III.1 Fourier expansions

In Part III, Computing with Functions, we work with approximating functions by expansions in bases: that is, instead of approximating at a grid (as in the Differential Equations chapter), we approximate functions by other, simpler, functions. The ultimate goal is to use such expansions to numerically approximate solutions to ordinary and partial differential equations.

1. Review of Fourier series
2. Trapezium rule and discrete orthogonality
3. Convergence of approximate Fourier expansions

1. Review of Fourier series

The most fundamental basis is (complex) Fourier: we have $\mathrm{e}^{ik\theta}$ are orthogonal with respect to the inner product

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} \bar{f}(\theta) g(\theta) \mathrm{d}\theta,$$

where we conjugate the first argument to be consistent with the vector inner product $\mathbf{x}^\star \mathbf{y}$. We can (typically) expand functions in this basis:

Definition 1 (Fourier) A function f has a Fourier expansion if

$$f(\theta) = \sum_{k=-\infty}^{\infty} \hat{f}_k \mathrm{e}^{ik\theta}$$

where

$$\hat{f}_k := \langle \mathrm{e}^{ik\theta}, f \rangle = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-ik\theta} f(\theta) \mathrm{d}\theta$$

A basic observation is if a Fourier expansion has no negative terms it is equivalent to a Taylor series if we write $z = \mathrm{e}^{\mathrm{i}\theta}$:

Definition 2 (Fourier–Taylor) A function f has a Fourier–Taylor expansion if

$$f(\theta) = \sum_{k=0}^{\infty} \hat{f}_k \mathrm{e}^{ik\theta}$$

where $\hat{f}_k := \langle \mathrm{e}^{ik\theta}, f \rangle$.

In numerical analysis we try to build on the analogy with linear algebra as much as possible. Therefore we can write this this as:

$$f(\theta) = \underbrace{[1 | \mathrm{e}^{\mathrm{i}\theta} | \mathrm{e}^{2\mathrm{i}\theta} | \cdots]}_{T(\theta)} \underbrace{\begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \vdots \end{bmatrix}}_{\hat{\mathbf{f}}}.$$

Essentially, expansions in bases are viewed as a way of turning *functions* into (infinite) *vectors*. And (differential) *operators* into *matrices*.

In analysis one typically works with continuous functions and relates results to continuity. In numerical analysis we inheritely have to work with *vectors*, so it is more natural to focus on the case where the *Fourier coefficients* $\mathfrak{f}_k$ are *absolutely convergent*:

Definition 3 (absolute convergent) We write $\mathbf{\mathfrak{f}} \in \ell^1$ if it is absolutely convergent, or in otherwords, the 1-norm of $\mathbf{\mathfrak{f}}$ is bounded:

$$\|\mathbf{\mathfrak{f}}\|_1 := \sum_{k=-\infty}^{\infty} |\mathfrak{f}_k| < \infty$$

We first state a couple basic results (whose proof is beyond the scope of this module):

Theorem 1 (2-norm convergence) A function f has bounded 2-norm

$$\|f\|_2 := \sqrt{\langle f, f\rangle} = \sqrt{\int_0^{2\pi} |f(\theta)|^2 \mathrm{d}\theta} < \infty$$

if and only if

$$\|\mathbf{\mathfrak{f}}\|_2 = \sqrt{\mathbf{\mathfrak{f}}^\star \mathbf{\mathfrak{f}}} = \sqrt{\sum_{k=-\infty}^{\infty} |\mathfrak{f}_k|^2} < \infty.$$

In this case

$$\|f - \lim_{m\to\infty} \sum_{k=-m}^{m} \mathfrak{f}_{\mathrm{k}} \mathrm{e}^{\mathrm{i}k\theta}\| \to 0$$

Theorem 2 (Absolute convergence) If $\mathbf{\mathfrak{f}} \in \ell^1$ then

$$f(\theta) = \sum_{k=-\infty}^{\infty} \mathfrak{f}_{\mathrm{k}} \mathrm{e}^{\mathrm{i}k\theta},$$

which converges uniformly.

Continuity gives us sufficient (though not necessary) conditions for absolute convergence:

Lemma 1 (differentiability and absolutely convergence) If $f : \mathbb{R} \to \mathbb{C}$ and f' are periodic and f'' is uniformly bounded, then $\mathbf{\mathfrak{f}} \in \ell^1$.

