COARSE GEOMETRY AND GROUPOIDS

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ABSTRACT. These are a personal note of a mini-course of Nick Wright on the coarse Baum–Connes conjecture, provided in the autumn school on large-scale geometry in Göttingen, October 9–13, 2023.

1. Introduction

During October 9–13, 2023, I attended an autumn school on large-scale geometry in Göttingen. There were four scheduled mini-courses. The courses of Cornelia Drutu (Oxford) and Alessandro Sisto (Heriot-Watt) were more on the geometry of groups (geometric group theory, property (T) and a-T-menability), which were interesting but yet not quite my field of research. Guoliang Yu (Texas A&M) was a plenary speaker for coarse index theory, but he was unfortunately sick before the autumn school. Instead, Thomas Schick (Göttingen) took over the lectures. Thomas's lectures were interesting, but I had been quite familiar with those materials before.

Nick Wright (Southampton) gave lecture series on the coarse Baum–Connes conjecture, a topic that I also have some knowledge on. Towards the end, he covered some old results of Skandalis, Tu and Yu [6], in which groupoid models of Roe C*-algebras were built. These results are both interesting and useful for me. My previous experience with topological insulators tells that the Roe C*-algebras are quite universal as a dynamical object, and in most cases serve as the universal target for doing index theory. A groupoid model for a Roe C*-algebra makes this more explicit. Moreover, a Roe C*-algebra has many concrete realisations depending on the choice of the ample module, and it is important in physics to keep in mind this choice. Reinterpreting a Roe C*-algebra as a groupoid crossed product sometimes fixes such a choice and might hence become useful for physical applications.

The following will be devoted to a non-faithful recording of Nick Wright's lectures. I will not cover the fundaments that I am already quite familiar with.

2. Coarse spaces

We start with the construction of uniform and non-uniform Roe C*-algebras.

Let X be a uniformly locally finite metric space. Uniformly locally finite means that for every $R > 0$, there exists N such that every R-ball in X has at most N points. An operator $T \in \mathbb{B}(\ell^2 X)$ can be described by an infinite matrix $(T_{x,y})_{x,y\in X}$. The *propagation* of T is

$$\mathrm{Prop}(T) := \sup\{d(x,y) \mid T_{x,y} \neq 0\}.$$

Write $\mathbb{C}_{\mathrm{u}}[X]$ for the *-algebra of operators with *finite* propagation. The *uniform Roe* C*-*algebra* on X is the closure of $\mathbb{C}_{\mathrm{u}}[X]$ inside $\mathbb{B}(\ell^2 X)$, denoted by $\mathrm{C}^*_{\mathrm{u}}(X)$.

The above construction actually describes a coarse structure from a metric space, specifying the controlled sets (=entourages) as sets of points with finite supremal distance. We may, instead, define in a more general setting.

Definition 2.1. A *coarse structure* on a set X is $\mathfrak{E} \subseteq \mathcal{P}(X \times X)$ consisting of so-called *controlled sets* or *entourages*, satisfying:

- If $\mathcal{E} \in \mathfrak{E}$, then $\mathcal{E}^{\mathrm{T}} := \{(y, x) \mid (x, y) \in \mathcal{E}\} \in \mathfrak{E}$.
- If $\mathcal{E} \in \mathfrak{E}$ and $\mathcal{F} \in \mathfrak{E}$, then $\mathcal{E} \circ \mathcal{F} := \{(x, y) \mid \exists z,\ (x, z) \in \mathcal{E}, (z, y) \in \mathcal{F}\} \in \mathfrak{E}$.
- If $\mathcal{E} \in \mathfrak{E}$ and $\mathcal{F} \in \mathfrak{E}$, then $\mathcal{E} \cup \mathcal{F} \in \mathfrak{E}$.
- If $\mathcal{E} \in \mathfrak{E}$ and $\mathcal{F} \subseteq \mathcal{E}$, then $\mathcal{F} \in \mathfrak{E}$.

A coarse structure $\mathfrak{E}$ is *unital* if $\Delta_X \in \mathfrak{E}$, and is *weakly connected* if $\{(x, y)\} \in \mathfrak{E}$ for every pair of points $x, y \in X$.

