III.5 Interpolation and quadrature

Polynomial interpolation is the process of finding a polynomial that equals data at a precise set of points. *Quadrature* is the act of approximating an integral by a weighted sum:

$$\int_a^b f(x)w(x)\mathrm{d}x \approx \sum_{j=1}^n w_j f(x_j).$$

In these notes we see that the two concepts are intrinsically linked: interpolation leads naturally to quadrature rules.

1. Polynomial Interpolation: we describe how to interpolate a function by a polynomial and a set of points.
2. Interpolatory quadrature rule: polynomial interpolation leads naturally to ways to integrate functions, but onely realisable in the simplest cases.

1. Polynomial Interpolation

We already saw a special case of polynomial interpolation, where we saw that the polynomial

$$f(z) \approx \sum_{k=0}^{n-1} f_k^n z^k$$

equaled f at evenly spaced points on the unit circle: $\mathrm{e}^{\mathrm{i}2\pi j/n}$. But here we consider the following:

Definition 1 (interpolatory polynomial) Given n distinct points $x_1, \ldots, x_n \in \mathbb{R}$ and n *samples* $f_1, \ldots, f_n \in \mathbb{R}$, a degree $n-1$ *interpolatory polynomial* $p(x)$ satisfies

$$p(x_j) = f_j$$

The easiest way to solve this problem is to invert the Vandermonde system:

Definition 2 (Vandermonde) The *Vandermonde matrix* associated with n distinct points $x_1, \ldots, x_n \in \mathbb{R}$ is the matrix

$$V := \begin{bmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{bmatrix}$$

Proposition 1 (interpolatory polynomial uniqueness) The interpolatory polynomial is unique, and the Vandermonde matrix is invertible.

Proof Suppose p and $\tilde{p}$ are both interpolatory polynomials. Then $p(x) - \tilde{p}(x)$ vanishes at n distinct points x_j. By the fundamental theorem of algebra it must be zero, i.e., $p = \tilde{p}$.

For the second part, if $V\mathbf{c} = 0$ for $\mathbf{c} \in \mathbb{R}$ then for $q(x) = c_1 + \cdots + c_n x^{n-1}$ we have

$$q(x_j) = \mathbf{e}_j^\top V \mathbf{c} = 0$$

hence q vanishes at n distinct points and is therefore 0, i.e., $\mathbf{c} = 0$.

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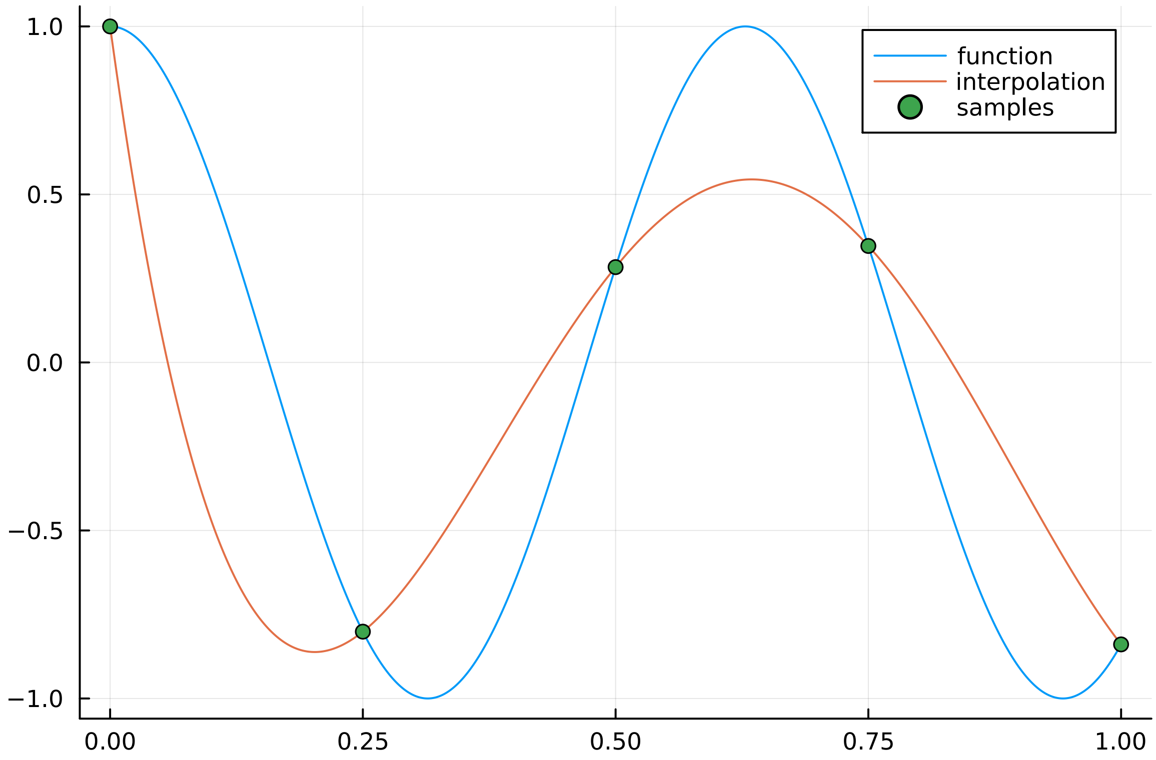
Thus a quick-and-dirty way to to do interpolation is to invert the Vandermonde matrix (which we saw in the least squares setting with more samples then coefficients):

