

The spectral localiser via E-theory

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Yuezhao Li

Mathematical institute, Leiden university



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The odd index pairing

Setup:

- ▶ A a unital C^* -algebra, $v \in A$ a unitary;
- ▶ (A, \mathcal{H}, D) an odd (=ungraded) spectral triple over A .

The **odd index pairing** is the Kasparov product

$$K_1(A) \times KK_1(A, \mathbb{C}) \rightarrow K_0(\mathbb{C}) \simeq \mathbb{Z}, \quad [v] \times [D] \mapsto \text{ind}(PvP + 1 - P),$$

where P is the positive spectral projection of D .

The odd spectral localiser, 1

Loring and Schulz-Baldes have introduced a method called **spectral localiser** to compute the odd index pairing $[v] \times [D]$.

Input:

- ▶ an odd spectral triple $(\mathcal{A}, \mathcal{H}, D)$;
- ▶ a unitary $v \in \mathcal{A}$;
- ▶ a tuning parameter $\kappa > 0$;
- ▶ a threshold $\lambda > 0$.

Output:

- ▶ a (family of) unbounded self-adjoint operators

$$L_{\kappa} := \begin{pmatrix} \kappa D & v \\ v^* & -\kappa D \end{pmatrix};$$

- ▶ a finite-dimensional matrix $L_{\kappa, \lambda}$;
- ▶ half-signature $\frac{1}{2} \operatorname{sig}(L_{\kappa, \lambda})$.

The odd spectral localiser, 2

Theorem (Loring–Schulz–Baldes 2017, 2019) For sufficiently **small** tuning parameter κ and sufficiently **large** threshold λ , then

$$\mathrm{ind}(P \vee P + 1 - P) = \frac{1}{2} \mathrm{sig}(L_{\kappa, \lambda})$$

where $L_{\kappa, \lambda}$ is the **truncation** of L_{κ} onto the spectral subspace

$$\mathcal{H}_{\lambda} := \chi_{|x| \leq \lambda} (D \oplus D) (\mathcal{H} \oplus \mathcal{H}).$$

(**finite-dimensional**, if D has discrete spectrum)

Examples from topological insulators:

- D is the **position** operator.
- ⇒ Finite-**volume** computation of index pairing.

Signature versus rank

Let \mathcal{K} be a fin.-dim'l Hilbert space. Then $\mathbb{B}(\mathcal{K}) \simeq \mathbb{M}_{\dim \mathcal{K}}(\mathbb{C})$.

- The **signature** of a self-adjoint matrix L is

$$\begin{aligned}\text{sig}(L) &= \#(\text{pos. eigenvalues of } L) - \#(\text{neg. eigenvalues of } L) \\ &= \text{rank}(\chi_{(0,+\infty)}(L)) - \text{rank}(\chi_{(-\infty,0)}(L)).\end{aligned}$$

- If p is a **projection** on \mathcal{K} , then

$$\text{sig}(2p - \text{id}_{\mathcal{K}}) = \text{rank}(p) - \text{rank}(\text{id}_{\mathcal{K}} - p).$$

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- If p is a **quasi-projection** on \mathcal{K} , then

$$\text{sig}(2p - \text{id}_{\mathcal{K}}) = \text{rank}(\kappa_0(p)) - \text{rank}(\kappa_0(\text{id}_{\mathcal{K}} - p)).$$

$$\kappa_0: \mathbb{C} \setminus \{z \in \mathbb{C} \mid \text{Re } z = 1/2\} \rightarrow \{0, 1\}, \quad \kappa_0(z) := \begin{cases} 1, & \text{Re } z > 1/2; \\ 0, & \text{Re } z < 1/2. \end{cases}$$

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⇒ $L_{\kappa,\lambda}$ may come from a **quasi-projection** representative of K-theory.

Quasi-projections and quasi-idempotents

- ▶ Quasi-projection: $\|p^2 - p\| < 1/4$, $p = p^*$;
- ▶ Quasi-idempotent: $\|e^2 - e\| < 1/4$.