Example 2.2. Let X be a metric space. We may equip it with two coarse structures. The first one, which is the usual one, consists of entourages $\mathcal{E}$ such that

$$\sup\{d(x, y) \mid (x, y) \in \mathcal{E}\} < +\infty.$$

Example 2.3. Another finer coarse structure on a metric space X is specified by the entourages $\mathcal{E}$ satisfying the property: for each pair of points $(x, y) \in \mathcal{E}$:

$$d(x, y) \to 0 \quad \text{at infinity}.$$

Formally, this means that for every $\epsilon > 0$, there exists a bounded set (Definition 2.5) $K \subseteq X$, such that $d(x, y) < \epsilon$ for every $(x, y) \in \mathcal{E} \setminus K \times K$.

Example 2.4. Let X be a topological space, and $\bar{X}$ be a compactification of it. Then a coarse structure on X is given by the entourages $\mathcal{E} \subseteq X \times X$ such that

$$\bar{\mathcal{E}} \setminus X \times X \subseteq \Delta_{\bar{X}}.$$

Example 2.3 is the special case where $\bar{X}$ is the one-point compactification of X equipped with the topology given by its metric.

Definition 2.5. A subset $K \subseteq X$ is *bounded* if $K \times K \in \mathfrak{E}$. A collection of subsets $\mathcal{C} \subseteq \mathcal{P}(X)$ is *uniformly bounded* if there exists $\mathcal{E} \in \mathfrak{E}$ such that $K \times K \subseteq \mathcal{E}$ for every $K \in \mathcal{C}$. Namely, there exists a uniform entourage such that every element in $\mathcal{C}$ is a bounded set by this uniform entourage.

If X is a topological space, then we also require that every $K \in \mathcal{C}$ is contained in some open set $U \in \mathcal{C}$. So uniformly bounded covers can be enlarged to open uniformly bounded covers.

Remark 2.6. I asked Nick Wright whether the bounded subsets defined in this way form a *bornology* (the definition I have in mind is from [1]). The answer I have in mind in no, because in Nick's definition of coarse structures, the diagonal is not assumed to be an entourage (when this happens, Nick Wright calls it a unital coarse structure, because then the uniform Roe C*-algebra is unital). Then the bounded sets of X do not in general cover X, and that is a condition required by Bunke and Engel's definition of a bornology. I wonder whether or not being non-unital in the sense of Nick Wright is interesting enough in some cases.

Definition 2.7. Let $(X, \mathfrak{E}_X)$ and $(Y, \mathfrak{E}_Y)$ be coarse spaces. A set-theoretic map $f\colon X \to Y$ is called *coarse* if:

- $f \times f$ is controlled, i.e. maps entourages to entourages.
- The preimage of a bounded set under f is also bounded.

Fix a very ample module $\mathcal{H}_X$ for X, which means that it is the infinite direct sum of some ample module. The uniform Roe C*-algebra $\mathrm{C}^*_{\mathrm{u}}(X)$ of a coarse space X is the closure of all controlled operators. The Roe C*-algebra $\mathrm{C}^*(X)$ is the closure of all locally compact and controlled operators.

Roe C*-algebras are functorial for coarse maps using covering isometries. A *covering isometry* for a coarse map $f\colon X \to Y$ is an isometry $V_f\colon \mathcal{H}_X \to \mathcal{H}_Y$ such that

$$\{(y, f(x)) \mid (y, x) \in \operatorname{Supp} V_f\} \in \mathcal{E}_Y.$$

Ampleness implies the existence of covering isometries. And on the K-theory level

$$\mathrm{Ad}_{V_f}\colon\ \mathrm{K}_*(\mathrm{C}^*(X)) \to \mathrm{K}_*(\mathrm{C}^*(Y))$$

is canonically defined.

Exercise 2.8. Find an covering isometry for the coarse equivalence $\mathbb{Z} \hookrightarrow \mathbb{R}$.