Proof Integrate by parts twice using the fact that $f(0) = f(2\pi)$, $f'(0) = f(2\pi)$:

$$\begin{aligned}
2\pi \mathfrak{f}_{\mathrm{k}} &= \int_0^{2\pi} f(\theta) \mathrm{e}^{-\mathrm{i}k\theta} \mathrm{d}\theta = [f(\theta)\mathrm{e}^{-\mathrm{i}k\theta}]_0^{2\pi} + \frac{1}{\mathrm{i}k} \int_0^{2\pi} f'(\theta)\mathrm{e}^{-\mathrm{i}k\theta} \mathrm{d}\theta \\
&= \frac{1}{\mathrm{i}k} [f'(\theta)\mathrm{e}^{-\mathrm{i}k\theta}]_0^{2\pi} - \frac{1}{k^2} \int_0^{2\pi} f''(\theta)\mathrm{e}^{-\mathrm{i}k\theta} \mathrm{d}\theta \\
&= -\frac{1}{k^2} \int_0^{2\pi} f''(\theta)\mathrm{e}^{-\mathrm{i}k\theta} \mathrm{d}\theta
\end{aligned}$$

thus uniform boundedness of f'' guarantees $|\mathfrak{f}_{\mathrm{k}}| \leq M|k|^{-2}$ for some M, and we have

$$\sum_{k=-\infty}^{\infty} |\mathfrak{f}_{\mathrm{k}}| \leq |\mathfrak{f}_0| + 2M \sum_{k=1}^{\infty} |k|^{-2} < \infty.$$

using the dominant convergence test.

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This condition can be weakened to Lipschitz continuity but the proof is beyond the scope of this module. Of more practical importance is the other direction: the more times differentiable a function the faster the coefficients decay, and thence the faster Fourier expansions converge. In fact, if a function is smooth and 2π-periodic its Fourier coefficients decay faster than algebraically: they decay like $O(k^{-\lambda})$ for any λ. This will be explored in the problem sheet.

Remark (advanced) Going further, if we let $z = \mathrm{e}^{\mathrm{i}\theta}$ then if $f(z)$ is *analytic* in a neighbourhood of the unit circle the Fourier coefficients decay *exponentially fast*. And if $f(z)$ is entire they decay even faster than exponentially.

2. Trapezium rule and discrete Fourier coefficients

Definition 4 (Trapezium Rule) Let $\theta_j = 2\pi j/n$ for $j = 0, 1, \ldots, n$ denote $n+1$ evenly spaced points over $[0, 2\pi]$. The *Trapezium rule* over $[0, 2\pi]$ is the approximation:

$$\int_0^{2\pi} f(\theta)\mathrm{d}\theta \approx \frac{2\pi}{n}\left[\frac{f(0)}{2} + \sum_{j=1}^{n-1} f(\theta_j) + \frac{f(2\pi)}{2}\right]$$

But if f is periodic we have $f(0) = f(2\pi)$ we get the *periodic Trapezium rule*:

$$\int_0^{2\pi} f(\theta)\mathrm{d}\theta \approx 2\pi \underbrace{\frac{1}{n}\sum_{j=0}^{n-1} f(\theta_j)}_{\Sigma_n[f]}$$

We know that $\mathrm{e}^{\mathrm{i}k\theta}$ are orthogonal with respect to the continuous inner product. The following says that this property is maintained (up to "aliasing") when we replace the continuous integral with a trapezium rule approximation:

Lemma 2 (Discrete orthogonality) We have:

$$\sum_{j=0}^{n-1} \mathrm{e}^{\mathrm{i}k\theta_j} = \begin{cases} n & k = \ldots, -2n, -n, 0, n, 2n, \ldots \\ 0 & \text{otherwise} \end{cases}$$

In other words,

$$\Sigma_n[\mathrm{e}^{\mathrm{i}(k-\ell)\theta}] = \begin{cases} 1 & k-\ell = \ldots, -2n, -n, 0, n, 2n, \ldots \\ 0 & \text{otherwise} \end{cases}.$$

Proof

Consider $\omega := \mathrm{e}^{\mathrm{i}\theta_1} = \mathrm{e}^{\frac{2\pi\mathrm{i}}{n}}$. This is an n th root of unity: $\omega^n = 1$. Note that $\mathrm{e}^{\mathrm{i}\theta_j} = \mathrm{e}^{\frac{2\pi\mathrm{i}j}{n}} = \omega^j$.

(Case 1: $k = pn$ for an integer p) We have

$$\sum_{j=0}^{n-1} \mathrm{e}^{\mathrm{i}k\theta_j} = \sum_{j=0}^{n-1} \omega^{kj} = \sum_{j=0}^{n-1} (\omega^{pn})^j = \sum_{j=0}^{n-1} 1 = n$$

(Case 2 $k \neq pn$ for an integer p) Recall that

$$\sum_{j=0}^{n-1} z^j = \frac{z^n - 1}{z - 1}.$$

Then we have

$$\sum_{j=0}^{n-1} \mathrm{e}^{\mathrm{i}k\theta_j} = \sum_{j=0}^{n-1} (\omega^k)^j = \frac{\omega^{kn} - 1}{\omega^k - 1} = 0.$$

where we use the fact that k is not a multiple of n to guarantee that $\omega^k \neq 1$.

