The spectral localiser via E-theory

Based on joint work with Bram Mesland, arXiv:2506.17143

Yuezhao Li

Mathematical institute, Leiden university

04-07-2025, Hannover

Sketch of the talk

- K-theory of C*-algebras can be used to classify the topology of a quantum system.
- The spectral localiser is a computational tool that allows for finite-dimensional computation of index pairings.
- We provide a (bivariant) K-theoretic framework for it.
- This may have impacts on index theorems.

C*-algebras and K-theory

C*-algebra: norm-closed *-subalgebra of $\mathbb{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$.

Theorem (Gelfand–Naimark) Every commutative C*-algebra is *-isomorphic to $C_0(X)$ for some locally compact Hausdorff space X.

$$\text{commutative C*-algebras} \quad \longleftrightarrow \quad \text{spaces.}$$

C*-algebras and K-theory

C*-algebra: norm-closed *-subalgebra of $\mathbb{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$.

Theorem (Gelfand–Naimark) Every commutative C*-algebra is *-isomorphic to $C_0(X)$ for some locally compact Hausdorff space X.

noncommutative C*-algebras $\longleftrightarrow$ "noncommutative" spaces.

C*-algebras and K-theory

C*-algebra: norm-closed *-subalgebra of $\mathbb{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$.

Theorem (Gelfand–Naimark) Every commutative C*-algebra is *-isomorphic to $C_0(X)$ for some locally compact Hausdorff space X.

noncommutative C*-algebras $\longleftrightarrow$ "noncommutative" spaces.

K-theory: a (co)homology theory for "noncommutative spaces":

$$I \rightarrowtail E \twoheadrightarrow Q \quad \text{short exact sequence of C*-algebras}$$

gives

$$\cdots \to \mathrm{K}_i(I) \to \mathrm{K}_i(E) \to \mathrm{K}_i(Q) \to \mathrm{K}_{i-1}(I) \to \dots$$

long exact sequence of abelian groups

C*-algebras and K-theory

C*-algebra: norm-closed *-subalgebra of $\mathbb{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$.

Theorem (Gelfand–Naimark) Every commutative C*-algebra is *-isomorphic to $C_0(X)$ for some locally compact Hausdorff space X.

noncommutative C*-algebras $\longleftrightarrow$ "noncommutative" spaces.

K-theory: a (co)homology theory for "noncommutative spaces":

$I \rightarrowtail E \twoheadrightarrow Q$ short exact sequence of C*-algebras

gives

$$\cdots \to K_i(I) \to K_i(E) \to K_i(Q) \to K_{i-1}(I) \to \dots$$

long exact sequence of abelian groups

Theorem (Bott periodicity) $K_i(A) \simeq K_{i+2}(A)$.

Kasparov's bivariant K-theory

Kasparov's KK-theory generalises both K-theory and its "dual theory".

A, B: (separable) C*-algebras $\Rightarrow$ abelian groups $\mathrm{KK}_j(A, B)$, $j \in \{0, 1\}$.

- $A = \mathbb{C}$: $\mathrm{KK}_j(\mathbb{C}, B) \simeq \mathrm{K}_j(B)$ K-theory.
- $B = \mathbb{C}$: $\mathrm{KK}_j(A, \mathbb{C})$ K-homology.

Kasparov's bivariant K-theory

Kasparov's KK-theory generalises both K-theory and its "dual theory".

A, B: (separable) C*-algebras $\Rightarrow$ abelian groups $\mathrm{KK}_j(A,B)$, $j \in \{0,1\}$.

- $A = \mathbb{C}$: $\mathrm{KK}_j(\mathbb{C},B) \simeq \mathrm{K}_j(B)$ K-theory.
- $B = \mathbb{C}$: $\mathrm{KK}_j(A,\mathbb{C})$ K-homology.