3. Assembly maps

In the following we describe the "controlled dual" approach (i.e. Paschke duality) to the coarse Baum–Connes conjecture. Fix an ample module $\mathcal{H}$ for X. An operator $T \in \mathbb{B}(\mathcal{H})$ is *pseudolocal* if:

- $\phi T - T\phi \in \mathbb{K}$ for all $\phi \in \mathrm{C}_0(X)$.
- $\phi T\psi \in \mathbb{K}$ for all $\phi, \psi \in \mathrm{C}_0(X)$ with $\phi\psi = 0$.

Let $\mathrm{D}^*(X)$ be the closure of all controlled, pseudolocal operators. It contains $\mathrm{C}^*(X)$ as an ideal. The K-homology of X is *defined* as

$$\mathrm{K}_*(X) := \mathrm{K}_{*+1}(\mathrm{D}^*(X)/\mathrm{C}^*(X)).$$

The *assembly map* is the boundary map $\partial\colon\ \mathrm{K}_*(X) \to \mathrm{K}_*(\mathrm{C}^*(X))$ in the K-theory long exact sequence for the inclusion of ideal $\mathrm{C}^*(X) \subseteq \mathrm{D}^*(X)$.

Higson and Roe [4] proved that the assembly map is an isomorphism if X is scalable, which we now define.

Definition 3.1. A metric space X is scalable if there exists a map $f\colon X \to X$ such that:

- $d(f(x), f(y)) \le \frac{1}{2} d(x, y)$.
- f is homotopic to the identity map.
- There exsts a sequence

  $$\{f_0 = \mathrm{id}, f_1, \ldots\}$$

  such that:
  - For every bounded set $K \subseteq X$: $f_n|_K = f|_K$ for $n \gg 0$.
  - f_n is uniformly close to f_{n+1}.

  These conditions say that $\{f_n\}$ is a *coarse homotopy* from id to f.

Remark 3.2. The usual definition of a coarse homotopy uses a map $f\colon [0,1]\times X \to X$ (see [5]). But there is no difference if we replace it by the countable sequence in coarse geometry.

We have the following coarse homotopy invariance for scalable spaces:

Lemma 3.3. *If X is scalable. Then coarse homotopic maps induce the same map in K-homology.*

Proof. Let $E^*(X)$ denote the closure of all controlled operators, and the double of $E^*(X)$ over $\mathrm{C}^*(X)$ is defined as

$$D := \{(S+T, S) \mid S \in E^*(X), T \in \mathrm{C}^*(X)\}.$$

Then the quotient map $D \to E^*(X)$ splits. So $\mathrm{K}_*(D) = \mathrm{K}_*(\mathrm{C}^*(X)) \oplus \mathrm{K}_*(E^*(X)) = \mathrm{K}_*(\mathrm{C}^*(X))$. Now let $[P] \in \mathrm{K}_*(\mathrm{C}^*(X))$ and define

$$Q := (f_{0*}P \oplus f_{1*}(P) \oplus \cdots, f_*P \oplus f_*P \oplus \cdots)$$

which lies in D (due to the very ampleness!) since on any bounded set only a finite number of i make $f_{i*}P$ differ from f_*P. Each f_i is uniformly close to f_{i+1}, so Q is equivalent to

$$Q' := (f_{1*}P \oplus f_{2*}(P) \oplus \cdots, f_*P \oplus f_*P \oplus \cdots)$$

which means that $(f_{0*}P, f_*P)$ is trivial in K-theory. So $[f_{0*}P] = [f_*P] = [f_{1*}P]$. $\square$

Theorem 3.4 (Higson–Roe)**.** *If X is scalable, then the assembly map for X is an isomorphism.*

Proof. One checks that $f_* = \mathrm{id}$ on $\mathrm{K}_*(\mathrm{D}^*(X))$ by using $f_* = \mathrm{id}$ in $\mathrm{K}_*(\mathrm{D}^*(X)/\mathrm{C}^*(X))$ and $\mathrm{K}_*(\mathrm{C}^*(X))$. Note that (pointed out by Thomas Schick) one cannot use the five lemma here: the five lemma only tells that f_* is necessarily an *isomorphism* on $\mathrm{K}_*(\mathrm{D}^*(X))$, which is not enough.

