高级算法

Advanced Topics in Algorithms

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Chapter 5：Finite Dynamic Games

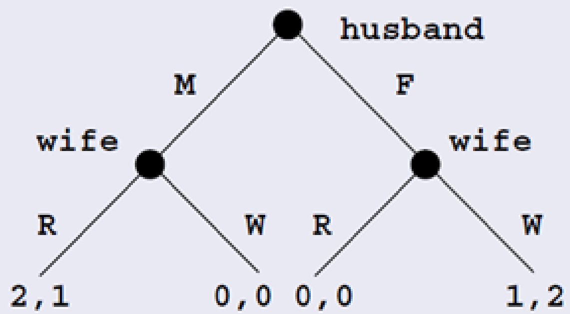
Outline

Introduction

- Previously, we studied static game in which decisions are assumed to be made *simultaneously*.
- In dynamic games, there is an explicit time-schedule that describes when players make their decisions.
- We use game tree: an extensive form of game representation, to examine dynamic games.
- In a game tree: we have (a) decision nodes; (b) branch due to an action; (c) payoff at the end of a path.

Example

- A husband and a wife are buying items for a dinner party. The husband buys either fish (F) or meat (M) for the main course; the wife buys either red wine (R) or white wine (W).
- Both stick to the convention: R goes with M, W goes with F.
- But the husband prefers M over F, while the wife prefers F over M.
- Utility: $\pi_h(M, R) = 2$, $\pi_h(F, W) = 1$, $\pi_h(F, R) = \pi_h(M, W) = 0$; $\pi_w(M, R) = 1$, $\pi_w(F, W) = 2$, $\pi_w(F, R) = \pi_w(M, W) = 0$.



- Husband moves first. Using backward induction, what is the solution of this game?

Analyzing the Nash Equilibria

- For the solution we found in the game between the husband and the wife, is it a Nash equilibrium?
- The set of actions are $\boldsymbol{A}_h = \{M, F\}$, $\boldsymbol{A}_w = \{R, W\}$.
- The set of pure strategies: $\boldsymbol{S}_h = \{M, F\}$, $\boldsymbol{S}_w = \{RR, RW, WR, WW\}$ where X, Y denote the "play X if husband chooses M, and Y if he chooses F".
- In normal (or strategic) form, the game has the following payoff table:

	R, R	R, W	W, R	W, W
M	$\underline{2}, \underline{1}$	$\underline{2}, \underline{1}$	$0, 0$	$0, 0$
F	$0, 0$	$1, \underline{2}$	$0, 0$	$\underline{1}, \underline{2}$

- Three pure strategies NE: $(M, RR), (M, RW), (F, WW)$
- Any pair (M, σ_2^*) where σ_2^* assigns probability p to RR and $(1 - p)$ to RW, is also a NE, or we have infinite number of NE.
- Solution by the backward induction is only one of the many NE.

Comment on having multiple NE

- Although there are many NE, not all are equally believable.
- Consider the option (W, W).
  - If the wife chooses (W, W) and informs her husband, what should her husband do?
  - If the husband informs his wife that he has bought "M", what should be the rational behavior of his wife?

  In other words, the NE (F, WW) is an unbelievable treat !!!
- Now consider the strategies RR and σ_2^*, are these strategies (infinite number of them), *believable*? Can these strategies by the wife be considered a believable threat?
- In conclusion, the NE found by the normal (or strategic) form may contain unbelievable threat, while the solution of backward induction is more reasonable.

Difference between Static vs. Dynamic Game

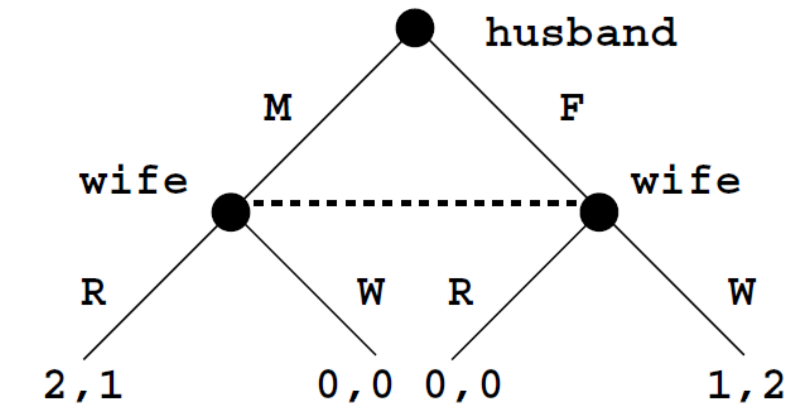
- The difference is not that static game is represented by normal (strategic) form while dynamic game is represented by extensive form (game tree). As a matter fact, both can be represented by normal form or extensive form.
- The main difference between them is what is known by the players when they make their decision !! In the previous game, the wife knew whether her husband had bought meat or fish when she needs to choose between red or white wine.

