Understanding strong and weak topological phases

A glimpse of groupoids and coarse geometry in topological insulators

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1. Topological phases of matter

Topological phases

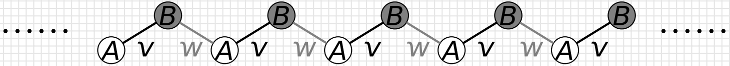
$$\boxed{\begin{array}{c}\text{phase 1}\\ \text{symmetry group } G_1\end{array}} \xRightarrow[\text{phase transition}]{\text{Landau-like}} \boxed{\begin{array}{c}\text{phase 2}\\ \text{symmetry group } G_2\end{array}}$$

$$\boxed{\begin{array}{c}\text{phase 1}\\ \text{symmetry group } G\\ \text{topological invariant } N_1\end{array}} \xRightarrow[\text{phase transition}]{\text{topological}} \boxed{\begin{array}{c}\text{phase 2}\\ \text{symmetry group } G\\ \text{topological invariant } N_2\end{array}}$$

Examples

- the quantum Hall effect;
- (symmetry-protected) topological insulators;
- Chern insulators;
- ...

Example: the Su–Schrieffer–Heeger model

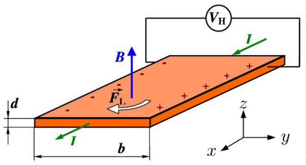


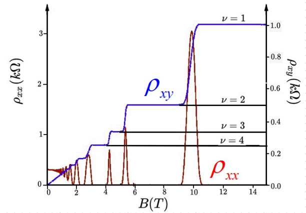
$$H = \sum_{m\in\mathbb{Z}} \Big(-v|m,A\rangle\langle m,B| - w|m+1,A\rangle\langle m,B|$$
$$-v^*|m,B\rangle\langle m,A| - w^*|m,B\rangle\langle m+1,A| \Big)$$

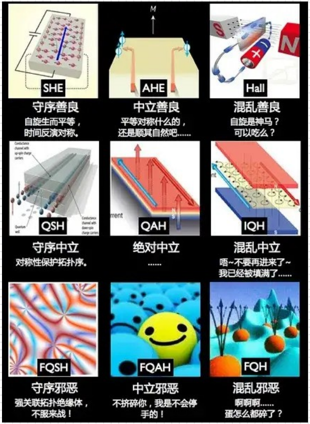
$$\Longrightarrow \quad \widehat{H} = \int_{k\in\mathbb{T}} \widehat{H}_k \mathrm{d}k, \quad \widehat{H}_k = -\begin{pmatrix} 0 & v + w\exp(-ik) \\ v + w\exp(ik) & 0 \end{pmatrix}.$$

- Chiral symmetry $\Longrightarrow$ $\widehat{H}_k$ is off-diagonal.
- $\widehat{H}_k$ invertible for all k $\Longrightarrow$ $v \neq w$.
- $v > w$ and $v < w$ characterises different topological phases.

Example: quantum Hall effects







Nobel prizes from QHE (and friends):

- Von Klitzing (1985): Integer QHE.
- Störmer–Tsui–Laughlin (1998): Fractional QHE.

The NCG framework of topological phases

1. We assume to work with free fermions. This allows us to apply the single-particle approximation.
2. The dynamics of the (single-particle) physical system is therefore be described by a (one-body) Hamiltonian H.
3. The observable C*-algebra is a C*-algebra A containing (the resolvent of) H, which describes the symmetries of the system.
4. A topological phase is represented by a K-theory class of A. Depending on the choice of A and the symmetry type, there are different versions of topological phases.

4'. If the system has anti-unitary symmetries to be preserved, then we must work with real K-theory. In many cases (e.g. the periodic model or the Roe C*-algebra model), this can be simplified to KO-theory or quaternionic K-theory.

Numerical index of topological phases

- A numerical index (topological invariant / topological index / generalised Chern number / . . .) is a map

$$\mathrm{K}_*(A) \to \mathbb{Z} \quad \text{or} \quad \mathbb{R}$$

  sending the topological phase to a number.
- Different sources of numerical indices:

  Kasparov theory index pairing with a Fredholm module / spectral triple;
  Semi-finite index theory index pairing with a semi-finite spectral triple;
  Cyclic homology pairing with cyclic cocycles;
  Coarse homology pairing with coarse cohomology classes [Ludewig–Thiang];
  . . . . . .

Robustness of topological phases

- A priori, topological phases and their numerical indexs should be robust under disorder.
- Which disorder?
- “Old-school” approach: study the continuity of certain numerical invariants;

$\Rightarrow$ automatic constancy if the range is quantised.