Now for $[P] \in \mathrm{K}_*(\mathrm{D}^*(X))$, we have that

$$Q := P \oplus f_*P \oplus f_*f_*P \oplus \ldots$$

lies in D because the propagation tends to 0. $f_* = \mathrm{id}$ on $\mathrm{K}_*(\mathrm{D}^*(X))$ implies that Q is equivalent to

$$f_*Q = f_*P \oplus f_*f_*P \oplus f_*f_*f_*P \oplus \ldots$$

which further implies that $[P] = 0$ in $\mathrm{K}_*(\mathrm{D}^*(X))$. This is fairly standard Eilenberg swindle argument. $\square$

4. Coarse Baum–Connes conjecture

It is useful to first look at the Baum–Connes conjecture, which states that the map

$$\mathrm{K}^G_{*,G\text{-cpt}}(\underline{E}G) \to \mathrm{K}_*(\mathrm{C}^*_{\mathrm{r}}(G))$$

is an isomorphism.

If G is torsion-free, then BG is compact and the left-hand side is equivalent to $\mathrm{K}_*(BG) = \mathrm{K}^G_*(EG)$, where EG is the universal *principal* G-space. In the general case, one needs to use $\underline{E}G$ which is the universal *proper* G-space. This space is the infinite simplex on G and we define

$$\mathrm{K}^G_{*,G\text{-cpt}}(\underline{E}G) := \varinjlim_{\substack{X \subseteq \underline{E}G \\ G\text{-cpt}}} \mathrm{K}^G_*(X).$$

The coarse Baum–Connes conjecture is defined following a similar spirit. Let $\underline{E}X$ be the infinite simplex on a coarse space $(X, \mathfrak{E})$. For each $\mathcal{E} \in \mathfrak{E}$, define the Rip complex $P_{\mathcal{E}}(X)$ as the subcomplex of $\underline{E}X$, generated by simplices of the form

$$[x_0, \ldots, x_n], \qquad d(x_i, x_j) \le d.$$

Definition 4.1. The *coarse* K-*homology* of X is defined to be

$$\mathrm{KX}_*(X) := \varinjlim \mathrm{K}_*(P_{\mathcal{E}}(X)).$$

Conjecture 4.2 (Coarse Baum–Connes conjecture, CBC)**.** *The map* $\mathrm{KX}_*(X) \to \mathrm{K}_*(\mathrm{C}^*(X))$ *is an isomorphism. The map is defined using the following non-trivial isomorphism*

$$\mathrm{K}_*(\mathrm{C}^*(X)) \simeq \varinjlim \mathrm{K}_*(\mathrm{C}^*(P_{\mathcal{E}}(X))).$$

Yu [7] proved that CBC holds for spaces with finite asymptotic dimension:

Definition 4.3. A coarse space X has asymptotic dimension less or equal than n, if there exists a sequence of uniformly bounded cover $\{\mathfrak{U}_k\}_{k\in\mathbb{N}}$ such that:

- The Lebesgue number of $\mathfrak{U}_k$ tends to 0 as $k \to \infty$.
- The nerve of every $\mathfrak{U}_k$ is less or equal than n.

Assume G is torsion-free and $X = EG$ is a finite G-CW-complex. Then the left-hand side of the Baum–Connes conjecture is just $\mathrm{K}_*^G(EG)$. In this setting, we have:

Theorem 4.4. *The CBC for X implies the injectivity of the Baum–Connes assembly map.*

The proof is based on a "descent principal" and uses equivariant Roe C*-algebras. I do not plan to cover this.

5. Roe C*-algebras and groupoids

The final part will be devoted to understanding the results mainly contained in the work [6] of Skandalis, Tu and Yu. They define the Roe C*-algebra of a coarse space X as a reduced crossed product by a so-called coarse groupoid $\mathcal{G}_X$.