Kasparov product: there is a natural group homomorphism

$$\mathrm{KK}_i(A,B) \times \mathrm{KK}_j(B,C) \to \mathrm{KK}_{i+j}(A,C), \quad (\alpha,\beta) \mapsto \alpha \times \beta.$$

In particular: let $\beta \in \mathrm{KK}_j(A,B)$, then it induces a group homomorphism

$$\mathrm{K}_i(A) \xrightarrow{\times\beta} \mathrm{K}_{i+j}(B).$$

- If $B = \mathbb{C}$ and $i + j = 0 \mod 2$:
  $\mathrm{K}_0(\mathbb{C}) \simeq \mathbb{Z}$ generated by finite-rank projections in $\mathbb{K}(\mathcal{H})$.

$\Rightarrow$ K-homology is a "dual" theory of K-theory.

Physical motivation

Quantum mechanics describes dynamics by Schrödinger equation.

Physical motivation

Quantum mechanics describes dynamics by Schrödinger equation.

- Hamiltonian H: (possibly) unbounded, self-adjoint operator acting on a Hilbert space $\mathcal{H}$.

Physical motivation

Quantum mechanics describes dynamics by Schrödinger equation.

- Hamiltonian H: (possibly) unbounded, self-adjoint operator acting on a Hilbert space $\mathcal{H}$.
- Observable algebra A: C*-algebra that contains the resolvent of H.

Physical motivation

Quantum mechanics describes dynamics by Schrödinger equation.

- Hamiltonian H: (possibly) unbounded, self-adjoint operator acting on a Hilbert space $\mathcal{H}$.
- Observable algebra A: C*-algebra that contains the resolvent of H.
- H describe an insulator if H is invertible. Then

$$[\chi_{(-\infty,0)}(H)]$$

  defines an element in $\mathrm{K}_0(A)$.
- $\mathrm{K}_0(A)$ classifies Hamiltonians "up to stable homotopy".
- In the presence of symmetries: replace $\mathrm{K}_0(A)$ by $\mathrm{K}_i(A)$ or $\mathrm{KO}_i(A)$.

Physical motivation

Quantum mechanics describes dynamics by Schrödinger equation.

- Hamiltonian H: (possibly) unbounded, self-adjoint operator acting on a Hilbert space $\mathcal{H}$.
- Observable algebra A: C*-algebra that contains the resolvent of H.
- H describe an insulator if H is invertible. Then

$$[\chi_{(-\infty,0)}(H)]$$

  defines an element in $\mathrm{K}_0(A)$.
- $\mathrm{K}_0(A)$ classifies Hamiltonians "up to stable homotopy".
- In the presence of symmetries: replace $\mathrm{K}_0(A)$ by $\mathrm{K}_i(A)$ or $\mathrm{KO}_i(A)$.
- Index pairing: extract numerical information from a K-theory class.

$\Leftarrow$ Kasparov product with an element in $\mathrm{KK}_j(A, B)$.

The odd index pairing

Setup:

- A a unital C*-algebra, $v \in A$ a unitary;
- $(\mathcal{A}, \mathcal{H}, D)$ an odd (=ungraded) spectral triple over A.
  - D is an unbounded, self-adjoint operator on $\mathcal{H}$ such that $[D, a]$ is bounded, and D has compact resolvent.
  - D may be thought of as an first-order differential operator.

The odd index pairing is the Kasparov product

$$\mathrm{K}_1(A) \times \mathrm{KK}_1(A, \mathbb{C}) \to \mathrm{K}_0(\mathbb{C}) \simeq \mathbb{Z}, \quad [v] \times [D] \mapsto \mathrm{ind}(\underbrace{PvP + 1 - P}_{\text{Fredholm}}),$$

where P is the positive spectral projection of D:

$$P := \chi_{(0,+\infty)}(D).$$

The odd spectral localiser, 1

Loring and Schulz-Baldes have introduced a method called spectral localiser to compute the odd index pairing $[v] \times [D]$.

Input:

- an odd spectral triple $(\mathcal{A}, \mathcal{H}, D)$;
- a unitary $v \in \mathcal{A}$;
- a tuning parameter $\kappa > 0$;
- a threshold $\lambda > 0$.

