \\
-\frac{1}{5}li(100^{\frac{1}{5}}) &= -0.33604670 \\
+\frac{1}{6}li(100^{\frac{1}{6}}) &= 0.20944495 \\
&= -4.34495202
\end{aligned}$$

Integral Term

$$\begin{aligned}
\frac{1}{6}\int_{\sqrt[6]{100}}^{\infty}\frac{1}{t(t^2-1)\ln t}dt &= \frac{0.109093824}{6} &= 0.018182304\\
-\frac{1}{5}\int_{\sqrt[5]{100}}^{\infty}\frac{1}{t(t^2-1)\ln t}dt &= -\frac{0.067237637}{5} &= -0.013447527\\
-\frac{1}{3}\int_{\sqrt[3]{100}}^{\infty}\frac{1}{t(t^2-1)\ln t}dt &= -\frac{0.012254312}{3} &= -0.004084771\\
-\frac{1}{2}\int_{\sqrt{100}}^{\infty}\frac{1}{t(t^2-1)\ln t}dt &= -\frac{0.001839687}{2} &= -0.000919843\\
&+\int_{100}^{\infty}\frac{1}{t(t^2-1)\ln t}dt &= 0.000009876\\
&&= -0.000259962
\end{aligned}$$

Log term

$$-\ln 2+\frac{\ln 2}{2}+\frac{\ln 2}{3}+\frac{\ln 2}{5}-\frac{\ln 2}{6} = -0.092419624$$

Periodic Term The terms discussed so far have been mathematically quite simple, though that is not the case for the periodic term. Unlike the other terms, each element consists of an infinite sum. This would suggest that simplifications might be necessary, but it is clear that the largest number possible of significant figures has been used so far – this has been the case due to the periodic term, which as will discussed below, requires as many significant figures as possible in order to provide appropriate results (Miroshnychenko, Y., personal communication, October 2017).

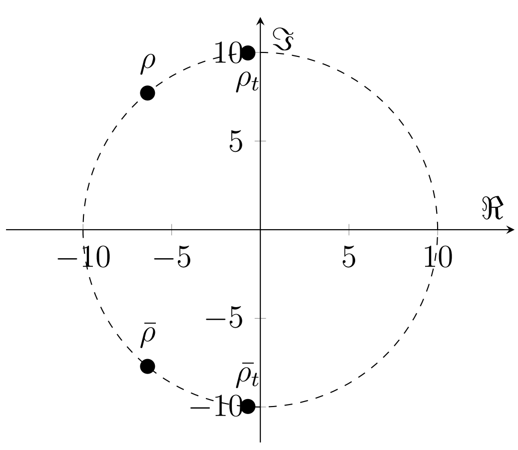
For our calculations, we will be utilising ρ with an accuracy of 9 significant figures, though for the sake of brevity the data displayed will be far more concise. The first value of ρ is $\frac{1}{2}+i14.134725$ (Odlyzko, n.d.), so for 100^{ρ}:

$$
\begin{aligned}
100^{\rho} &= 100^{\frac{1}{2}+14.134725i} = 100^{\frac{1}{2}} \cdot 100^{14.134725i} = 10 \cdot e^{\ln 100^{14.134725i}} \\
&= 10 \cdot e^{4.605170 \cdot 14.134725i} = 10 \cdot e^{65.092812i} = 10 \cdot e^{10.359842 \cdot 2\pi i} \\
&= 10 \cdot 1e^{10 \cdot 2\pi i} \cdot e^{0.359842 \cdot 2\pi i} \\
\therefore 100^{\rho} &= 10 \cdot e^{2.260954i}
\end{aligned} \tag{8}
$$