We first define the (uniform) Roe C*-algebra of a group. Let G be a discrete group with a left-invariant word-length metric ρ. The *uniform Roe* C*-*algebra of* G, denoted by $\mathrm{C}^*_{\mathrm{u}}(G)$ is generated by

$$(\rho(g)\delta_x \mapsto \delta_{xg^{-1}})_{g\in G} \qquad \text{and} \qquad \ell^\infty X.$$

The uniform Roe C*-algebra $\mathrm{C}^*_{\mathrm{u}}(G)$ can be also defined as $\ell^\infty G \rtimes_{\mathrm{r}} G$. Similarly, the Roe C*-algebra of G is defined as $\ell^\infty(G,\mathbb{K}) \rtimes_{\mathrm{r}} G$. The key point is that $d(x,y) \le R$ iff $x = yg^{-1}$ for some g with $\rho(g) \le R$.

Example 5.1. If $G = \mathbb{Z}^d$ equipped with the obvious word-length metric. Then we recover the well-known result (at least well-known to me)

$$\mathrm{C}^*_{\mathrm{u}}(\mathbb{Z}^d) = \ell^\infty(\mathbb{Z}^d) \rtimes_{\mathrm{r}} \mathbb{Z}^d \qquad \text{and} \qquad \mathrm{C}^*(\mathbb{Z}^d) = \ell^\infty(\mathbb{Z}^d,\mathbb{K}) \rtimes_{\mathrm{r}} \mathbb{Z}^d.$$

A crossed product is the C*-algebra of an action groupoid. Since $\ell^\infty(G)$ can be identified with $\mathrm{C}_{\mathrm{b}}(G)$ and therefore with $\mathrm{C}(\beta G)$ where βG is the Stone-Čech compactification of G. So

$$\mathrm{C}^*_{\mathrm{u}}(G) = \mathrm{C}(\beta G) \rtimes_{\mathrm{r}} G = \mathrm{C}^*_{\mathrm{r}}(\beta G \rtimes G).$$

The space βX can be identified with the space of *ultrafilters* on X.

Definition 5.2. An *ultrafilter* on a set X is a collection of subsets $\omega \in \mathcal{P}(X)$ such that:

- If $A, B \in \omega$, then $A \cap B \in \omega$.
- If $A \in \omega$ and $A \subseteq B$, then $B \in \omega$.
- For every $A \subseteq X$, either $A \in \omega$ or $X \setminus A \in \omega$.

If $A \in \omega$, then we say the ultrafilter ω *chooses* A.

Example 5.3. Let $x \in X$. Then

$$\omega_x := \{A \subseteq X \mid x \in A\}$$

is an ultrafilter, called the *principal filter* with principal element x.

Proposition 5.4. *There is a bijection between points in βX and ultrafilters on X. A point $x \in X \subseteq \beta X$ is identified with the principal ultrafilter ω_x.*

More generally, if $A \subseteq X$ is a subset. Then there is a bijection between βA and $\bar{A}$, which is the set of ultrafilters on X which choose A. $\bar{A}$ is clopen and is the closure of A in βX. These clopen sets form a topology for βX.

Remark 5.5. Ultrafilter chooses between subsets of X. If $A \in \omega$ and $A = \coprod_i A_i$ is a finite disjoint union. Then ω chooses exactly one of A_i: suppose first that ω chooses none of A_i's, then ω chooses $X \setminus A_i$ and hence ω chooses

$$\bigcap_i X \setminus A_i = X \setminus \coprod_i A_i = X \setminus A$$

contradicting to the condition that X chooses exactly one of A and $X \setminus A$.

Now assume ω chooses at least two of A_i's, say, A_i and A_j. Then ω chooses $\emptyset = A_i \cap A_j$ as well. Then $X = X \setminus \emptyset$ is not chosen by ω. But if $A \in \omega$ and $A \subseteq X$, then $X \in \omega$. This is a contradiction.