Output:

- a (family of) unbounded self-adjoint operators

$$L_\kappa := \begin{pmatrix} \kappa D & v \\ v^* & -\kappa D \end{pmatrix};$$

- a finite-dimensional matrix $L_{\kappa,\lambda}$;
- half-signature $\frac{1}{2}\operatorname{sig}(L_{\kappa,\lambda})$.

The odd spectral localiser, 2

Theorem (Loring–Schulz-Baldes 2017, 2019) For sufficiently small tuning parameter κ and sufficiently large threshold λ, then

$$\operatorname{ind}(PvP + 1 - P) = \frac{1}{2}\operatorname{sig}(L_{\kappa,\lambda})$$

where $L_{\kappa,\lambda}$ is the truncation of L_κ onto the spectral subspace

$$\mathcal{H}_\lambda := \chi_{|x|\leq\lambda}(D \oplus D)(\mathcal{H} \oplus \mathcal{H}).$$

(finite-dimensional, if D has discrete spectrum)

Examples from topological insulators:

- D is the position operator.

$\Rightarrow$ Finite-volume computation of index pairing.

Signature versus rank

Let $\mathcal{K}$ be a fin.-dim'l Hilbert space. Then $\mathbb{B}(\mathcal{K}) \simeq \mathbb{M}_{\dim \mathcal{K}}(\mathbb{C})$.

- The signature of a self-adjoint matrix L is

$$\begin{aligned} \mathrm{sig}(L) &= \#(\text{pos. eigenvalues of } L) - \#(\text{neg. eigenvalues of } L) \\ &= \mathrm{rank}(\chi_{(0,+\infty)}(L)) - \mathrm{rank}(\chi_{(-\infty,0)}(L)). \end{aligned}$$

- If p is a projection on $\mathcal{K}$, then

$$\mathrm{sig}(2p - \mathrm{id}_{\mathcal{K}}) = \mathrm{rank}(p) - \mathrm{rank}(\mathrm{id}_{\mathcal{K}} - p).$$

Signature versus rank

Let $\mathcal{K}$ be a fin.-dim'l Hilbert space. Then $\mathbb{B}(\mathcal{K}) \simeq \mathbb{M}_{\dim \mathcal{K}}(\mathbb{C})$.

- The signature of a self-adjoint matrix L is

$$\begin{aligned}\operatorname{sig}(L) &= \#(\text{pos. eigenvalues of } L) - \#(\text{neg. eigenvalues of } L)\\ &= \operatorname{rank}(\chi_{(0,+\infty)}(L)) - \operatorname{rank}(\chi_{(-\infty,0)}(L)).\end{aligned}$$

- If p is a quasi-projection on $\mathcal{K}$, then

$$\operatorname{sig}(2p - \mathrm{id}_{\mathcal{K}}) = \operatorname{rank}(\kappa_0(p)) - \operatorname{rank}(\kappa_0(\mathrm{id}_{\mathcal{K}} - p)).$$

$$\kappa_0 : \mathbb{C} \setminus \{z \in \mathbb{C} \mid \operatorname{Re} z = 1/2\} \to \{0, 1\}, \quad \kappa_0(z) := \begin{cases} 1, & \operatorname{Re} z > 1/2; \\ 0, & \operatorname{Re} z < 1/2. \end{cases}$$

Signature versus rank

Let $\mathcal{K}$ be a fin.-dim'l Hilbert space. Then $\mathbb{B}(\mathcal{K}) \simeq \mathbb{M}_{\dim \mathcal{K}}(\mathbb{C})$.