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1. Convergence of Approximate Fourier expansions

We will now use the Trapezium rule to approximate Fourier coefficients and expansions:

Definition 5 (Discrete Fourier coefficients) Define the Trapezium rule approximation to the Fourier coefficients by:

$$\mathfrak{f}_k^n := \Sigma_n[\mathrm{e}^{-ik\theta} f(\theta)] = \frac{1}{n} \sum_{j=0}^{n-1} \mathrm{e}^{-ik\theta_j} f(\theta_j)$$

A remarkable fact is that the discete Fourier coefficients can be expressed as a sum of the true Fourier coefficients:

Theorem 3 (discrete Fourier coefficients) If $\mathbf{\mathfrak{f}} \in \ell^1$ (absolutely convergent Fourier coefficients) then

$$\mathfrak{f}_k^n = \cdots + \mathfrak{f}_{k-2n} + \mathfrak{f}_{k-n} + \mathfrak{f}_k + \mathfrak{f}_{k+n} + \mathfrak{f}_{k+2n} + \cdots$$

Proof

$$\begin{aligned}
\mathfrak{f}_k^n &= \Sigma_n[f(\theta)\mathrm{e}^{-\mathrm{i}k\theta}] = \sum_{\ell=-\infty}^{\infty} \mathfrak{f}_\ell \Sigma_n[\mathrm{e}^{\mathrm{i}(\ell-k)\theta}] \\
&= \sum_{\ell=-\infty}^{\infty} \mathfrak{f}_\ell \begin{cases} 1 & \ell - k = \ldots, -2n, -n, 0, n, 2n, \ldots \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

∎

Note that there is redundancy:

Corollary 1 (aliasing) For all $p \in \mathbb{Z}$, $\mathfrak{f}_k^n = \mathfrak{f}_{k+pn}^n$.

In other words if we know $\mathfrak{f}_0^n, \ldots, \mathfrak{f}_{n-1}^n$, we know $\mathfrak{f}_k^n$ for all k via a permutation, for example if $n = 2m + 1$ we have

$$\begin{bmatrix} \mathfrak{f}_{-m}^n \\ \vdots \\ \mathfrak{f}_m^n \end{bmatrix} = \underbrace{\begin{bmatrix} & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \\ 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \end{bmatrix}}_{P_\sigma} \begin{bmatrix} \mathfrak{f}_0^n \\ \vdots \\ \mathfrak{f}_{n-1}^n \end{bmatrix}$$

where σ has Cauchy notation (*Careful*: we are using 1-based indexing here):

$$\begin{pmatrix} 1 & 2 & \cdots & m & m+1 & m+2 & \cdots & n \\ m+2 & m+3 & \cdots & n & 1 & 2 & \cdots & m+1 \end{pmatrix}.$$

We first discuss the case when all negative coefficients are zero. That is, $\mathfrak{f}_0^n, \ldots, \mathfrak{f}_{n-1}^n$ are approximations of the Fourier–Taylor coefficients by evaluating on the boundary.

We can prove *convergence* whenever of this approximation whenever f has absolutely summable coefficients. We will prove the result here in the special case where the negative coefficients are zero.

Theorem 4 (Approximate Fourier–Taylor expansions converge) If $0 = \mathfrak{f}_{-1} = \mathfrak{f}_{-2} = \cdots$ and $\mathbf{\mathfrak{f}}$ is absolutely convergent then

$$\mathfrak{f}_n(\theta) = \sum_{k=0}^{n-1} \mathfrak{f}_k^n \mathrm{e}^{\mathrm{i}k\theta}$$

converges uniformly to $f(\theta)$.

Proof

$$|f(\theta) - \mathfrak{f}_n(\theta)| = |\sum_{k=0}^{n-1} (\mathfrak{f}_k - \mathfrak{f}_k^n)\mathrm{e}^{\mathrm{i}k\theta} + \sum_{k=n}^{\infty} \mathfrak{f}_k \mathrm{e}^{\mathrm{i}k\theta}| = |\sum_{k=n}^{\infty} \mathfrak{f}_k(\mathrm{e}^{\mathrm{i}k\theta} - \mathrm{e}^{\mathrm{i}\mathrm{mod}(k,n)\theta})| \leq$$

which goes to zero as $n \to \infty$. ∎

For the general case we need to choose a range of coefficients that includes roughly an equal number of negative and positive coefficients (preferring negative over positive in a tie as a convention):

$$\mathfrak{f}_n(\theta) = \sum_{k=-\lceil n/2 \rceil}^{\lfloor n/2 \rfloor} \mathfrak{f}_\mathrm{k} \mathrm{e}^{\mathrm{i}k\theta}$$

In the problem sheet we will prove this converges provided the coefficients are absolutely convergent.