- “Modern” approach? Factors through Roe C*-algebras.

2. Topological phases of "generic" aperiodic systems

Modelling QHE systems

- Space $(\mathbb{R}^2, dx \wedge dy)$
  electromagnetic potential A, $dA = \theta dx \wedge dy$.

$\Rightarrow$ 2-cocycle $\sigma \colon \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{T}$,
$\sigma((m,n),(m',n')) := \exp(-2\pi i \theta m' n)$.

- The Hamiltonian

$$H_{A,V} = \frac{1}{2}(d + iA)^*(d + iA) + V, \quad V \in L^\infty(\mathbb{R}^2)$$

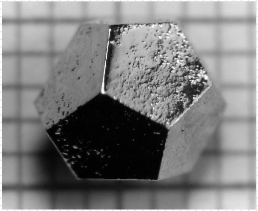
- If V is translation-invariant for $\mathbb{Z}^2$, and λ lies in a spectral gap. Then the Fermi projection p_λ defines a K_0-class:

$$p_\lambda := \chi_{(-\infty,\lambda)}(H_{A,V}) \in (\mathbb{C} \rtimes_\sigma \mathbb{Z}^2) \otimes \mathbb{K}(L^2[0,1)).$$

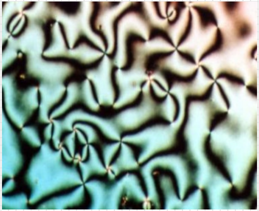
- If V is aperiodic, then Bellissard describes the system by a crossed product $C^*(\Omega) \rtimes_\sigma \mathbb{Z}^2$.
- This still requires a $\mathbb{Z}^2$-labelling of the sites.

Emergence of generic aperiodic systems

We would like to have a general description for these materials:



quasi-crystal



liquid crystal

glass

Definition Let $0 < r < R$. A discrete infinite set $\Lambda \subseteq \mathbb{R}^d$ is called an (r, R)-Delone set if for all $x \in \mathbb{R}^d$:

$$\#(\mathrm{B}(x, r) \cap \Lambda) \leq 1 \quad \text{and} \quad \#(\mathrm{B}(x, R) \cap \Lambda) \geq 1.$$

i.e. Λ is “uniformly discrete” and “relatively dense”.

Observable C*-algebras: two approaches

How to model an observable C*-algebra from a Delone set Λ?
It should be:

- large enough to contain all possible Hamiltonians;
- small enough to have useful homotopy theory (K-theory).

"Dynamical" approach describes a crossed product C*-algebra, covariant for the groupoid actions on the aperiodic point pattern;

$\Longrightarrow$ groupoid C*-algebras of Delone sets.
(Bellissard, Prodan, Bourne, Mesland, . . .)

"Universal" approach describes a C*-algebra which is stable under all "local" perturbations;

$\Longrightarrow$ uniform or non-uniform Roe C*-algebras.
(Kubota, Ewert, Meyer, Ludewig, Thiang, . . .)

Topological groupoids

Definitions

- A groupoid is a small category $\mathcal{G}$ whose all arrows are isomorphisms. Equivalently, it is given by a set of arrows $\mathcal{G}$ and a set of objects $\mathcal{G}^0$, together with structure maps

$$s, r\colon \mathcal{G} \to \mathcal{G}^0, \quad \text{␣}^{-1}\colon \mathcal{G} \to \mathcal{G}, \quad \mathrm{id}\colon \mathcal{G}^0 \to \mathcal{G}$$

  satisfying a collection of properties.
- A (locally compact) topological groupoid is a groupoid $\mathcal{G}$, such that $\mathcal{G}$ is locally compact, $\mathcal{G}^0$ is Hausdorff, and all structure maps are continuous.
- An étale groupoid is a topological groupoid whose range and source maps are local homeomorphisms.

Dynamics of Delone sets

Delone set $\Lambda \Rightarrow$ atomic measure $\sum_{x\in\Lambda}\delta_x$.

Equip the set of (r,R)-Delone set $\mathrm{Del}_{(r,R)}(\mathbb{R}^d) \subseteq \mathrm{C_c}(\mathbb{R}^d)'$ with the weak*-topology.

Theorem $\mathrm{Del}_{(r,R)}(\mathbb{R}^d)$ is a compact, metrisable space, which carries a continuous action of $\mathbb{R}^d$ by translations.