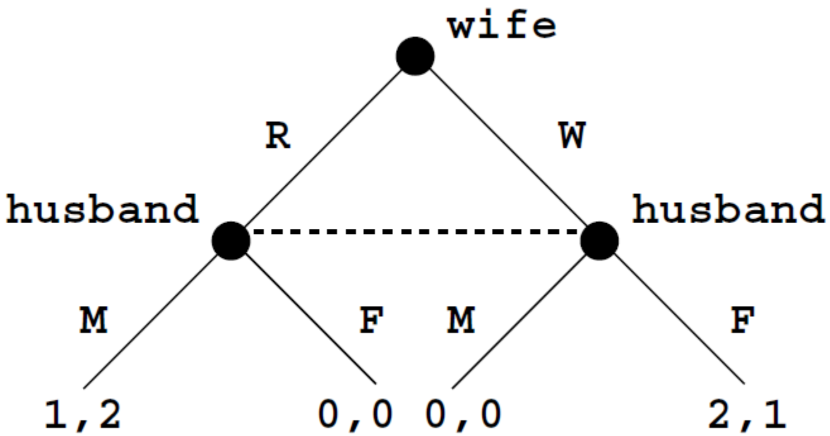
Definition

An information set for a player is a set of decision nodes in a game tree such that

1. the player concerned (and no other) is making a decision;
2. the player does not know which node has been reached (only that it is one of the nodes in the set). So a player must have the same choices at all nodes in an information set.

Example of Information Set





	R	W
M	2,1	0,0
F	0,0	1,2

Recall the previous definition:

Definition

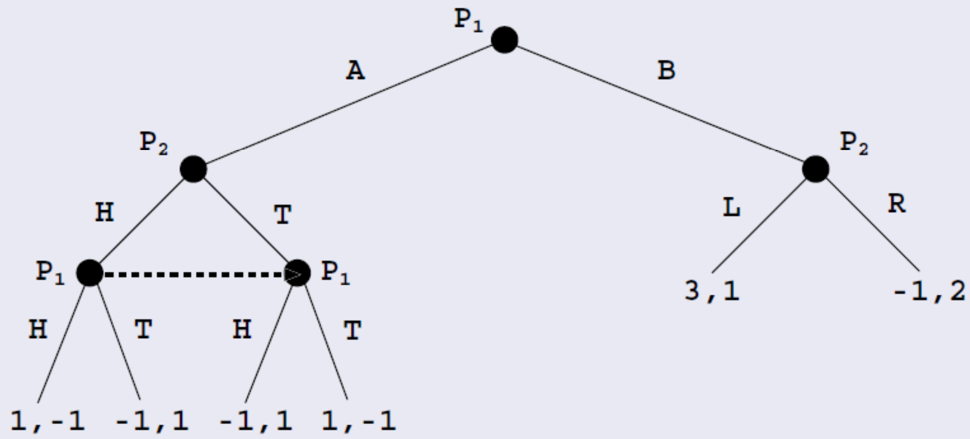
Let the decision nodes be labelled by an indicator set $\boldsymbol{I} = \{1, \ldots, n\}$. At each node i, the action set is $\boldsymbol{A}_i = \{a_1^i, a_2^i, \ldots, a_{k_i}^i\}$. An individual's behavior at node i is determined by the probability vector $\boldsymbol{p}_i = (p(a_1^i), p(a_2^i), \ldots, p(a_{k_i}^i))$. A behavioral strategy β is the collection of probability vectors:

$$\beta = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n\}.$$

- For a dynamic game, behavioral strategy performs *randomization* at the information set.
- In working backward through the game tree, we found a best response at each information set. So the end result is an equilibrium in "behavioral strategies".

Example of using behavioral strategies

- Consider a game: player 1 chooses between actions A and B.
- If A is chosen, then player 1 and 2 play a game of "matching pennies".
- If B is chosen, then player 2 chooses L or R. Game tree is:



- Let σ be the strategy "Play H with probability 1/2", then (σ, σ) is the unique NE of the "matching pennies" game.
- Show by reduction, the solution of the above game is $(A\sigma, \sigma R)$.