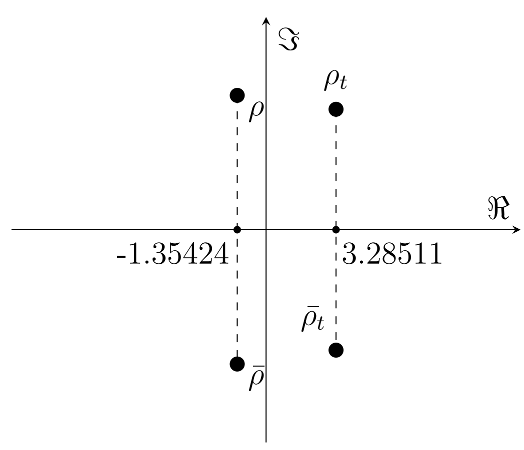
This is compared to the calculation for $\rho_t = \frac{1}{2} + i14$:

$$
\begin{aligned}
100^{\rho_t} &= 100^{\frac{1}{2}} \cdot e^{4.605170 \cdot 14i} = 10 \cdot e^{64.472380i} \\
&= 10 \cdot 1e^{10 \cdot 2\pi i} \cdot e^{0.261098 \cdot 2\pi i} \\
\therefore 100^{\rho_t} &= 10 \cdot e^{1.640527i}
\end{aligned} \tag{9}
$$

By plotting both 100^{ρ} and 100^{ρ_t} into an Argand diagram, as can be seen in figure 5a. Finally, figure 5b makes it really clear that simplifications will indeed lead to a very large change in value for the periodic term.



(a) Argand diagram of $\rho = \frac{1}{2} + 14.134725i$, $\rho_t = \frac{1}{2} + 14i$ and their respective conjugates



(b) Argand diagram of $li(\rho)$ and $li(\rho_t)$, and respective conjugates

Calculations Despite the need for infinitely many terms, for the sake of demonstration we will take only the first forty zeros (Odlyzko, n.d.) and their conjugates for the first element of the periodic term. The calculations are done through Mathematica.

$$
\begin{aligned}
1a &: li(100^{\frac{1}{2}+i14.134725}) = li(-6.36664+i7.71141) = 1.35421+i6.31436\\
1b &: li(100^{\frac{1}{2}-i14.134725}) = li(-6.36664-i7.71141) = 1.35421-i6.31436\\
2a &: li(100^{\frac{1}{2}+i21.022039}) = li(-8.36843+i5.47442) = 0.452923+i6.28610\\
2b &: li(100^{\frac{1}{2}-i21.022039}) = li(-8.36843-i5.47442) = 0.452923-i6.28610
\end{aligned} \tag{10}
$$

The addition of the first forty zeros (ρ) for the very first term of the periodic term (that is, $li(n^{\rho})$) and their respective conjugates can be seen in Appendix 7.1, Table 7.1.2. Though what is important is that they add up to about 255. Thus, we use our known value of $\pi(100)$ from earlier sieving in order to verify whether our findings are logical. We do so by utilising equation 7.

$$
\begin{aligned}
&\pi(100) = \text{Principal} + \text{Secondary} + \text{Integral} + \text{Log} + \text{Periodic}\\
&25 = 30.1261416 - 4.34495202 - 0.000259962 - 0.092419624 + \text{ Periodic}\\
&\text{Periodic} = -0.688509994
\end{aligned}
$$

This means that somehow our calculations are completely nonsensical, as we are expecting the sum of all periodic terms to be -0.7 and the very first term is already 255! In attempting to uncover what is generating this error, we compare Derbyshire's provided example calculations for $li(20)$ (Derbyshire, 2003, p.340) to our results with Mathematica (Appendix 7.2.2.1), as seen below.