Definition 5.6. Let $(X, \mathfrak{E})$ be a coarse space of bounded geometry. The *coarse groupoid* $\mathcal{G}_X$ is the set of all ultrafilters on $X \times X$ which choose an entourage. Equivalently, we may write

$$\mathcal{G}_X := \bigcup_{\mathcal{E} \in \mathfrak{E}} \bar{\mathcal{E}}.$$

$\mathcal{G}_X$ contains $X \times X$ (view points in X as principal ultrafilters). The composition on $\mathcal{G}_X$ extends the pair groupoid $X \times X \rightrightarrows X$.

Theorem 5.7. *We have*

$$\mathrm{C}^*_{\mathrm{r}}(\mathcal{G}_X) \simeq \ell^\infty X \rtimes_{\mathrm{r}} \mathcal{G}_X \simeq \mathrm{C}^*_{\mathrm{u}}(X) \quad \text{and} \quad \ell^\infty(X, \mathbb{K}) \rtimes_{\mathrm{r}} \mathcal{G}_X \simeq \mathrm{C}^*(X).$$

An alternative description of $\mathcal{G}_X$ is as follows.

A *partial translation* on X is an entourage t such that both coordinate projections are injective. Then t gives a partial bijection

$$\mathrm{pr}_1(t) \mapsto \mathrm{pr}_2(t), \qquad \text{or} \qquad x \mapsto y \text{ iff } (x, y) \in t.$$

Let $\mathcal{E}$ be an entourage and $\bar{\mathcal{E}}$ be the ultrafilters on $X \times X$ choosing $\mathcal{E}$. Since X is assumed to have bounded geometry, we have $\mathcal{E} = t_1 \cup \ldots \cup t_n$ is a finite union of partial translations. So every ultrafilter in $\bar{\mathcal{E}}$ chooses between $t_1, \ldots, t_n$.

Then $\mathcal{G}_X$ is the groupoid of germ of partial translations, i.e.

- $\mathcal{G}_X$ consists of equivalence classes $[\omega, t]$ with

  $$\omega \in \beta X, \quad t \text{ a partial translation such that } \omega \text{ chooses } \mathrm{pr}_1(t).$$

  The equivalence relation is

  $$[\omega, t] \sim [\omega', t'] \quad \text{if} \quad \omega = \omega',\ \mathrm{pr}_1(t \cap t') \in \omega.$$

- The composition is

  $$[\omega, t] \cdot [\eta, s] = [\omega, t \circ s].$$

Remark 5.8. Partial translations are partial bijections, and hence give rise to semigroup actions. This makes it possible to describe a Roe C*-algebra as a semigroup crossed product. As Nick Wright indicated, this is an ongoing work of one of his students.

6. Boundary coarse Baum–Connes conjecture

The followings are completely new to me. It might be interesting to see whether they have certain physical applications. Another question I have in mind would be how they are related to (stable) Higson coronas and therefore to the coarse co-assembly maps of Emerson and Meyer [2]. The latter have potential applications in index theory.

Definition 6.1. The *boundary coarse groupoid* is the restriction of $\mathcal{G}_X$ to $\partial\beta X$, say,

$$\partial\mathcal{G}_X = \mathcal{G}_X \setminus X \times X \rightrightarrows \partial\beta X.$$

Let $\partial\ell^\infty(X, \mathbb{K}) := \ell^\infty(X, \mathbb{K})/\mathrm{C}_0(X, \mathbb{K})$. The *boundary Roe* C^**-algebra* is

$$\mathrm{C}^*_\partial X := \partial\ell^\infty(X, \mathbb{K}) \rtimes_\mathrm{r} \partial\mathcal{G}_X.$$

Example 6.2. If G is a group. Then $\mathcal{G}_X = \beta G \rtimes G$, $X \times X = G \times G = G \rtimes G$ and $\partial\mathcal{G}_X = \partial\beta G \rtimes G$.

The boundary CBC asserts a "boundary assembly map", from a suitable coarse K-homology group to the K-theory of the boundary Roe C*-algebra, is an isomorphism. The coarse K-homology is defined using groupoid-equivariant Kasparov theory. Finn-Sell and Wright [3] proved that the boundary coarse Baum-Connes conjecture holds for sequences of bounded-degree graphs of large girth. Unfortunately, I do not understand most of the terms in this statement.

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