Comment

- The previous randomizing strategy we show is a *behavioral strategy*.
- Note that NE is defined using "mixed strategy", which is formed by taking a weighted combination of pure strategies:

$$\sigma = \sum_{s \in \boldsymbol{S}} p(s)\, s \quad \text{with} \quad \sum_{s \in \boldsymbol{S}} p(s) = 1.$$

- We let σ (β) to denote mixed (behavioral) strategy.

Theorem

Let (β_1^*, β_2^*) be an equilibrium in behavioral strategies. Then there exist mixed strategies σ_1^* and σ_2^* such that

1. $\pi_i(\sigma_1^*, \sigma_2^*) = \pi_i(\beta_1^*, \beta_2^*)$ for $i = 1, 2$ and
2. the pair of strategies (σ_1^*, σ_2^*) is a NE.

In other words, the equilibria in behavioral strategies are EQUIVALENT to NE in the mixed strategy

Exercise

- Find the strategic form of the game from previous example.
- Find mixed strategies σ_1^* and σ_2^* that give both players the same payoff they achieve by using the behavioral strategies found by backward induction.
- Show that the pair σ_1^* and σ_2^* is a Nash equilibrium.

Solution

- The strategic form is:

	HL	HR	TL	TR
AH	$+1, -1$	$+1, -1$	$-1, +1$	$-1, +1$
AT	$-1, +1$	$-1, +1$	$+1, -1$	$+1, -1$
BH	$+3, +1$	$-1, +2$	$+3, +1$	$-1, +2$
BT	$+3, +1$	$-1, +2$	$+3, +1$	$-1, +2$

- The mixed strategy $\sigma_1^* = \frac{1}{2}AH + \frac{1}{2}AT$ and $\sigma_2^* = \frac{1}{2}HR + \frac{1}{2}TR$ give $\pi_1(\sigma_1^*, \sigma_2^*) = \pi_2(\sigma_1^*, \sigma_2^*) = 0$.
- Because $\pi_1(AH, \sigma_2^*) = \pi_1(AT, \sigma_2^*) = 0$ and $\pi_1(BH, \sigma_2^*) = \pi_1(BT, \sigma_2^*) = -1$, we have $\pi_1(\sigma_1^*, \sigma_2^*) \geq \pi_1(\sigma_1, \sigma_2^*)\, \forall \sigma_1 \in \sum_1$.
- Because $\pi_2(\sigma_1^*, s_2) = 0\, \forall s_2 \in \boldsymbol{S}_2$, we have $\pi_2(\sigma_1^*, \sigma_2^*) \geq \pi_2(\sigma_1^*, \sigma_2)\, \forall \sigma_2 \in \sum_2$. Hence, (σ_1^*, σ_2^*) is a Nash equilibrium.

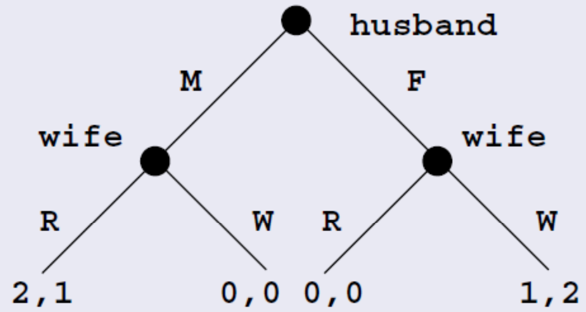
Definition

A subgame is a part (sub-tree) of a game tree that satisfies the following conditions:

1. It begins at a decision node (for any player).
2. The information set containing the initial decision node contains no other decision nodes. That is, the player knows all the decisions that have been made up until that time.
3. The sub-tree contains all the decision nodes that follow the initial node.