Let A be a unital C^* -algebra. Then (cf. [Willett–Yu]):

$$\begin{aligned} K_0(A) &= \text{Gr}(\text{homotopy classes of projections in } \mathbb{M}_n A) \\ &= \text{Gr}(\text{homotopy classes of quasi-projections in } \mathbb{M}_n A). \end{aligned}$$

Class of quasi-projection p = class of the projection $\kappa_0(p)$.

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Physical intuition:

$$\begin{aligned} \text{Quasi-projections} &\Leftarrow \text{self-adjoint invertibles} \\ &\Leftarrow \text{gapped Hamiltonians.} \end{aligned}$$

A quasi-idempotent may arise as the image of a projection under an asymptotic morphism.

Willett's finite-dimensional K-homology representative

Willett [Willett 2021] has constructed a fin.-dim'l representative of "the" inverse Bott map $K_1(SA) \rightarrow K_0(A \otimes \mathbb{K})$ based on Loring's (quasi-)projection and Yu's localisation algebra.

Review on MathSciNet (by Meyer):

..... This idea is reminiscent of the finite volume calculations of K-theory elements in [Loring and Schulz-Baldes 2017].

Source of this reminiscence: both of them can be viewed as an E-theoretic index pairing.

Blueprint

Goal Compute the index pairing $K_1(A) \times KK_1(A, B) \rightarrow K_0(B)$ between a unitary $v \in A$ and an odd unbounded Kasparov A - B -module (\mathcal{A}, E, D) .

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2. Pair $(\Phi_t^D)_t$ with v to obtain a **quasi-projection** representative $[e_t] - [f_t]$ of $\langle [v], [D] \rangle \in K_0(\mathbb{K}_B(E))$ when $t \gg 0$.

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3. Spectrally **truncate** $[e_t] - [f_t]$ to a submodule to get yet another quasi-projection representative $[p_{t,\lambda}^e] - [p_{t,\lambda}^f]$.
This depends on another parameter λ .

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4. The spectral localiser $L_{t^{-1},\lambda}$ is congruent to $2p_{t,\lambda}^e - 1$, and $p_{t,\lambda}^f$ contributes to zero signature.

Asymptotic morphism, 1

Definition An **asymptotic morphism** $A \dashrightarrow B$ is a family of maps $(\Phi_t: A \rightarrow B)_{t \in [1, \infty)}$ such that:

1. $t \mapsto \Phi_t(a)$ is continuous for all $a \in A$;
2. for all $a, a' \in A$ and $\lambda \in \mathbb{C}$,

$$\left. \begin{array}{l} \Phi_t(a^*) - \Phi_t(a)^* \\ \Phi_t(aa') - \Phi_t(a)\Phi_t(a') \\ \Phi_t(a + \lambda a') - \Phi_t(a) - \lambda \Phi_t(a') \end{array} \right\} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

An asymptotic morphism $A \dashrightarrow B$ represents a class in $E(SA, B) =: E_1(A, B)$ in **E-theory** of Connes and **Higson**.

This is another **bivariant** K-theory, characterised by different universal properties from Kasparov's KK-theory.

Asymptotic morphism, 2

There is a natural transformation $KK_1 \Rightarrow E_1$, making the following diagram commutes:

$$\begin{array}{ccc} K_1(A) & \xrightarrow{KK_1(A,B)} & K_0(B) \\ \downarrow \text{ind} & \Downarrow & \downarrow \pm 1 \\ K_0(SA) & \xrightarrow{E_1(A,B)} & K_0(B). \end{array}$$

The following result is folklore.

Theorem (Higson–Kasparov 2001; L–Mesland 2025) Let (\mathcal{A}, E, D) be an **essential** odd unbounded Kasparov A - B -module. The asymptotic morphism

$$\Phi_t^D : C_0(\mathbb{R}) \otimes A \rightarrow \mathbb{K}_B(E), \quad f \otimes a \mapsto f(t^{-1}D)a,$$

represents the class of the image of $[D] \in KK_1(A, B)$ in $E_1(A, B)$ under the natural transformation $KK_1 \rightarrow E_1$.