	Derbyshire	Mathematica
$20^{(\frac{1}{2}+i14.134725)}$	$-0.302303-i4.46191$	$-0.302305-i4.46191$
$li(20^{(\frac{1}{2}+i14.134725)})$	$-0.105384-i3.14749$	$0.952805-i3.91384$

Table 2: Comparison of known values (Derbyshire, 2003, p.340) for complex exponentiation and logarithmic integral. (Mathematica, Appendix 7.2.2.1)

So why is this happening? In solving $\ln(n^{\rho})$, Mathematica is not returning $\rho \ln n$, but rather $\rho \ln n - 2k\pi i$ where k is a value such that the imaginary part of the solution lie between

$-\pi$ and π (Wagon, 1999, p.548), leading to the incorrect sum we are observing. As this is a quirk of the software, we won't be discussing it in great deal, but the solution, according to Wagon, lies in utilising the Exponential Integral function, $Ei(n)$, whose relationship with $li(n)$ is as follows.

$$li(n^\rho) = Ei(\ln n^\rho)$$

We repeat the calculations in Mathematica utilising $Ei(n)$ (Algorithm: Appendix 7.2.2.2), add each respective conjugate like done previously (Appendix 7.1, table 7.1.3). The process is then done for the entire periodic term, as per the algorithm in Appendix 7.2.2.3, which yields the value of -0.573055632. Incredibly close to the expected -0.688509994, especially since we only used forty terms! Thus, we can affirm the value of $\pi(100)$, and have shown that equation 6 can indeed tell us how many primes there are between 1 and n.

$$\pi(100) = 30.1261416 - 4.34495202 - 0.573055632 - 0.000259962 - 0.092419624$$
$$\therefore \pi(100) = 25$$

5 Conclusion

The research question of "how many primes there are between 1 and n?" did not invite an answer that is a number, but rather an expression – which mathematicians call $\pi(n)$ and later on, the prime number theorem. The study began by contrasting trial-and-error and the sieve of Eratosthenes. The former consisted of checking the primality of each number as we counted up to n, whilst the latter consisted of removing the composites of every prime smaller than $\sqrt{n}$. The sieving technique led us to a preliminary distribution of primes and a prime counting function, called $Ln(n)$ that is equivalent to that of 18th and early 19th century Mathematicians. This function was later improved into $li(n)$. At this stage, we had a reasonably good expression for $\pi(n)$ – and consequently an answer – though only for large values of n, and we sought the answer for any value of n.

Thus Riemann's zeta function and the Riemann Hypothesis were introduced and discussed as deeply as possible, with the objective of perfecting what we called the remainder term, the difference between $li(n)$ and known values of $pi(n)$. This led us to the answer to our question, namely:

$$\pi(n) = \sum_k \frac{\mu(k)}{k} J(\sqrt[k]{n})$$

This expression, however, was so complicated that it had to be broken down and each portion individually examined. The largest amount of focus put on the so-called periodic term, which presented the largest challenge in its calculation. Not only was it mathematically complicated, consisting of an infinite sum of complex integrals, but also generated the most amount of discussion: This showed a limitation in the software used to make the calculations (Mathematica), though a solution was found in utilising $Ei(\ln n)$ instead of $li(n)$.

Further considerations Originally, instead of $li(n)$, the function used was the so-called "European notation" (Li(n)):

$$Li(n) := \int_2^n \frac{1}{\ln x} dx = li(n) - li(2)$$

The final equation that describes both function are similar to each other, so while performing the final calculations to show the example of $\pi(100)$, an interesting problem arose with the periodic term. If, according to the equation, the periodic term (P) with respect to $Li(n)$ could be given by (Grime, 2014):

$$P = \sum_\rho Li(n^\rho)$$
$$\therefore P = \sum_\rho (li(n^\rho) - li(2)) = \sum_\rho li(n^\rho) - \sum_\rho li(2)$$

This implies that if we count forty zeros, then the first term alone would be offset by $40 \cdot li(2)$, or if we count to 10^{100} zeros, the term will be offset by $10^{100} \cdot li(2)$. Granted that some terms are negative and others positive and that compensates somewhat, but the second term, for instance, only has the weight of $\frac{1}{2}$, the third, $\frac{1}{3}$ and so on. The hypothesis is that when calculating the periodic term, the sum must be offset by $li(2)$ only, however no literature was found on the subject, and no amount of discussion led to a mathematically rigorous proof. Instead, the American notation was ultimately utilised in place of the European.