- The signature of a self-adjoint matrix L is

$$\begin{aligned}\operatorname{sig}(L) &= \#(\text{pos. eigenvalues of } L) - \#(\text{neg. eigenvalues of } L)\\ &= \operatorname{rank}(\chi_{(0,+\infty)}(L)) - \operatorname{rank}(\chi_{(-\infty,0)}(L)).\end{aligned}$$

- If p is a quasi-projection on $\mathcal{K}$, then

$$\operatorname{sig}(2p - \mathrm{id}_{\mathcal{K}}) = \operatorname{rank}(\kappa_0(p)) - \operatorname{rank}(\kappa_0(\mathrm{id}_{\mathcal{K}} - p)).$$

$$\kappa_0 : \mathbb{C} \setminus \{z \in \mathbb{C} \mid \operatorname{Re} z = 1/2\} \to \{0, 1\}, \quad \kappa_0(z) := \begin{cases} 1, & \operatorname{Re} z > 1/2; \\ 0, & \operatorname{Re} z < 1/2. \end{cases}$$

$\Rightarrow$ $L_{\kappa,\lambda}$ may come from a quasi-projection representative of K-theory.

Quasi-projections and quasi-idempotents

- Quasi-projection: $\|p^2 - p\| < 1/4$, $p = p^*$;
- Quasi-idempotent: $\|e^2 - e\| < 1/4$.

Let A be a unital C*-algebra. Then

$$\begin{aligned} \mathrm{K}_0(A) &= \mathrm{Gr}(\text{homotopy classes of projections in } \mathbb{M}_n A) \\ &= \mathrm{Gr}(\text{homotopy classes of quasi-projections in } \mathbb{M}_n A). \end{aligned}$$

Class of quasi-projection p = class of the projection $\kappa_0(p)$.

Quasi-projections and quasi-idempotents

- Quasi-projection: $\|p^2 - p\| < 1/4$, $p = p^*$;
- Quasi-idempotent: $\|e^2 - e\| < 1/4$.

Let A be a unital C*-algebra. Then

$$\begin{aligned} \mathrm{K}_0(A) &= \mathrm{Gr}(\text{homotopy classes of projections in } \mathbb{M}_n A) \\ &= \mathrm{Gr}(\text{homotopy classes of quasi-projections in } \mathbb{M}_n A). \end{aligned}$$

Class of quasi-projection p = class of the projection $\kappa_0(p)$.

Physical intuition:

$$\begin{aligned} \text{Quasi-projections} &\Leftarrow \text{self-adjoint invertibles} \\ &\Leftarrow \text{gapped Hamiltonians.} \end{aligned}$$

A quasi-idempotent may arise as the image of a projection under an asymptotic morphism.

Blueprint

Goal Compute the index pairing $\mathrm{K}_1(A) \times \mathrm{KK}_1(A, B) \to \mathrm{K}_0(B)$ between a unitary $v \in A$ and an odd unbounded Kasparov A-B–module $(\mathcal{A}, E, D)$.

Blueprint

Goal Compute the index pairing $\mathrm{K}_1(A) \times \mathrm{KK}_1(A, B) \to \mathrm{K}_0(B)$ between a unitary $v \in A$ and an odd unbounded Kasparov A-B–module $(\mathcal{A}, E, D)$.

1. Construct an asymptotic morphism $(\Phi_t^D)_{t\in[1,\infty)}$ which gives the same index pairing $\mathrm{K}_1(A) \to \mathrm{K}_0(\mathbb{K}_B(E))$.

Blueprint

Goal Compute the index pairing $\mathrm{K}_1(A) \times \mathrm{KK}_1(A, B) \to \mathrm{K}_0(B)$ between a unitary $v \in A$ and an odd unbounded Kasparov A-B–module $(\mathcal{A}, E, D)$.

1. Construct an asymptotic morphism $(\Phi_t^D)_{t \in [1,\infty)}$ which gives the same index pairing $\mathrm{K}_1(A) \to \mathrm{K}_0(\mathbb{K}_B(E))$.
2. Pair $(\Phi_t^D)_t$ with v to obtain a quasi-projection representative $[e_t] - [f_t]$ of $\langle [v], [D] \rangle \in \mathrm{K}_0(\mathbb{K}_B(E))$ when $t \gg 0$.