In [ ]:
```julia
using Plots, LinearAlgebra
f = x -> cos(10x)
n = 5

x = range(0, 1; length=n)# evenly spaced points (BAD for interpolation)
V = x .^ (0:n-1)' # Vandermonde matrix
c = V \ f.(x) # coefficients of interpolatory polynomial
p = x -> dot(c, x .^ (0:n-1))

g = range(0,1; length=1000) # plotting grid
plot(g, f.(g); label="function")
plot!(g, p.(g); label="interpolation")
scatter!(x, f.(x); label="samples")
```

Out[ ]:



But it turns out we can also construct the interpolatory polynomial directly. We will use the following which equal 1 at one grid point and zero at the others:

Definition 3 (Lagrange basis polynomial) The *Lagrange basis polynomial* is defined as

$$\ell_k(x) := \prod_{j \neq k} \frac{x - x_j}{x_k - x_j} = \frac{(x - x_1) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_1) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)}$$

Plugging in the grid points verifies the following:

Proposition 2 (delta interpolation)

$$\ell_k(x_j) = \delta_{kj}$$

We can use these to construct the interpolatory polynomial:

Theorem 1 (Lagrange interpolation) The unique polynomial of degree at most $n-1$ that interpolates f at n distinct points x_j is:

$$p(x) = f(x_1)\ell_1(x) + \cdots + f(x_n)\ell_n(x)$$

Proof Note that

$$p(x_j) = \sum_{j=1}^n f(x_j)\ell_k(x_j) = f(x_j)$$

so we just need to show it is unique. Suppose $\tilde{p}(x)$ is a polynomial of degree at most $n-1$ that also interpolates f. Then $\tilde{p} - p$ vanishes at n distinct points. Thus by the fundamental theorem of algebra it must be zero.

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Example 1 We can interpolate $\exp(x)$ at the points $0, 1, 2$:

$$\begin{aligned} p(x) &= \ell_1(x) + \mathrm{e}\ell_2(x) + \mathrm{e}^2\ell_3(x) = \frac{(x-1)(x-2)}{(-1)(-2)} + \mathrm{e}\frac{x(x-2)}{(-1)} + \mathrm{e}^2\frac{x(x-1)}{2} \\ &= (1/2 - \mathrm{e} + \mathrm{e}^2/2)x^2 + (-3/2 + 2\mathrm{e} - \mathrm{e}^2/2)x + 1 \end{aligned}$$

Remark Interpolating at evenly spaced points is a really **bad** idea: interpolation is inheritely ill-conditioned. The labs have explored this issue experimentally.

2. Interpolatory quadrature rules

By integrating an interpolant exactly we get a simple approach to approximating integrals. Using the Lagrange basis we can rewrite this procedure as a simple weighted sum:

