MATH50003 Numerical Analysis (2022–23)

Problem Sheet 4

This problem sheet concerns matrix multiplication, permutation matrices, and properties of orthogonal/unitary matrices.

Problem 1.1 True or false: `mul_rows(A, x)` and `mul_cols(A, x)` from the lectures/notes will always return the exact same result if the input are floating point? Explain your answer.

Problem 1.2 Express the vector that `mul_rows(A, x)` returns in terms of $\bigoplus_{j=1}^n$ and $\otimes$, using the notation $A_{k,j}$ for the k, j entry of A and x_j for the j-th entry of $\mathbf{x}$, where $A \in F_{\sigma,Q,S}^{m\times n}$ and $\mathbf{x} \in F_{\sigma,Q,S}^n$.

Problem 1.3 Show when implemented in floating point arithmetic for $A \in F_{\sigma,Q,S}^{m\times n}$ and $\mathbf{x} \in F_{\sigma,Q,S}^n$ that

$$\backslash\text{tt mul}\backslash\_\text{rows}(A, \mathbf{x}) = A\mathbf{x} + \boldsymbol{\delta}$$

where

$$\|\delta\|_\infty \leq 2\|A\|_\infty\|\mathbf{x}\|_\infty E_{n-1,\epsilon_{\rm m}/2}$$

and

$$E_{n,\varepsilon} := \frac{n\varepsilon}{1 - n\varepsilon},$$

assuming that $n\epsilon_{\rm m} < 2$. We use the notation (to be discussed in detail later):

$$\|A\|_\infty := \max_k \sum\nolimits_{j=1}^n |a_{kj}|, \|\mathbf{x}\|_\infty := \max_k |x_k|.$$

You may assume all operations are in the normalised range and use Problem 2.3 from PS2.

Problem 2 What are the permutation matrices corresponding to the following permutations?

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix}.$$

Problem 3.1 Show for a unitary matrix $Q \in U(n)$ and a vector $\mathbf{x} \in \mathbb{C}^n$ that multiplication by Q preserve the 2-norm:

$$\|Q\mathbf{x}\| = \|\mathbf{x}\|.$$

Problem 3.2 Show that the eigenvalues λ of a unitary matrix Q are on the unit circle: $|\lambda| = 1$. Hint: recall for any eigenvalue λ that there exists a unit vector $\mathbf{v} \in \mathbb{C}^n$ (satisfying

$\|\mathbf{v}\| = 1$). Combine this with Problem 3.1.

Problem 3.3 Show for an orthogonal matrix $Q \in O(n)$ that $\det Q = \pm 1$. Give an example of $Q \in U(n)$ such that $\det Q \neq \pm 1$. Hint: recall for any real matrices A and B that $\det A = \det A^\top$ and $\det(AB) = \det A \det B$.

Problem 3.4 Show that $Q \in U(n)$ is a normal matrix (that it commutes with its adjoint).

Problem 3.5 Explain why $Q \in U(n)$ must be equal to I if all its eigenvalues are 1. Hint: use the spectral theorem, which says that any normal matrix is diagonalisable with unitary eigenvectors (see notes on C. Adjoints and Normal Matrices).