Blueprint

Goal   Compute the index pairing $\mathrm{K}_1(A) \times \mathrm{KK}_1(A,B) \to \mathrm{K}_0(B)$ between a unitary $v \in A$ and an odd unbounded Kasparov A-B–module $(\mathcal{A}, E, D)$.

1. Construct an asymptotic morphism $(\Phi_t^D)_{t\in[1,\infty)}$ which gives the same index pairing $\mathrm{K}_1(A) \to \mathrm{K}_0(\mathbb{K}_B(E))$.
2. Pair $(\Phi_t^D)_t$ with v to obtain a quasi-projection representative $[e_t] - [f_t]$ of $\langle [v], [D] \rangle \in \mathrm{K}_0(\mathbb{K}_B(E))$ when $t \gg 0$.
3. Spectrally truncate $[e_t] - [f_t]$ to a submodule to get yet another quasi-projection representative $[p^e_{t,\lambda}] - [p^f_{t,\lambda}]$.
   This depends on another parameter λ.

Blueprint

Goal Compute the index pairing $\mathrm{K}_1(A) \times \mathrm{KK}_1(A, B) \to \mathrm{K}_0(B)$ between a unitary $v \in A$ and an odd unbounded Kasparov A-B–module $(\mathcal{A}, E, D)$.

1. Construct an asymptotic morphism $(\Phi_t^D)_{t\in[1,\infty)}$ which gives the same index pairing $\mathrm{K}_1(A) \to \mathrm{K}_0(\mathbb{K}_B(E))$.
2. Pair $(\Phi_t^D)_t$ with v to obtain a quasi-projection representative $[e_t] - [f_t]$ of $\langle [v], [D] \rangle \in \mathrm{K}_0(\mathbb{K}_B(E))$ when $t \gg 0$.
3. Spectrally truncate $[e_t] - [f_t]$ to a submodule to get yet another quasi-projection representative $[p^e_{t,\lambda}] - [p^f_{t,\lambda}]$. This depends on another parameter λ.
4. The spectral localiser $L_{t^{-1},\lambda}$ is congruent to $2p^e_{t,\lambda} - 1$, and $p^f_{t,\lambda}$ contributes to zero signature.

Unbounded Kasparov module, 1

Definition   Let A and B be C*-algebras. An odd unbounded Kasparov A-B–module is a triple $(\mathcal{A}, E, D)$, where:

- E is a Hilbert B-module;
- $\varrho\colon A \to \operatorname{End}_B^*(E)$ is a *-homomorphism onto the C*-algebra of adjointable operators; and $\mathcal{A} \subseteq A$ is a dense *-subalgebra;
- $D\colon\ \operatorname{dom} D \subseteq E \to E$ is a self-adjoint and regular operator.

such that:

- $\varrho(a)(D + \mathrm{i})^{-1} \in \mathbb{K}_B(E)$ for all $a \in \mathcal{A}$ (and hence for all $a \in A$);
- For every $a \in \mathcal{A}$, $\varrho(a)$ maps $\operatorname{dom} D$ into $\operatorname{dom} D$, and $[D, \varrho(a)]$ extends to an element of $\operatorname{End}_B^*(E)$.

An unbounded Kasparov module is essential if $\overline{\varrho(A)E} = E$.

Unbounded Kasparov module, 2

Let $\chi\colon \mathbb{R} \to [-1,1]$ be any chopping function (i.e.$\lim_{x\to\pm\infty}\chi(x) = \pm 1$). Then the bounded transform

$$(\mathcal{A}, E, D) \quad \longmapsto \quad (A, E, \chi(D))$$

gives a odd, bounded Kasparov A-B–module.

Every class in $\mathrm{KK}_1(A,B)$ can be represented by an essential odd, unbounded Kasparov A-B–module.