- This yields a topological dynamical system

$$\mathrm{Del}_{(r,R)}(\mathbb{R}^d) \curvearrowleft \mathbb{R}^d \quad \text{or} \quad \Omega_\Lambda \curvearrowleft \mathbb{R}^d.$$

  where Ω_Λ is the closure of the orbit of Λ.
- Every $\omega \in \Omega_\Lambda$ may be viewed as a "limit configuration".
- It generates the action groupoid

$$\Omega_\Lambda \rtimes \mathbb{R}^d \rightrightarrows \Omega_\Lambda.$$
$$s(\omega,x) := \omega - x, \qquad r(\omega,x) := \omega.$$

Tight-binding by restricting to the transversal

Let $\mathcal{G}$ be a groupoid and $X, Y \subseteq \mathcal{G}^0$. Denote:

$$\mathcal{G}_X := s^{-1}(X), \quad \mathcal{G}^Y := r^{-1}(Y), \quad \mathcal{G}_X^Y := \mathcal{G}_X \cap \mathcal{G}^Y.$$

Definition A closed subset $X \subseteq \mathcal{G}^0$ is called a transversal, if X meets every orbit of $\mathcal{G}^0$ under the translations by $\mathcal{G}$, and the restrctions of r and s to $\mathcal{G}^X$ are local homeomorphisms.

Lemma If $X \subseteq \mathcal{G}^0$ is a transversal, then $\mathcal{G}$ is Morita equivalent to $\mathcal{G}_X^X$, the restriction of $\mathcal{G}$ to X.

- For $\Omega_\Lambda \rtimes \mathbb{R}^d \rightrightarrows \Omega_\Lambda$, there is an abstract transversal

$$\Omega_0 := \{\omega \in \Omega_\Lambda \mid 0 \in \omega\}.$$

- Tight-binding: Restricts to a transversal gives an étale groupoid

$$\mathcal{G}_\Lambda \rightrightarrows \Omega_0, \qquad \mathcal{G}_\Lambda := \Omega_\Lambda \rtimes \mathbb{R}^d\big|_{\Omega_0}^{\Omega_0}.$$

C*-algebra of an étale groupoid

Let $\mathcal{G}$ be an étale groupoid.

- The convolution groupoid *-algebra $C_c(\mathcal{G})$ consists compactly supported functions on $\mathcal{G}$, equipped with
  - $(f_1 * f_2)(\eta) = \sum_{\gamma \in \mathcal{G}^{\eta}} f(\gamma) g(\gamma^{-1}\eta)$;
  - $f^*(\gamma) := \overline{f(\gamma^{-1})}$.
- $C_c(\mathcal{G})$ can be completed into a right Hilbert $C_0(\mathcal{G}^0)$-module, denoted by $L^2(\mathcal{G})$:
  - $(f \cdot \phi)(\gamma) := f(\gamma)\phi(s(\gamma))$, for $f \in L^2(\mathcal{G})$ and $\phi \in C_0(\mathcal{G}^0)$;
  - $\langle f_1, f_2 \rangle(x) := \sum (f_1^* * f_2)|_{\mathcal{G}^0}(x)$, for $f_1, f_2 \in L^2(\mathcal{G})$.
- $C_c(\mathcal{G}) \curvearrowright C_c(\mathcal{G})$ extends to $C_c(\mathcal{G}) \curvearrowright L^2(\mathcal{G})$. This completes $C_c(\mathcal{G})$ into the reduced groupoid C*-algebra $C^*(\mathcal{G})$. $x \in \Lambda$.

C*-algebra of a Delone set

- $C^*(\mathcal{G}_\Lambda)$ =: the C*-algebra of the Delone set Λ.
- It consists of copies of the Hamiltonians on the sites of Λ, which are distinguished in transversal Ω_0.
- Morita equivalent topological groupoids give Morita–Rieffel equivalent C*-algebras.

Example

- $\Lambda = \mathbb{Z}^d \subseteq \mathbb{R}^d$.
- $\Omega_\Lambda = \mathbb{R}^d/\mathbb{Z}^d \simeq \mathbb{T}^d$.
- Ω_0 can be chosen to be any point in $\mathbb{T}^d$.
- The Morita equivalence and *-isomorphism

$$C^*(\mathbb{T}^d \rtimes \mathbb{R}^d) \simeq C(\mathbb{T}^d) \rtimes \mathbb{R}^d \sim C(pt) \rtimes \mathbb{R}^d/\mathbb{T}^d \simeq C(\mathbb{T}^d).$$

- This is a special case of the Connes–Thom isomorphism, which plays a special role in noncommutative T-duality.
  Cf. work of Mathai, Rosenberg and Thiang.