Example 1

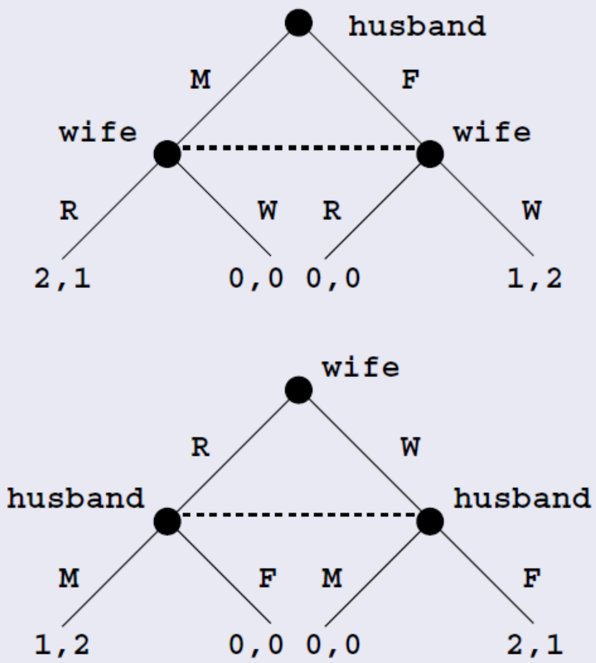
- In the sequential decision dinner party game:



- The subgames are:
  1. the parts of the game tree beginning at each of the wife's decision nodes and
  2. the whole game tree.

Example 2

- For the previous "simultaneous" decision dinner party game



	R	W
M	2,1	0,0
F	0,0	1,2

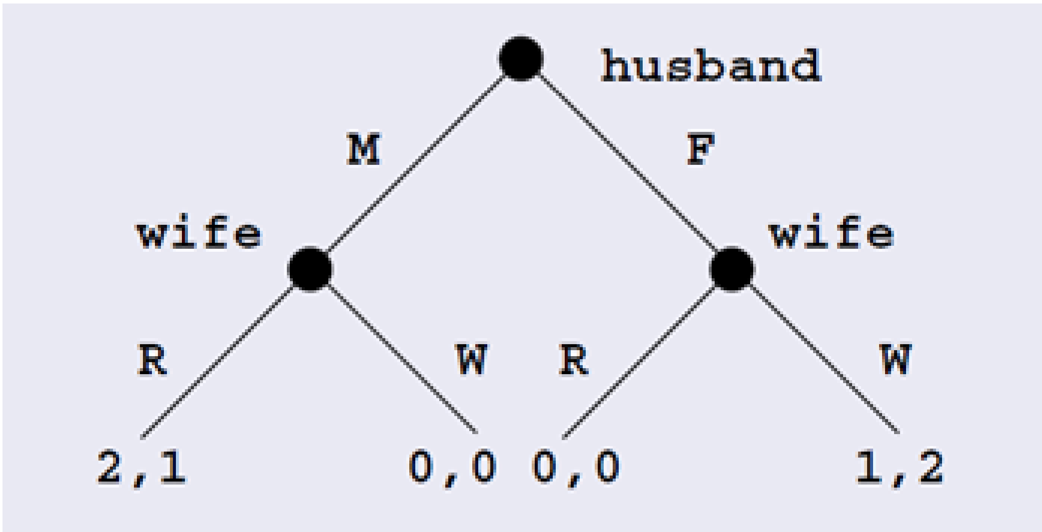
- The only subgame is the whole game.

Definition

A subgame perfect Nash equilibrium is a Nash equilibrium in which the behavior specified in *every subgame* is a Nash equilibrium for the subgame. Note that this applies even to subgames that are *not* reached during a player of the game using the Nash equilibrium strategies.

Example

In the dinner party game (first example), the Nash equilibrium is (M, RW) is a subgame perfect Nash equilibrium



Example

In the dinner party game (first example), the Nash equilibrium is (M, RW) is a subgame perfect Nash equilibrium because

- the wife's decision in response to a choice of meat is to choose red wine, which is a Nash equilibrium in that subgame
- the wife's decision in response to a choice of fish is to choose white wine, which is a Nash equilibrium in that subgame
- the husband's decision is to choose meat, which (together with his wife's strategy RW), constitutes a Nash equilibrium.
- The Nash equilibrium (F, WW) is NOT subgame perfect because it specifies a behavior (choosing W) that is not a Nash equilibrium for the subgame beginning at the wife's decision node following a choice of meat by her husband.

Comments

- It follows from the definition of subgame perfect Nash equilibrium that *any* Nash equilibrium that is found by backward induction is subgame perfect.
- If a simultaneous decision subgame occurs, then all possible Nash equilibria of this subgame may appear in some subgame perfect Nash equilibrium for the whole game.

