III.4 Classical Orthogonal Polynomials

Classical orthogonal polynomials are special families of orthogonal polynomials with a number of beautiful properties, for example

1. Their derivatives are also OPs
2. They are eigenfunctions of simple differential operators

As stated above orthogonal polynomials are uniquely defined by the weight $w(x)$ and the constant k_n. The classical orthogonal polynomials are:

1. Chebyshev polynomials (1st kind) $T_n(x)$: $w(x) = 1/\sqrt{1-x^2}$ on $[-1,1]$.
2. Chebyshev polynomials (2nd kind) $U_n(x)$: $\sqrt{1-x^2}$ on $[-1,1]$.
3. Legendre polynomials $P_n(x)$: $w(x) = 1$ on $[-1,1]$.
4. Ultrapsherical polynomials (my fav!): $C_n^{(\lambda)}(x)$: $w(x) = (1-x^2)^{\lambda-1/2}$ on $[-1,1]$, $\lambda \neq 0$, $\lambda > -1/2$.
5. Jacobi polynomials: $P_n^{(a,b)}(x)$: $w(x) = (1-x)^a(1+x)^b$ on $[-1,1]$, $a, b > -1$.
6. Laguerre polynomials: $L_n(x)$: $w(x) = \exp(-x)$ on $[0,\infty)$.
7. Hermite polynomials $H_n(x)$: $w(x) = \exp(-x^2)$ on $(-\infty,\infty)$.

In the notes we will discuss:

1. Chebyshev polynomials: These are closely linked to Fourier series and are one of the most powerful tools in numerics.
2. Legendre polynomials: These have no simple closed-form expression but can be defined in terms of a Rodriguez formula, a feature that applies to all other classical families.

1. Chebyshev

Definition 1 (Chebyshev polynomials, 1st kind) $T_n(x)$ are orthogonal with respect to $1/\sqrt{1-x^2}$ and satisfy:

$$\begin{aligned} T_0(x) &= 1, \\ T_n(x) &= 2^{n-1}x^n + O(x^{n-1}) \end{aligned}$$

Definition 2 (Chebyshev polynomials, 2nd kind) $U_n(x)$ are orthogonal with respect to $sqrt1 - x^2$.

$$U_n(x) = 2^n x^n + O(x^{n-1})$$

Theorem 1 (Chebyshev T are cos)

$$T_n(x) = \cos n \operatorname{acos} x$$

In other words

$$T_n(\cos\theta) = \cos n\theta.$$

Proof

We need to show that $p_n(x) := \cos n \mathrm{acos}\, x$ are

1. graded polynomials
2. orthogonal w.r.t. $1/\sqrt{1-x^2}$ on $[-1,1]$, and
3. have the right normalisation constant $k_n = 2^{n-1}$ for $n = 2, \ldots$.

Property (2) follows under a change of variables:

$$\int_{-1}^{1} \frac{p_n(x)p_m(x)}{\sqrt{1-x^2}} \mathrm{d}x = \int_{-\pi}^{\pi} \frac{\cos(n\theta)\cos(m\theta)}{\sqrt{1-\cos^2\theta}} \sin\theta \mathrm{d}\theta = \int_{-\pi}^{\pi} \cos(n\theta)\cos(m\theta) \mathrm{d}x = 0$$

if $n \neq m$.

To see that they are graded we use the fact that

$$xp_n(x) = \cos\theta \cos n\theta = \frac{\cos(n-1)\theta + \cos(n+1)\theta}{2} = \frac{p_{n-1}(x) + p_{n+1}(x)}{2}$$

In other words $p_{n+1}(x) = 2xp_n(x) - p_{n-1}(x)$. Since each time we multiply by $2x$ and $p_0(x) = 1$ we have

$$p_n(x) = (2x)^n + O(x^{n-1})$$

which completes the proof.

∎

Buried in the proof is the 3-term recurrence:

Corollary

$$\begin{aligned} xT_0(x) &= T_1(x) \\ xT_n(x) &= \frac{T_{n-1}(x) + T_{n+1}(x)}{2} \end{aligned}$$

Chebyshev polynomials are particularly powerful as their expansions are cosine series in disguise: for

$$f(x) = \sum_{k=0}^{\infty} f_k T_k(x)$$

we have

$$f(\cos\theta) = \sum_{k=0}^{\infty} f_k \cos k\theta.$$

Thus the coefficients can be recovered fast using FFT-based techniques as discussed in the problem sheet.

In the problem sheet we will also show the following:

Theorem 2 (Chebyshev U are sin) For $x = \cos\theta$,

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}$$

which satisfy:

$$
\begin{aligned}
xU_0(x) &= U_1(x)/2 \\
xU_n(x) &= \frac{U_{n-1}(x)}{2} + \frac{U_{n+1}(x)}{2}.
\end{aligned}
$$

2. Legendre

Definition 3 (Legendre) Legendre polynomials $P_n(x)$ are orthogonal polynomials with respect to $w(x) = 1$ on $[-1, 1]$, with

$$
k_n = \frac{1}{2^n}\binom{2n}{n} = \frac{(2n)!}{2^n (n!)^2}
$$

The reason for this complicated normalisation constant is both historical and that it leads to simpler formulae for recurrence relationships.