Quasi-projection representative

The image of $[v] \in K_1(A)$ in $K_0(SA)$ is given by

$$\left[\begin{pmatrix} s(x)^2 \otimes 1 & c(x)s(x) \otimes v \\ c(x)s(x) \otimes v^* & c(x)^2 \otimes 1 \end{pmatrix} \right] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right].$$

where $c, s: \mathbb{R} \rightarrow [0, 1]$ are continuous functions satisfying

$$c^2(x) + s^2(x) = 1, \quad \lim_{x \rightarrow \infty} s(x) = 1, \quad \lim_{x \rightarrow -\infty} c(x) = 1.$$

Theorem The odd index pairing $\langle [v], [D] \rangle$ is represented by:
(t sufficiently large + certain commutators being bounded):

$$\underbrace{\left[\begin{pmatrix} s_t^2 & \sqrt{c_t s_t} v \sqrt{c_t s_t} \\ \sqrt{c_t s_t} v^* \sqrt{c_t s_t} & c_t^2 \end{pmatrix} \right]}_{\text{quasi-projection representative, } =: e_t} - \underbrace{\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right]}_{\text{projection, } =: f_t}.$$

where $c_t := c(t^{-1}D)$ and $s_t := s(t^{-1}D)$.

Spectral truncation, 1

Setup:

- ▶ B a C^* -algebra;
- ▶ \mathcal{E} a Hilbert B -module;
- ▶ \mathcal{D} an unbounded, self-adjoint and regular operator on \mathcal{E} .
- ▶ λ a positive real number (“spectral threshold”)

Definition A **spectral decomposition** of \mathcal{E} is a pair of mutually complemented submodules $(\mathcal{E}_\lambda^\downarrow, \mathcal{E}_\lambda^\uparrow)$ of \mathcal{E} , such that

$$\langle \mathcal{D}\xi, \mathcal{D}\xi \rangle \geq \lambda^2 \langle \xi, \xi \rangle \quad \text{for all } \xi \in \text{dom } \mathcal{D} \cap \mathcal{E}_\lambda^\uparrow.$$

Remark For any $\lambda > 0$, a spectral decomposition $\mathcal{E} = \mathcal{E}_\lambda^\downarrow \oplus \mathcal{E}_\lambda^\uparrow$ **always** exists if $B = \mathbb{C}$; may **not** exist for general B .

Spectral truncation, 2

Let $(E_\lambda^\downarrow, E_\lambda^\uparrow)$ be a spectral decomposition of $\widehat{E} := E \oplus E$ for the operator $D \oplus D$.

Then e_t decomposes as

$$e_t = \begin{pmatrix} p_{t,\lambda}^e & m_{t,\lambda}^{e,*} \\ m_{t,\lambda}^e & q_{t,\lambda}^e \end{pmatrix} \curvearrowright \begin{pmatrix} E_\lambda^\downarrow \\ E_\lambda^\uparrow \end{pmatrix}.$$

Theorem For $t \gg 0$ and $\lambda \gg 0$, the following quasi-projections are homotopic:

$$\begin{pmatrix} p_{t,\lambda}^e & m_{t,\lambda}^{e,*} \\ m_{t,\lambda}^e & q_{t,\lambda}^e \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} p_{t,\lambda}^e & \\ & q_{t,\lambda}^e \end{pmatrix}.$$

Therefore, as classes in $K_0(\mathbb{K}_B(E)^+)$:

$$[e_t] = [p_{t,\lambda}^e] + [q_{t,\lambda}^e].$$

Spectral truncation, 3

We have $[f_t] = [p_{t,\lambda}^f] + [q_{t,\lambda}^f]$, and

$$q_{t,\lambda}^e \sim q_{t,\lambda}^f$$

are asymptotically unitarily equivalent. So $[q_{t,\lambda}^e] = [q_{t,\lambda}^f]$.

Summing up:

$$\begin{aligned}[e_t] - [f_t] &= [p_{t,\lambda}^e] + [q_{t,\lambda}^e] - [p_{t,\lambda}^f] - [q_{t,\lambda}^f] \\ &= [p_{t,\lambda}^e] - [p_{t,\lambda}^f],\end{aligned}$$

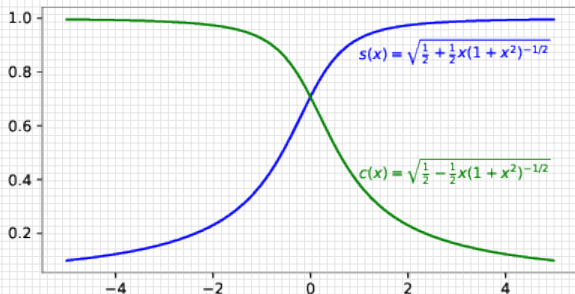
both $p_{t,\lambda}^e$ and $p_{t,\lambda}^f$ are quasi-projections on E_λ^\downarrow .