Another fruitful further investigation regards Riemann's number theory work. The requirement of understanding analytic continuation and other highly complex mathematics meant they had to be oversimplified or ignored. The most fascinating of them comes from the calculation of $\zeta(-1)$, which is actually used in physics and implies that the sum of all natural numbers is negative as seen below. The interesting dialogue comes from whether

$\zeta(-1)$ before and after the analytical continuation do indeed refer to the same thing.

$$\begin{aligned}
\zeta(-1) &= \sum_{n=1}^{\infty} \frac{1}{n^{-1}} = \sum_{n=1}^{\infty} n \\
\zeta(s) &:= 2^s \pi^{s-1} \ \sin\left(\frac{\pi s}{2}\right) \ \Gamma(1-s) \ \zeta(1-s) \\
\Gamma(2) &= 1; \quad \zeta(2) = \frac{\pi^2}{6} \\
\therefore \zeta(-1) &= 2^{-1}\pi^{-2} \ \sin\left(\frac{-\pi}{2}\right) \ \Gamma(2) \ \zeta(2) \\
&= \frac{-1}{2\pi^2} \ \zeta(2) = \frac{-1}{2\pi^2} \ \frac{\pi^2}{6} = -\frac{1}{12}
\end{aligned}$$

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7 Appendix

7.1 Data

7.1.1 Primes up to 10^8

n	Number of Primes less than n
10	4
10^2	25
10^3	168
10^4	1229
10^5	9592
10^6	78498
10^7	664579
10^8	5761455

7.1.2 $Li(100^{\rho}) + Li(100^{\bar{\rho}})$

2.7084200	9.5016400	10.2658000	12.2747000
0.9058290	2.2602700	7.6452200	5.6277900
3.8253000	−1.9335100	11.2720000	−0.6114760
5.0836800	0.6758890	11.7614000	2.0299400
10.5207000	5.9727300	11.5349000	12.3303000
−0.5936720	9.8246200	8.4361300	8.3938400
12.3230000	12.2608000	2.1885100	9.1784900
7.2079200	9.4031500	6.3455400	11.9124000
9.2406500	−1.6384500	−0.1075450	12.1482000
−1.4794400	−0.7849060	6.6322500	11.1421000

Table 3: $Li(100^{\rho}) + Li(100^{\bar{\rho}})$ for the first forty ρ. Algorithm: Appendix 7.2.2.1

7.1.3 $Ei(100^{\rho}) + Ei(100^{\bar{\rho}})$

0.2328730	-0.0731235	0.0442353	0.0064389
0.1107380	0.0553308	-0.0520266	0.0401361
0.1504590	$7.725 \cdot 10^{-4}$	0.0317150	-0.0120642
0.1354770	-0.0369247	0.0232752	0.0271491
0.1019350	-0.0658092	0.0266850	$-6.52 \cdot 10^{-4}$
-0.0353206	0.0559016	-0.0453980	-0.0368101
-0.0054084	-0.0102795	0.0326849	0.0344818
-0.1001350	-0.0543295	0.0450270	0.0148083
0.0831643	0.0045880	0.0168529	-0.0095816
0.0098914	-0.0149513	0.0427812	0.0229364

Table 4: First forty values for $Ei(100^{\rho}) + Ei(100^{\bar{\rho}})$ (Algorithm: Appendix 7.2.2.2)

7.2 Algorithms

7.2.1 Sieve of Eratosthenes in Typescript

```
function sieveOfErathosthenes(max: number): Array<number> {
 let flags: Array<boolean> = [];
 let primes: Array<number> = [];
 let prime = 2;

 let n: number = max;
 while(n--) { flags[max-n] = true;}

 for (prime = 2; prime < Math.sqrt(max); prime++) {
  if (flags[prime]) {
   for (let j = prime + prime; j < max; j += prime) { flags[j] = false;}
  }}

 for (let i = 2; i < max; i++) {
```

```
  if (flags[i]) {primes.push(i);}
 }
 return primes;
}
```

7.2.2 Mathematica

The variable `data` refers to the imaginary portion of the zeros(Odlyzko, n.d.).