Example

- $\left(C_c^\infty(\mathbb{R}), L^2(\mathbb{R}), -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\right)$: odd, unbounded Kasparov $C_0(\mathbb{R})$-$\mathbb{C}$–module.
- $\left(\mathbb{C}, C_0(\mathbb{R}), x\right)$: odd, unbounded Kasparov $\mathbb{C}$-$C_0(\mathbb{R})$–module.
- A spectral triple is an odd unbounded Kasparov A-$\mathbb{C}$–module.

Asymptotic morphism, 1

Definition An asymptotic morphism $A \dashrightarrow B$ is a family of maps $(\Phi_t : A \to B)_{t\in[1,\infty)}$ such that:

1. $t \mapsto \Phi_t(a)$ is continuous for all $a \in A$;
2. for all $a, a' \in A$ and $\lambda \in \mathbb{C}$,

$$\left.\begin{array}{c} \Phi_t(a^*) - \Phi_t(a)^* \\ \Phi_t(aa') - \Phi_t(a)\Phi_t(a') \\ \Phi_t(a+\lambda a') - \Phi_t(a) - \lambda\Phi_t(a') \end{array}\right\} \to 0 \quad \text{as } t \to \infty.$$

An asymptotic morphism $A \dashrightarrow B$ represents a class in $\mathrm{E}(\mathrm{S}A, B) =: \mathrm{E}_1(A, B)$ in E-theory of Connes and Higson.

This is another bivariant K-theory, characterised by different universal properties from Kasparov's KK-theory.

Asymptotic morphism, 2

There is a natural transformation $\mathrm{KK}_1 \Rightarrow \mathrm{E}_1$, making the following diagram commutes:

$$\begin{tikzcd}
\mathrm{K}_1(A) \arrow[r, "\mathrm{KK}_1(A,B)"] \arrow[d, "\mathrm{ind}"] \arrow[dr, phantom, "\Downarrow"] & \mathrm{K}_0(B) \arrow[d, "\pm 1"] \\
\mathrm{K}_0(\mathrm{S}A) \arrow[r, "\mathrm{E}_1(A,B)"'] & \mathrm{K}_0(B).
\end{tikzcd}$$

The following result is folklore.

Theorem (Higson–Kasparov 2001; L–Mesland 2025) Let $(\mathcal{A}, E, D)$ be an essential odd unbounded Kasparov A-B–module. The asymptotic morphism

$$\Phi_t^D : \mathrm{C}_0(\mathbb{R}) \otimes A \to \mathbb{K}_B(E), \quad f \otimes a \mapsto f(t^{-1}D)a,$$

represents the class of the image of $[D] \in \mathrm{KK}_1(A,B)$ in $\mathrm{E}_1(A,B)$ under the natural transformation $\mathrm{KK}_1 \to \mathrm{E}_1$.

Quasi-projection representative

The image of $[v] \in \mathrm{K}_1(A)$ in $\mathrm{K}_0(SA)$ is given by

$$\left[\begin{pmatrix} s(x)^2 \otimes 1 & c(x)s(x) \otimes v \\ c(x)s(x) \otimes v^* & c(x)^2 \otimes 1 \end{pmatrix}\right] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right].$$

where $c, s \colon \mathbb{R} \to [0,1]$ are continuous functions satisfying

$$c^2(x) + s^2(x) = 1, \quad \lim_{x\to\infty} s(x) = 1, \quad \lim_{x\to-\infty} c(x) = 1.$$

Theorem The odd index pairing $\langle [v], [D] \rangle$ is represented by:
(t sufficiently large + certain commutators being bounded):

$$\underbrace{\left[\begin{pmatrix} s_t^2 & \sqrt{c_t s_t}\, v \sqrt{c_t s_t} \\ \sqrt{c_t s_t}\, v^* \sqrt{c_t s_t} & c_t^2 \end{pmatrix}\right]}_{\text{quasi-projection representative, } =: e_t} - \underbrace{\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]}_{\text{projection, } =: f_t}.$$

where $c_t := c(t^{-1}D)$ and $s_t := s(t^{-1}D)$.