If (\mathcal{A}, E, D) is a spectral triple and D has discrete spectrum, then $p_{t,\lambda}^e$ and $p_{t,\lambda}^f$ are **finite**-dimensional quasi-projections.

Emergence of the spectral localiser, 1

A choice of $c(x)$ and $s(x)$:

$$c(x) := \sqrt{\frac{1}{2} - \frac{1}{2}x(1+x^2)^{-1/2}}, \quad s(x) := \sqrt{\frac{1}{2} + \frac{1}{2}x(1+x^2)^{-1/2}}.$$



They satisfy the prescribed **commutator estimates**.

Emergence of the spectral localiser, 2

- $2p_{t,\lambda}^e - 1$ is given by $(D_t := t^{-1}D)$:

$$\begin{aligned} & \begin{pmatrix} D_t(1+D_t^2)^{-1/2} & (1+D_t^2)^{-1/4}v(1+D_t^2)^{-1/4} \\ (1+D_t^2)^{-1/4}v^*(1+D_t^2)^{-1/4} & -D_t(1+D_t^2)^{-1/2} \end{pmatrix} \\ &= \begin{pmatrix} (1+D_t^2)^{-1/4} & 0 \\ 0 & (1+D_t^2)^{-1/4} \end{pmatrix} \underbrace{\begin{pmatrix} D_t & v \\ v^* & -D_t \end{pmatrix}}_{L_{t^{-1},\lambda}} \begin{pmatrix} (1+D_t^2)^{-1/4} & 0 \\ 0 & (1+D_t^2)^{-1/4} \end{pmatrix}. \end{aligned}$$

- $2p_{t,\lambda}^f - 1$ is given by

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and has no signature.

Emergence of the spectral localiser, 3

Theorem (Loring–Schulz–Baldes 2017, 2019; L–Mesland 2025)

Assume that $(E_\lambda^\downarrow, E_\lambda^\uparrow)$ is a spectral decomposition of $\widehat{E} := E \oplus E$ (with spectral threshold λ). Define

$$L_{t^{-1}} := \begin{pmatrix} t^{-1}D & v \\ v^* & -t^{-1}D \end{pmatrix},$$

and $L_{t^{-1}, \lambda}$ be its truncation onto E_λ^\downarrow . Let:

- ▶ $\varepsilon, \delta > 0$ satisfy $\varepsilon + \delta < \frac{1}{400}$;
- ▶ t, λ satisfy $t > 4\varepsilon^{-1} \|[D, v]\|$ and $\lambda > t\delta^{-1}$,

Then

$$\frac{1}{2} \operatorname{sig}(L_{t^{-1}, \lambda}) = \langle [v], [D] \rangle \in K_0(\mathbb{K}_B(E)).$$

Further remarks

The data of an **unbounded Kasparov module** yield several **commutator estimates** that are used to construct:

- ▶ homotopies of quasi-projections/idempotents;
- ▶ homotopies of asymptotic morphisms.

The spectral truncation technique also applies to computing index pairings associated to certain **semi-finite spectral triples**, and yields the **semi-finite** spectral localiser of **Schulz-Baldes** and Stoiber.

Summary and outlook

- ▶ The index pairing $\langle [\nu], [D] \rangle$ may be computed via an **E-theoretic** index pairing.
- ▶ This yields **quasi-projection** representatives of K-theory. A **spectral truncation** may be applied, providing the existence of a suitable spectral decomposition.
- ▶ The **spectral localiser** occurs as such an index pairing. Similar story for its semi-finite analogue.

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- ▶ The **spectral localiser** occurs as such an index pairing. Similar story for its semi-finite analogue.
- ▶ Possible application to the **spectral flow**.
- ▶ Possible application to defining K-homology of **operator systems**, dual to their K-theory (à la van Suijlekom).