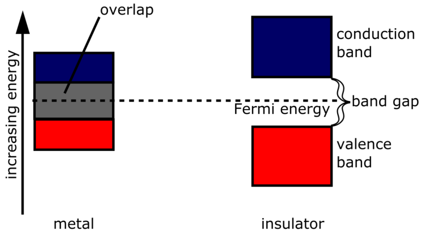
On robustness of topological phases

Yuezhao Li
Mathematical institute, Leiden university
CIRM, 08-04-2025

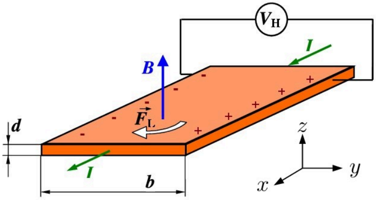
Universiteit Leiden
The Netherlands

Topological insulators + integer quantum Hall effect



Topological insulators

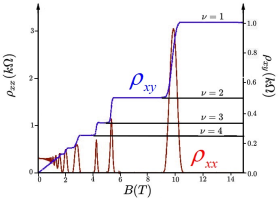
- Topological insulators are materials that are insulating in the bulk, but permit a current to flow on the boundary.
- The current flowing on the current is usually quite robust under disorder, protected by a topological invariant.



$$\mathbf{J} = \boldsymbol{\sigma} \cdot \boldsymbol{E}$$

$$\boldsymbol{\rho} = \boldsymbol{\sigma}^{-1}$$

$$= \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{yy} \end{pmatrix}$$



NC framework of topological insulators

- Single-particle Hamiltonian H.
  It describes an insulator if the Fermi energy μ belongs to a spectral gap.
- The observable C*-algebra:
  a C*-algebra containing the resolvent of H.
- Topological phases are described by K-theory classes of A.
  No chiral/time-reversal/particle-hole symmetries: K_0-class, given by the Fermi projection $p_\mu := \chi_{(-\infty,\mu)}(H)$.
- Presence of (real) symmetries: replace A by $A \otimes \mathrm{C}\ell_{p,q}$ and K by KO.

Example: IQHE system

- Space $\mathbb{R}^2$ + Electromagnetic potential A, $\mathrm{d}A = \theta \mathrm{d}x \wedge \mathrm{d}y$.
- Hamiltonian

$$H = \frac{1}{2}(\mathrm{d} + \mathrm{i}A)^*(\mathrm{d} + \mathrm{i}A) + V.$$

- If V is translation-invariant for $\mathbb{Z}^2$:

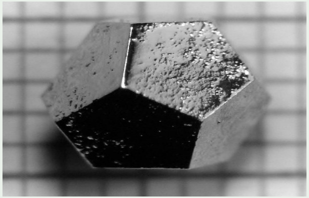
$$p_\mu := \chi_{(-\infty,\mu)}(H) \in (\mathbb{C} \rtimes_\sigma \mathbb{Z}^2) \otimes \mathbb{K}.$$

- If V is aperiodic, then Bellissard describes the system by a crossed product $\mathrm{C}^*(\Omega) \rtimes_\sigma \mathbb{Z}^2$.
- This still requires a $\mathbb{Z}^2$-labelling of the lattice.

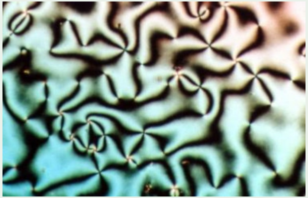
Modelling aperiodic systems, 1

Question

1. What if Λ is not a periodic lattice?



quasi-crystal



liquid crystal

glass

2. How to understand the “robustness” of topological phases under disorder?

Both questions suggest to look for the “correct” observable C*-algebra.

Modelling aperiodic systems, 2

Delone sets

Let $0 < r < R$. A discrete infinite set $\Lambda \subseteq \mathbb{R}^d$ is called an (r, R)-Delone set if for all $x \in \mathbb{R}^d$:

$$\#(\mathrm{B}(x, r) \cap \Lambda) \leq 1 \quad \text{and} \quad \#(\mathrm{B}(x, R) \cap \Lambda) \geq 1.$$

i.e. Λ is "uniformly discrete" and "relatively dense".