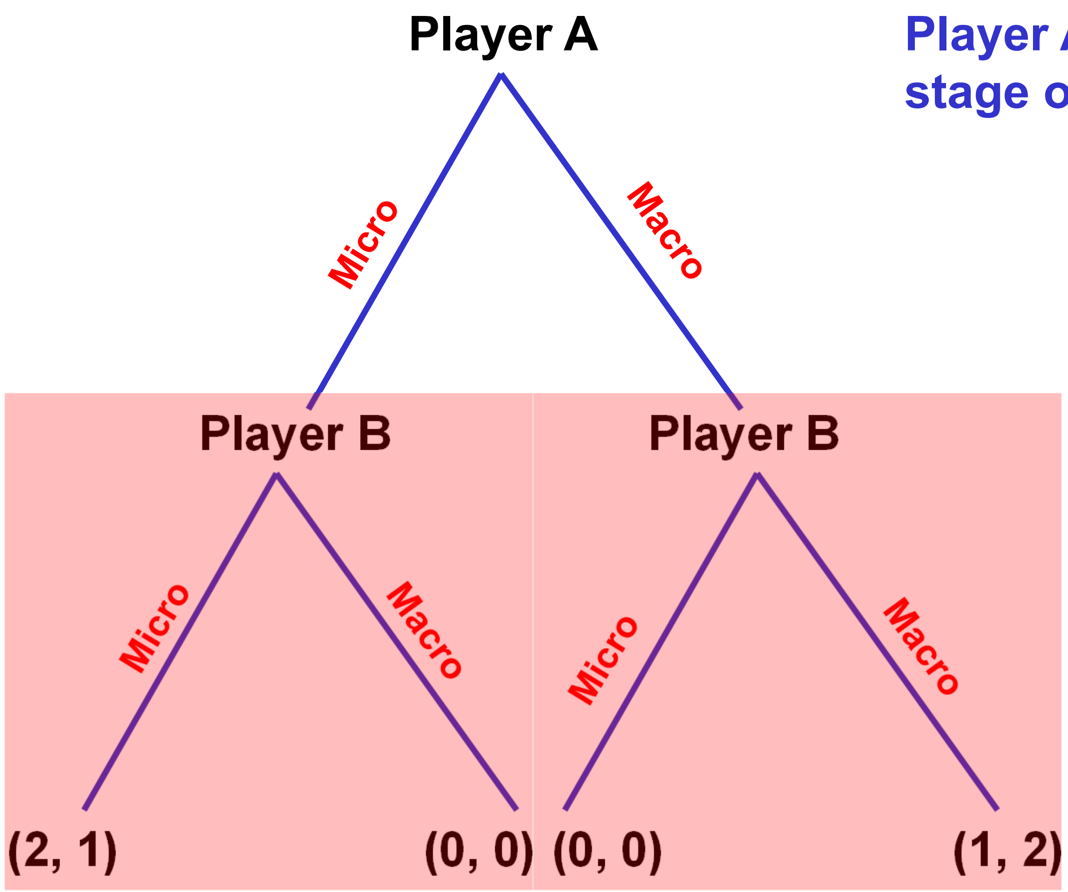
In many games of interest, *some* of the choices are made sequentially. That is, one player may know the opponents choice before she makes her decision.

Let Player A choose first.