Spectral truncation, 1

Setup:

- B a C*-algebra;
- $\mathcal{E}$ a Hilbert B-module;
- $\mathcal{D}$ an unbounded, self-adjoint and regular operator on $\mathcal{E}$.
- λ a positive real number ("spectral threshold")

Definition A spectral decomposition of $\mathcal{E}$ is a pair of mutually complemented submodules $(\mathcal{E}_\lambda^\downarrow, \mathcal{E}_\lambda^\uparrow)$ of $\mathcal{E}$, such that

$$\langle \mathcal{D}\xi, \mathcal{D}\xi \rangle \geq \lambda^2 \langle \xi, \xi \rangle \quad \text{for all } \xi \in \operatorname{dom} \mathcal{D} \cap \mathcal{E}_\lambda^\uparrow.$$

Remark For any $\lambda > 0$, a spectral decomposition $\mathcal{E} = \mathcal{E}_\lambda^\downarrow \oplus \mathcal{E}_\lambda^\uparrow$ always exists if $B = \mathbb{C}$; may not exist for general B.

Spectral truncation, 2

Let $(E^{\downarrow}_{\lambda}, E^{\uparrow}_{\lambda})$ be a spectral decomposition of $\widehat{E} := E \oplus E$ for the operator $D \oplus D$.

Then e_t decomposes as

$$e_t = \begin{pmatrix} p^e_{t,\lambda} & {m^e_{t,\lambda}}^* \\ m^e_{t,\lambda} & q^e_{t,\lambda} \end{pmatrix} \quad \curvearrowright \quad \begin{pmatrix} E^{\downarrow}_{\lambda} \\ E^{\uparrow}_{\lambda} \end{pmatrix}.$$

Theorem For $t \gg 0$ and $\lambda \gg 0$, the following quasi-projections are homotopic:

$$\begin{pmatrix} p^e_{t,\lambda} & {m^e_{t,\lambda}}^* \\ m^e_{t,\lambda} & q^e_{t,\lambda} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} p^e_{t,\lambda} & \\ & q^e_{t,\lambda} \end{pmatrix}.$$

Therefore, as classes in $\mathrm{K}_0(\mathbb{K}_B(E)^+)$:

$$[e_t] = [p^e_{t,\lambda}] + [q^e_{t,\lambda}].$$

Spectral truncation, 3

We have $[f_t] = [p^f_{t,\lambda}] + [q^f_{t,\lambda}]$, and

$$q^e_{t,\lambda} \sim q^f_{t,\lambda}$$

are asymptotically unitarily equivalent. So $[q^e_{t,\lambda}] = [q^f_{t,\lambda}]$.

Summing up:

$$\begin{aligned}[e_t] - [f_t] &= [p^e_{t,\lambda}] + [q^e_{t,\lambda}] - [p^f_{t,\lambda}] - [q^f_{t,\lambda}] \\ &= [p^e_{t,\lambda}] - [p^f_{t,\lambda}],\end{aligned}$$

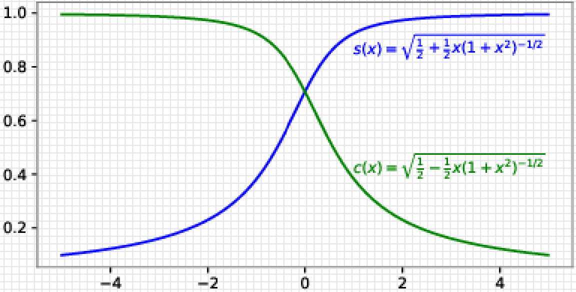
both $p^e_{t,\lambda}$ and $p^f_{t,\lambda}$ are quasi-projections on $E^{\downarrow}_{\lambda}$.

If $(\mathcal{A}, E, D)$ is a spectral triple and D has discrete spectrum, then $p^e_{t,\lambda}$ and $p^f_{t,\lambda}$ are finite-dimensional quasi-projections.