Definition 4 (interpolatory quadrature rule) Given a set of points $\mathbf{x} = [x_1, \ldots, x_n]$ the interpolatory quadrature rule is:

$$\Sigma_n^{w,\mathbf{x}}[f] := \sum_{j=1}^n w_j f(x_j)$$

where

$$w_j := \int_a^b \ell_j(x) w(x) \mathrm{d}x$$

Proposition 3 (interpolatory quadrature is exact for polynomials) Interpolatory quadrature is exact for all degree $n-1$ polynomials p:

$$\int_a^b p(x) w(x) \mathrm{d}x = \Sigma_n^{w,\mathbf{x}}[f]$$

Proof The result follows since, by uniqueness of interpolatory polynomial:

$$p(x) = \sum_{j=1}^n p(x_j)\ell_j(x)$$

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Example 2 (arbitrary points) Find the interpolatory quadrature rule for $w(x) = 1$ on $[0, 1]$ with points $[x_1, x_2, x_3] = [0, 1/4, 1]$? We have:

$$\begin{aligned} w_1 &= \int_0^1 w(x)\ell_1(x)\mathrm{d}x = \int_0^1 \frac{(x-1/4)(x-1)}{(-1/4)(-1)}\mathrm{d}x = -1/6 \\ w_2 &= \int_0^1 w(x)\ell_2(x)\mathrm{d}x = \int_0^1 \frac{x(x-1)}{(1/4)(-3/4)}\mathrm{d}x = 8/9 \\ w_3 &= \int_0^1 w(x)\ell_3(x)\mathrm{d}x = \int_0^1 \frac{x(x-1/4)}{3/4}\mathrm{d}x = 5/18 \end{aligned}$$

That is we have

$$\Sigma_n^{w,\mathbf{x}}[f] = -\frac{f(0)}{6} + \frac{8f(1/4)}{9} + \frac{5f(1)}{18}$$

This is indeed exact for polynomials up to degree 2 (and no more):

$$\Sigma_n^{w,\mathbf{x}}[1] = 1, \Sigma_n^{w,\mathbf{x}}[x] = 1/2, \Sigma_n^{w,\mathbf{x}}[x^2] = 1/3, \Sigma_n^{w,\mathbf{x}}[x^3] = 7/24 \neq 1/4.$$

Example 3 (Chebyshev roots) Find the interpolatory quadrature rule for $w(x) = 1/\sqrt{1-x^2}$ on $[-1, 1]$ with points equal to the roots of $T_3(x)$. This is a special case of Gaussian quadrature which we will approach in another way below. We use:

$$\int_{-1}^1 w(x)\mathrm{d}x = \pi, \int_{-1}^1 xw(x)\mathrm{d}x = 0, \int_{-1}^1 x^2 w(x)\mathrm{d}x = \pi/2$$

Recall from before that $x_1, x_2, x_3 = \sqrt{3}/2, 0, -\sqrt{3}/2$. Thus we have:

$$\begin{aligned} w_1 &= \int_{-1}^1 w(x)\ell_1(x)\mathrm{d}x = \int_{-1}^1 \frac{x(x+\sqrt{3}/2)}{(\sqrt{3}/2)\sqrt{3}\sqrt{1-x^2}}\mathrm{d}x = \frac{\pi}{3} \\ w_2 &= \int_{-1}^1 w(x)\ell_2(x)\mathrm{d}x = \int_{-1}^1 \frac{(x-\sqrt{3}/2)(x+\sqrt{3}/2)}{(-3/4)\sqrt{1-x^2}}\mathrm{d}x = \frac{\pi}{3} \\ w_3 &= \int_{-1}^1 w(x)\ell_3(x)\mathrm{d}x = \int_{-1}^1 \frac{(x-\sqrt{3}/2)x}{(-\sqrt{3})(-\sqrt{3}/2)\sqrt{1-x^2}}\mathrm{d}x = \frac{\pi}{3} \end{aligned}$$

(It's not a coincidence that they are all the same but this will differ for roots of other OPs.) That is we have

$$\Sigma_n^{w,\mathbf{x}}[f] = \frac{\pi}{3}(f(\sqrt{3}/2) + f(0) + f(-\sqrt{3}/2))$$

This is indeed exact for polynomials up to degree $n-1 = 2$, but it goes all the way up to $2n-1 = 5$:

$$\begin{aligned} \Sigma_n^{w,\mathbf{x}}[1] &= \pi, \Sigma_n^{w,\mathbf{x}}[x] = 0, \Sigma_n^{w,\mathbf{x}}[x^2] = \frac{\pi}{2}, \\ \Sigma_n^{w,\mathbf{x}}[x^3] &= 0, \Sigma_n^{w,\mathbf{x}}[x^4] &= \frac{3\pi}{8}, \Sigma_n^{w,\mathbf{x}}[x^5] = 0 \\ \Sigma_n^{w,\mathbf{x}}[x^6] &= \frac{9\pi}{32} \neq \frac{5\pi}{16} \end{aligned}$$

We shall explain this miracle in the next chapter.