Player B

Player A		Micro	Macro
	Micro	2 1	0 0
	Macro	0 0	1 2

Game Tree



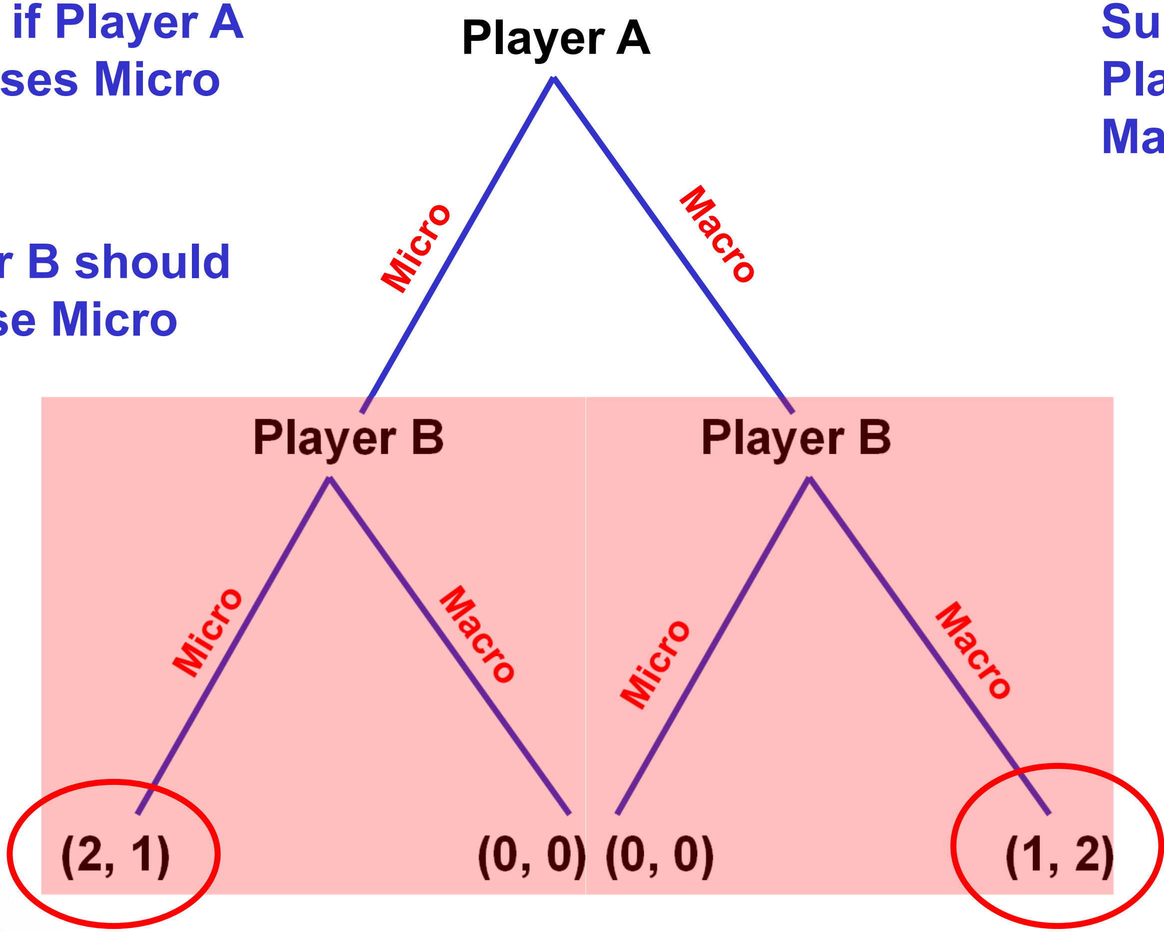
Player A moves first in stage one.