Numerical indices of the groupoid model

- $K_*(C^*(\mathcal{G}_\Lambda))$ is in general very complicated.
- Instead: [Bourne–Mesland] defines an unbounded Kasparov module, which represents a class

$$ {}_d\lambda_{\Omega_0} \in KK_d(C^*(\mathcal{G}_\Lambda), C(\Omega_0)), $$

  henceforth induces a map

$$ K_*(C^*(\mathcal{G}_\Lambda)) \to K_{*-d}(C(\Omega_0)). $$

- Maps $K_{*-d}(C(\Omega_0)) \to \mathbb{Z}$ or $\mathbb{R}$ can be constructed from:
  - point evaluation at a limit configuration $\omega \in \Omega_0$;
  - "trace" map on $C(\Omega_0)$ $\Leftrightarrow$ $\mathcal{G}_\Lambda$-invariant measure on Ω_0.
- Composition yields a numerical index

$$ K_*(C^*(\mathcal{G}_\Lambda)) \to \mathbb{Z} \quad \text{or} \quad \mathbb{R}. $$

Question Are these invariants robust under disorder?

The coarse-geometric approach

- $\Lambda \subseteq \mathbb{R}^d$ as a discrete metric space with bounded geometry.

$\Rightarrow$ coarse-geometric C*-algebras.

Definitions

- The uniform Roe C*-algebra $C^*_{u,\mathrm{Roe}}(\Lambda)$ consists of all operators on $\ell^2(\Lambda)$ with finite propagation.
- The Roe C*-algebra $C^*_{\mathrm{Roe}}(\Lambda)$ consists of operators on $\ell^2(\Lambda, \mathcal{K})$, which are locally compact and has finite propagation.

Remark

- $\mathcal{K}$ can be chosen as any separable Hilbert space.
  But we should consider them as the “fundamental domain”.
- It was explained in [Ewert–Meyer] why non-uniform Roe C*-algebras are better models.

Numerical invariants of Roe C*-algebras

- Roe C*-algebras and uniform Roe C*-algebras are coarsely invariant. So for any Delone set $\Lambda \subseteq \mathbb{R}^d$:

$$C^*_{\mathrm{Roe}}(\Lambda) \simeq C^*_{\mathrm{Roe}}(\mathbb{R}^d).$$

- As opposed to $C^*(\mathcal{G}_\Lambda)$, K-theory of $C^*_{\mathrm{Roe}}(\Lambda)$ is very simple:

Theorem

$$K_i(C^*_{\mathrm{Roe}}(\Lambda)) = \begin{cases} \mathbb{Z} & \text{if } i-d \text{ is even;} \\ 0 & \text{if } i-d \text{ is odd.} \end{cases}$$

- This can be computed using either a Mayer–Vietoris argument, or using the position operator to build a spectral triple ξ_Λ.
- Topological phases in $C^*_{\mathrm{Roe}}(\Lambda)$ are considered strong in [Ewert–Meyer]. They are “universally robust”.

Groupoid C*-algebras VS Roe C*-algebras

- We wish to compare the topological phases in $C^*(\mathcal{G}_\Lambda)$ and $C^*_{\mathrm{Roe}}(\Lambda)$.
- This comes from a family of *-homomorphisms

$$\pi_\omega : C^*(\mathcal{G}_\Lambda) \to C^*_{\mathrm{Roe}}(\omega) \simeq C^*_{\mathrm{Roe}}(\Lambda), \qquad \omega \in \Omega_0.$$

- K-theory implies that topological phases in $C^*(\mathcal{G}_\Lambda)$ are not always “strong”.

Question What are the strong topological phases / indices in the groupoid model?

Strong phases in the groupoid model

Theorem (L) For every $\omega \in \Omega_0$, The following diagram commutes:

$$\begin{tikzcd} K_*(C^*(\mathcal{G}_\Lambda) \hat{\otimes} C\ell_{0,d}) \arrow[r, "{}_d\lambda_{\Omega_0}"] \arrow[d, "\pi_\omega \otimes \mathrm{id}"'] & K_{*-d}(C(\Omega_0)) \arrow[d, "(\mathrm{ev}_\omega)_*"] \\ K_*(C^*_{\mathrm{Roe}}(\omega) \hat{\otimes} C\ell_{0,d}) \arrow[r, "\xi_\omega", "\sim"'] & \mathbb{Z}. \end{tikzcd}$$

- Strong topological phases of the groupoid model all come from “point evaluations” at a single “limit configuration”.