Observable C*-algebra from a Delone set Λ: it should be

- large enough to contain all possible Hamiltonians;
- small enough to have interesting K-theory.

Robustness

Stability under short-range + locally–finite-rank perturbations $\Rightarrow$

Study their images in the Roe C*-algebras through a "comparison map".

Modelling aperiodic systems, 3

Dynamical approach

- $\mathcal{G}_\Lambda$: a topological groupoid, which encodes the "generalised symmetries" of Λ.
- $\mathrm{C}^*_\mathrm{r}(\mathcal{G}_\Lambda)$: C*-algebra generated by regular representations of the convolution algebra $\mathrm{C}_\mathrm{c}(\mathcal{G}_\Lambda)$.
  It consists of (covariant families of) operators that are covariant for the *groupoid* action.
- If $\Lambda = \mathbb{Z}^d$: $\mathcal{G}_\Lambda \simeq \mathbb{Z}^d$, $\mathrm{C}^*_\mathrm{r}(\mathcal{G}_\Lambda) \simeq \mathrm{C}^*_\mathrm{r}(\mathbb{Z}^d)$.

Coarse-geometric approach

- $\mathrm{C}^*_\mathrm{Roe}(\Lambda)$: C*-algebra generated by operators that are "short-range" and "locally–finite-rank".
- Viewpoint of [Ewert–Meyer]: it is generated by the "position operators" on Λ.
- Eilenberg swindle argument.

The (localised) regular representations provide "comparison" maps

$$\pi_\omega : \mathrm{C}^*_\mathrm{r}(\mathcal{G}_\Lambda) \to \mathrm{C}^*_\mathrm{Roe}(\omega),$$

where ω's are translated copies of Λ, or their weak*-limits.

Stacked phases

- Eilenberg swindle argument:

$$0 = (1-1) + (1-1) + \cdots = 1 + (-1+1) + (-1+1) + \cdots = 1.$$

- This can be used to show the K-theory of certain C*-algebras vanish, e.g. $\mathbb{B}(\mathcal{H})$, where $\mathcal{H}$ is an infinite-dimensional Hilbert space.
- $\mathrm{K}_*(\mathrm{C}^*_{\mathrm{Roe}}(X)) = 0$ if X is a flasque space, e.g. $X = Y \times \mathbb{N}$.

Theorem (L)

If $\Lambda = \Lambda_1 \times \Lambda_2$ is a product Delone set, then the regular representation $\mathrm{C}^*_{\mathrm{r}}(\mathcal{G}_\Lambda) \to \mathrm{C}^*_{\mathrm{Roe}}(\Lambda_1 \times \Lambda_2)$ factors through a flasque space, hence "weak".

- Special case: $\Lambda_1 = \mathbb{Z}^d$ and $\Lambda_2 = \mathbb{Z}$.
  This can be thought of as "stacking" topological phases on $\mathbb{Z}^d$ along $\mathbb{Z}$.
  Such topological phases (and their invariants) are weak.

Position spectral triples

- Spectral triples represent classes in K-homology, and hence can be paired with K-theory to obtain numerical invariants.
- Call a spectral triple of the form

$$\xi := \left(\mathcal{A} \otimes \mathrm{C}\ell_{0,d}, \quad \ell^2(\Lambda, \mathcal{K})_{\mathbb{R}} \otimes \bigwedge^* \mathbb{R}^d, \quad \sum_{j=1}^{d} \mathsf{X}_j \otimes \gamma^j\right)$$

a position spectral triple.
($\mathcal{K}$ is a separable real Hilbert space, X_j are the "position" operators on $\ell^2(\Lambda)$.)

Theorem (L)

If a C-algebra A possesses a position spectral triple, then:*

1. *The image of A is contained in $\mathrm{C}^*_{\mathrm{Roe}}(\Lambda)$.*
2. *The K-homology class of ξ is the pullback of a position spectral triple $\xi_\Lambda^{\mathrm{Roe}}$ over $\mathrm{C}^*_{\mathrm{Roe}}(\Lambda)$.*
3. *$\xi_\Lambda^{\mathrm{Roe}}$ generates the K-homology of $\mathrm{C}^*_{\mathrm{Roe}}(\Lambda)$.*

$\Rightarrow$ Topological phases detected by position spectral triples are "strong".