The second stage (after the first decision is made) is known as the *subgame.*

Game Tree

Now, if Player A chooses Micro

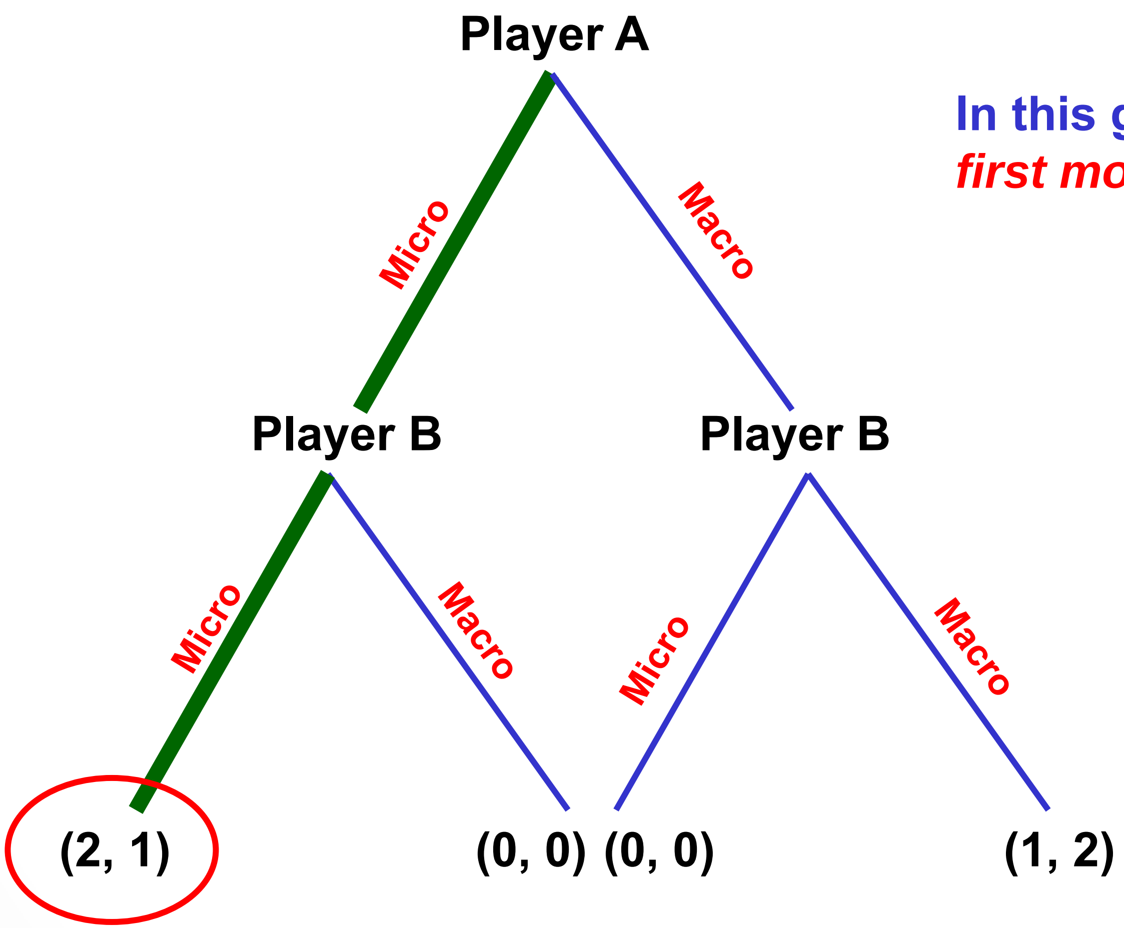
Player B should choose Micro



Suppose that Player A chooses Macro.

Player B should choose Macro

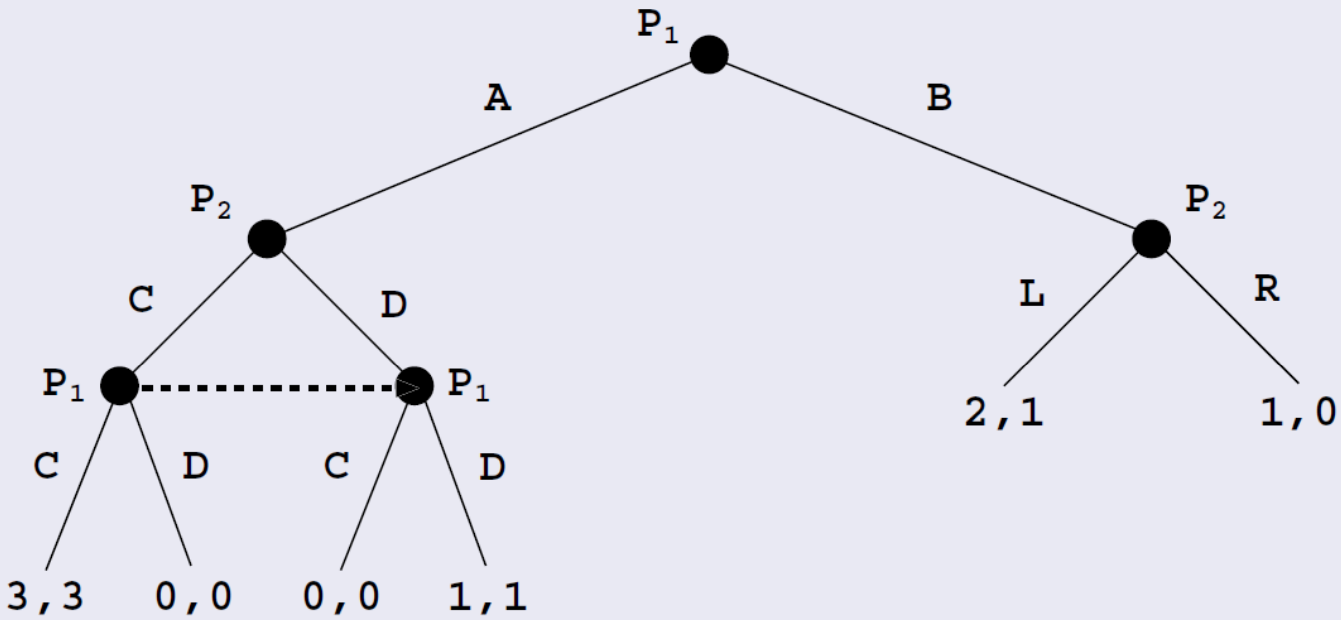
Player A knows how player B will respond, and therefore will always choose Micro (and a utility level of 2) over Macro (and a utility level of 1)



In this game, player A has a *first mover advantage*

Example

- Consider the dynamic game with multiple subgame perfect NE:



- What are the subgames?
- For the simultaneous subgame, what are the Nash equilibria?
- What are the subgame perfect Nash equilibria?