Classical orthogonal polynomials have *Rodriguez formulae*, defining orthogonal polynomials as high order derivatives of simple functions. In this case we have:

Lemma 1 (Legendre Rodriguez formula)

$$
P_n(x) = \frac{1}{(-2)^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(1-x^2)^n
$$

Proof We need to verify:

1. graded polynomials
2. orthogonal to all lower degree polynomials on $[-1, 1]$, and
3. have the right normalisation constant $k_n = \frac{1}{2^n}\binom{2n}{n}$.

(1) follows since its a degree n polynomial (the n-th derivative of a degree $2n$ polynomial). (2) follows by integration by parts. Note that $(1-x^2)^n$ and its first $n-1$ derivatives vanish at ± 1. If r_m is a degree $m < n$ polynomial we have:

$$
\int_{-1}^{1} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(1-x^2)^n r_m(x) \mathrm{d}x = -\int_{-1}^{1} \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(1-x^2)^n r_m'(x) \mathrm{d}x = \cdots = (-)^n \int_{-1}^{1} (1-x^2) r_
$$

(3) follows since:

$$
\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}x^n}[(-)^n x^{2n} + O(x^{2n-1})] &= (-)^n 2n \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} x^{2n-1} + O(x^{2n-1})] \\
&= (-)^n 2n(2n-1) \frac{\mathrm{d}^{n-2}}{\mathrm{d}x^{n-2}} x^{2n-2} + O(x^{2n-2})] = \cdots \\
&= (-)^n 2n(2n-1)\cdots(n+1)x^n + O(x^{n-1}) = (-)^n \frac{(2n)!}{n!} x^n +
\end{aligned}
$$

∎

This allows us to determine the coefficients $k_n^{(\lambda)}$ which are useful in proofs. In particular we will use $k_n^{(2)}$:

Lemma 2 (Legendre monomial coefficients)

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_n(x) &= \underbrace{\frac{(2n)!}{2^n(n!)^2}}_{k_n} x^n - \underbrace{\frac{(2n-2)!}{2^n(n-2)!(n-1)!}}_{k_n^{(2)}} x^{n-2} + O(x^{n-4})
\end{aligned}
$$

(Here the $O(x^{n-4})$ is as $x \to \infty$, which implies that the term is a polynomial of degree $\leq n-4$. For $n = 2, 3$ the $O(x^{n-4})$ term is therefore precisely zero.)

Proof

The $n = 0$ and 1 case are immediate. For the other case we expand $(1-x^2)^n$ to get:

$$
\begin{aligned}
(-)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n}(1-x^2)^n &= \frac{\mathrm{d}^n}{\mathrm{d}x^n}[x^{2n} - nx^{2n-2} + O(x^{2n-4})] \\
&= (2n)\cdots(2n-n+1)x^n - n(2n-2)\cdots(2n-2-n+1)x^{n-2} + O \\
&= \frac{(2n)!}{n!}x^n - \frac{n(2n-2)!}{(n-2)!}x^{n-2} + O(x^{n-4})
\end{aligned}
$$

Multiplying through by $\frac{1}{2^n(n!)}$ completes the derivation.

∎

Theorem 3 (Legendre 3-term recurrence)

$$
\begin{aligned}
xP_0(x) &= P_1(x) \\
(2n+1)xP_n(x) &= nP_{n-1}(x) + (n+1)P_{n+1}(x)
\end{aligned}
$$

Proof The $n = 0$ case is immediate (since $w(x) = w(-x)$ $a_n = 0$, from PS8). For the other cases we match terms:

$$
(2n+1)xP_n(x) - nP_{n-1}(x) - (n+1)P_{n+1}(x) = [(2n+1)k_n - (n+1)k_{n+1}]x^{n+1} + [(2n
$$

Using the expressions for k_n and $k_n^{(2)}$ above we have (leaving the manipulations as an exercise):

$$
\begin{aligned}
(2n+1)k_n - (n+1)k_{n+1} &= \frac{(2n+1)!}{2^n(n!)^2} \\
(2n+1)k_n^{(2)} - nk_{n-1} - (n+1)k_{n+1}^{(2)} &= -(2n+1)\frac{(2n-2)!}{2^n(n-2)!(n-1)!} - n\frac{(2n-2)!}{2^{n-1}((n-1)!)^2}
\end{aligned}
$$

Thus

$$
(2n+1)xP_n(x) - nP_{n-1}(x) - (n+1)P_{n+1}(x) = O(x^{n-3})
$$

But as it is orthogonal to $P_k(x)$ for $0 \leq k \leq n-3$ it must be zero. ∎