7.2.2.1 Li

```
f1 = 100.0^(0.5 + Part[data]I)
f2 = 100.0^(0.5 - Part[data]I)
TableForm[LogIntegral[f1] + LogIntegral[f2] - 2LogIntegral[2]]
```

7.2.2.2 Ei

```
f1 = (0.5 + Part[data]I)*Log[100.0]
f2 = (0.5 - Part[data]I)*Log[100.0]
TableForm[ExpIntegralEi[f1] + ExpIntegralEi[f2]]
```

7.2.2.3 Sum of forty zeros of $Ei(\ln 100^{\rho})$ to $Ei(\ln 100^{\frac{\rho}{6}})$.

```
f1a = (0.5 + Part[data]I)*Log[100.0^(1/1)]
f1b = (0.5 - Part[data]I)*Log[100.0^(1/1)]
f2a = (0.5 + Part[data]I)*Log[100.0^(1/2)]
f2b = (0.5 - Part[data]I)*Log[100.0^(1/2)]
f3a = (0.5 + Part[data]I)*Log[100.0^(1/3)]
f3b = (0.5 - Part[data]I)*Log[100.0^(1/3)]
f5a = (0.5 + Part[data]I)*Log[100.0^(1/5)]
f5b = (0.5 - Part[data]I)*Log[100.0^(1/5)]
```

```
f6a = (0.5 + Part[data]I)*Log[100.0^(1/6)]

f6b = (0.5 - Part[data]I)*Log[100.0^(1/6)]


TableForm[(1/1)(ExpIntegralEi[f1a] + ExpIntegralEi[f1b])]

TableForm[(-1/2)(ExpIntegralEi[f2a] + ExpIntegralEi[f2b])]

TableForm[(-1/3)(ExpIntegralEi[f3a] + ExpIntegralEi[f3b])]

TableForm[(-1/5)(ExpIntegralEi[f5a] + ExpIntegralEi[f5b])]

TableForm[(1/6)(ExpIntegralEi[f6a] + ExpIntegralEi[f6b])]
```

7.3 Equations

7.3.1 Complex plain zeta function

$$\zeta(s) := (1-2^{1-s})^{-1} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s},\ 0 < \Re(s) < 1 \tag{11}$$

$$\begin{gathered} \Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x}\, dx \\ \zeta(s) := 2^s \pi^{s-1}\ \sin\left(\frac{\pi s}{2}\right)\ \Gamma(1-s)\ \zeta(1-s),\ \Re(s) < 0 \end{gathered} \tag{12}$$

7.3.2 Weighted prime counting function

$$\begin{gathered} J(n) = \sum_{k=1}^{\epsilon} \frac{1}{k} \pi(\sqrt[k]{n}),\ \sqrt[k]{\epsilon} \geq 2 \text{ and } n \in \mathbb{R}^+ \\ \pi(n) = 0 \text{ for } n < 2, \text{ hence the stipulation of} \sqrt[k]{\epsilon} \geq 2 \end{gathered} \tag{13}$$

$$\begin{aligned} J(100) &= \pi(100) + \frac{1}{2}\pi(\sqrt[2]{100}) + \frac{1}{3}\pi(\sqrt[3]{100}) + \frac{1}{4}\pi(\sqrt[4]{100}) + \frac{1}{5}\pi(\sqrt[5]{100}) + \frac{1}{6}\pi(\sqrt[6]{100}) \\ &= 25 + \frac{1}{2}\pi(10) + \frac{1}{3}\pi(4.642) + \frac{1}{4}\pi(3.162) + \frac{1}{5}\pi(2.512) + \frac{1}{6}\pi(2.154) \\ &= 25 + \frac{1}{2}4 + \frac{1}{3}2 + \frac{1}{4}2 + \frac{1}{5} + \frac{1}{6} = 28.53\bar{3} \end{aligned} \tag{14}$$