3. Understanding the robustness of topological phases

A first comparison in the periodic case

Let $\Lambda = \mathbb{Z}^d$, considered as a group and a discrete metric space. Then there is an injective *-homomorphism

$$C^*(\Lambda) \to C^*_{\mathrm{Roe}}(\Lambda)$$

which induces group homomorphisms in K-theory:

$$K_i(C^*(\Lambda)) \to K_i(C^*_{\mathrm{Roe}}(\Lambda))$$
$$= \begin{cases} \mathbb{Z}^{2^{d-1}} \to \mathbb{Z} & \text{if } i-d \text{ is even;} \\ \mathbb{Z}^{2^{d-1}} \to 0 & \text{if } i-d \text{ is odd.} \end{cases}$$

Question How shall we understand these maps?

Stacked topological phases are weak

Theorem (Ewert–Meyer) If $\varphi\colon \mathbb{Z}^{d-1} \to \mathbb{Z}^d$ is an injective group homomorphism. Then the map

$$\mathrm{K}_i(\mathrm{C}^*(\mathbb{Z}^d)) \to \mathrm{K}_i(\mathrm{C}^*_{\mathrm{Roe}}(\mathbb{Z}^d))$$

vanishes on the image of

$$\varphi_*\colon \mathrm{K}_i(\mathrm{C}^*(\mathbb{Z}^{d-1})) \to \mathrm{K}_i(\mathrm{C}^*(\mathbb{Z}^d)).$$

- The theorem says that "stacking" lower-dimensional topological phases along a direction always gives weak invariants.
- The proof of Ewert and Meyer is based on the fact that φ_* factors through the K-theory of a flasque space.
- This can be understood in a physical way.

Equivariant Roe C*-algebras

- The equavariant Roe C*-algebra $C^*_{\mathrm{Roe}}(\mathbb{R}^d)^{\mathbb{Z}^d}$ consists of operators in $C^*(\mathbb{R}^d)$ that are equivariant for the $\mathbb{Z}^d$-action.
- It is isomorphic to the stablised group C*-algebra:

$$C^*_{\mathrm{Roe}}(\mathbb{R}^d)^{\mathbb{Z}^d} \simeq C^*(\mathbb{Z}^d) \otimes \mathbb{K}(L^2 \underbrace{[0,1) \times \cdots \times [0,1)}_{\text{fundamental domain}})$$

- Let $m = (m_1, \ldots, m_d) \in \mathbb{N}^d$. Then

$$m\mathbb{Z}^d := m_1\mathbb{Z} \times \cdots \times m_d\mathbb{Z} \subseteq \mathbb{Z}^d$$

is a subgroup which also acts properly on $\mathbb{R}^d$.

$\Rightarrow$ forgetful / descent map $\phi_m : C^*_{\mathrm{Roe}}(\mathbb{R}^d)^{\mathbb{Z}^d} \to C^*_{\mathrm{Roe}}(\mathbb{R}^d)^{m\mathbb{Z}^d}$.

Question What is its induced map in K-theory?

Renormalising-invariant phases are strong

Theorem (L–Thiang)

- $d = 1$: the map is multiplication with m on K_0, and identity on K_1.
- $d \geq 1$, the map is multiplication by a number (depending on the generator itself and m), on each generator of

$$K_*(C^*_{\mathrm{Roe}}(\mathbb{R}^d)^{\mathbb{Z}^d}) \simeq K_*(C^*(\mathbb{Z}^d)) \simeq \mathbb{Z}^{2^{d-1}}.$$

- There is only one generator in $K_0 \oplus K_1$ for each d, that is invariant under ϕ_m (the "Bott" generator).
- In other words: there is a unique topological phase in the periodic lattice model, invariant under "lattice renormalisation".

"Symmetry-breaking" Roe C*-algebras

- We may take the direct limit over all these forgetful maps.
- The resulting C*-algebra is a C*-subalgebra of $C^*_{\mathrm{Roe}}(\mathbb{R}^d)$, which embodies both "strong" and "weak" topological phases and distinguishes them.

Definition

$$C^*_{\mathrm{Roe}}(\mathbb{R}^d)^{\mathfrak{S}} := \varinjlim C^*(\mathbb{R}^d)^{m\mathbb{Z}^d}.$$

Theorem (L–Thiang)

$$K_i(C^*_{\mathrm{Roe}}(\mathbb{R}^d)^{\mathfrak{S}}) \simeq \begin{cases} \mathbb{Q}^{2^{n-1}-1} \oplus \mathbb{Z} & \text{if } i-d \text{ is even;} \\ \mathbb{Q}^{2^{n-1}} & \text{if } i-d \text{ is odd.} \end{cases}$$

In particular, the natural map $K_*(C^*_{\mathrm{Roe}}(\mathbb{R}^d)^{\mathfrak{S}}) \to K_*(C^*_{\mathrm{Roe}}(\mathbb{R}^d))$ survives on the unique $\mathbb{Z}$-factor.