Solution

- Three subgames: (1) the whole tree; (2) the right sub-tree; (3) the left sub-tree.
- For the simultaneous subgame, (C, C) and (D, D) are NE. For mixed strategy, let p (or q) for player 1 (or player 2) to play C and $1-p$ (or $1-q$) to play D. The payoff for player 1 (or player 2) is:

$$\pi_1 = \pi_2 = 3pq + (1-p)(1-q) = 1 - q + p(4q-1).$$

so $\sigma_1^* = \sigma_2^* = \frac{1}{4}\text{CC} + \frac{3}{4}\text{DD}$, and $\pi_1(\sigma_1^*) = \pi_2(\sigma_2^*) = \frac{3}{4}$.
- Subgame perfect NE are: (a) (AC, CL); (b) (BD, DL) (c) $(B\sigma_1^*, \sigma_2^* L)$.

Why not (a) (AD, DL), (b)$(A\sigma_1^*, \sigma_2^* L)$?, because they are illogical.

Theorem

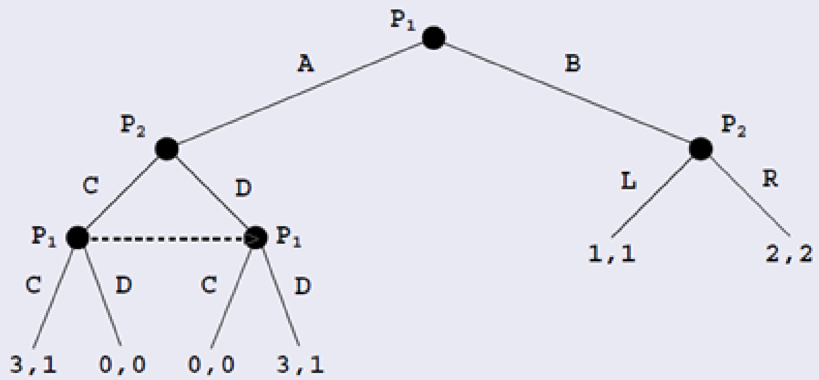
Every finite dynamic game has a subgame perfect Nash equilibrium.

Proof

This is straight forward from the definition of subgame perfect Nash equilibrium. In other words, solve the dynamic game via backward induction, the solution is also a subgame perfect Nash equilibrium.

Exercise

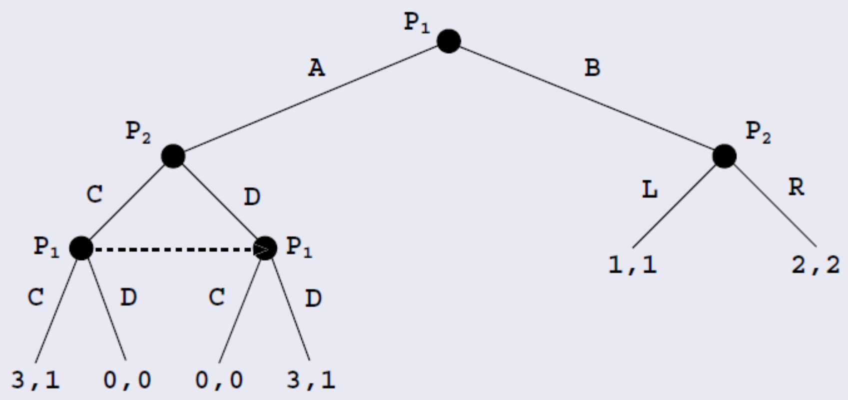
- Consider the following dynamic game:



- How many subgames?
- What is the NE of the right subtree:
- What are the NE of the left subtree:
- Subgame perfect NE are:

Exercise

- Consider the following dynamic game:



- How many subgames? Ans: three.
- What is the NE of the right subtree: Ans: *R*.
- What are the NE of the left subtree: Ans: (a) (C,C), with payoff (3,1), (b) (D,D), with payoff (3,1), (c) (σ,σ) where $\sigma = (1/2, 1/2)$, with payoff (1.5,0.5).
- Subgame perfect NE are: (a) (AC, CR), (b) (AD, DR), (c) $(B\sigma, \sigma R)$.

Assignments

Exercise 5.8—Text Book, P101

Find all the subgame perfect Nash equilibria of the game shown in Figure 5.8.

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