Emergence of the spectral localiser, 1

A choice of $c(x)$ and $s(x)$:

$$c(x) := \sqrt{\frac{1}{2} - \frac{1}{2}x(1+x^2)^{-1/2}}, \quad s(x) := \sqrt{\frac{1}{2} + \frac{1}{2}x(1+x^2)^{-1/2}}.$$



They satisfy the prescribed commutator estimates.

Emergence of the spectral localiser, 2

- $2p^e_{t,\lambda} - 1$ is given by ($D_t := t^{-1}D$):

$$\begin{pmatrix} D_t(1+D_t^2)^{-1/2} & (1+D_t^2)^{-1/4} v (1+D_t^2)^{-1/4} \\ (1+D_t^2)^{-1/4} v^* (1+D_t^2)^{-1/4} & -D_t(1+D_t^2)^{-1/2} \end{pmatrix}$$
$$= \begin{pmatrix} (1+D_t^2)^{-1/4} & 0 \\ 0 & (1+D_t^2)^{-1/4} \end{pmatrix} \underbrace{\begin{pmatrix} D_t & v \\ v^* & -D_t \end{pmatrix}}_{L_{t^{-1},\lambda}} \begin{pmatrix} (1+D_t^2)^{-1/4} & 0 \\ 0 & (1+D_t^2)^{-1/4} \end{pmatrix}.$$

- $2p^f_{t,\lambda} - 1$ is given by

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

  and has no signature.

Emergence of the spectral localiser, 3

Theorem (Loring–Schulz-Baldes 2017, 2019; L–Mesland 2025)
Assume that $(E^{\downarrow}_{\lambda}, E^{\uparrow}_{\lambda})$ is a spectral decomposition of $\widehat{E} := E \oplus E$ (with spectral threshold λ). Define

$$L_{t^{-1}} := \begin{pmatrix} t^{-1}D & v \\ v^* & -t^{-1}D \end{pmatrix},$$

and $L_{t^{-1},\lambda}$ be its truncation onto $E^{\downarrow}_{\lambda}$. Let:

- $\varepsilon, \delta > 0$ satisfy $\varepsilon + \delta < \frac{1}{400}$;
- t, λ satisfy $t > 4\varepsilon^{-1}\|[D, v]\|$ and $\lambda > t\delta^{-1}$,

Then

$$\frac{1}{2}\operatorname{sig}(L_{t^{-1},\lambda}) = \langle [v], [D] \rangle \in \mathrm{K}_0(\mathbb{K}_B(E)).$$

Further remarks

Why unbounded Kasparov modules?

The data of an unbounded Kasparov module yield several commutator estimates that are used to construct:

- homotopies of quasi-projections/idempotents;
- homotopies of asymptotic morphisms.

Numerical index pairings with traces

The spectral truncation technique also applies to computing index pairings associated to certain semi-finite spectral triples, arising as

$$\mathrm{K}_1(A) \xrightarrow{\times[D]} \mathrm{K}_0(B) \xrightarrow{\tau_*} \mathbb{R},$$

and yields the semi-finite spectral localiser of Schulz-Baldes and Stoiber.

Summary and outlook

Summary

- The index pairing $\langle [v], [D] \rangle$ may be computed via an E-theoretic index pairing.
- This yields quasi-projection representatives of K-theory.
  A spectral truncation may be applied, providing the existence of a suitable spectral decomposition.
- The spectral localiser occurs as such an index pairing. Similar story for its semi-finite analogue.

Summary and outlook

Summary

- The index pairing $\langle [v], [D] \rangle$ may be computed via an E-theoretic index pairing.
- This yields quasi-projection representatives of K-theory.
  A spectral truncation may be applied, providing the existence of a suitable spectral decomposition.
- The spectral localiser occurs as such an index pairing. Similar story for its semi-finite analogue.

Outlook

- Asymptotic morphisms arise also from deformations of C*-algebras. Possible application to index theorems?
- Van Suijlekom has defined a variant of K-theory for operator systems. Possible application to a definition of their K-homology?