Tightening Up the Incentive Ratio for Resource Sharing Over the Rings

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***Abstract*—Fundamental issues in resource sharing over large scale networks have gained much attention from the research community, in response to the growth of sharing economy over the Internet and mobile networks. We are particularly interested in the fundamental file sharing and subsequently P2P network bandwidth sharing developed by BitTorrent and later formalized by Wu and Zhang [15] as the *proportional response protocol*. It is of practical importance in the design to provide agent incentives to follow the distributed protocol out of their own rationality. We study the robustness of the distributed protocol in this incentive issue against a Sybil attack, a common type of grave threat in P2P network. For the resource sharing on rings, and we characterize the utility gain from a Sybil attack in the concept of *incentive ratio*. Previous works proved the incentive ratio is lower bounded by two and upper bounded by four, and later the upper bound is improved to three. It has been listed in [5] and [9] as an open problem to tighten them. In this paper, we completely resolve this open problem with a better understanding on the influence from different class agents to the resource allocation under the distributed protocol.**

***Keywords*-Resource sharing, Proportional response dynamics, Incentive ratio, Sybil attack, Distributed protocol.**

I. INTRODUCTION

Sharing economic resource over distributed networks has attracted venture capital investment and Internet entrepreneurship to make P2P network based sharing economies highly visible in recent years. Core to the success to such P2P networks, incentives have been regarded as among the most important factors [8]. It would build robustness [10], foster cooperations [12], promote voluntary contributions by participating agents [6], [7].

Formulating the key idea of the success of the BitTorrent protocol [10] as the proportional response algorithm, Wu and Zhang modeled each node as an agent to contribute its resource to each neighbor in proportion to the amount received from the corresponding neighbors. The protocol was further shown to converge and reach a proportional response equilibrium [15]. It was shown about ten years late that the protocol is indeed truthful [6], [7], if an agent manipulates its report on its resource available to share or the connections with its cooperative peers in the network.

It is desirable to investigate in further details on what operations an agent can make to incur changes of the resource allocation in the system. These changes would lead the system to improve the utility of this agent. Past progress in this direction has been slow. Some progress has been made but not enough to cover most of the network structures. In this work we make the next necessary step toward a full understanding in the effect of individual manipulative behavior on the general networks.

A. Related Work

Incentive design in the BitTorrent network for users to follow the resource sharing protocol has been widely regarded as the most successful applications in P2P network with an incentive design encouraging agents' voluntary participation [10], [12]. Cheng, et al., [7] have confirmed it neither capable of manipulative behavior of communication channel breaking nor capable of available resource misreporting by proving the truthfulness of each individual node.

Alternatively, there is another question whether it is possible for a user to deviate to improve its utility (possibly at the expenses of other peers in the system). Collective efforts of peers were known to improve their performance [13], [14]. The successfulness in an agent's improvement is commonly measured by the ratio of the new utility after deviation divided by its utility without deviation. This ratio is proposed by Chen, et al., [2], and is referred to as the *incentive ratio*.

There has been recent studies against the Sybil attack of any agent [3], [4]. In particular, for a network of a ring, several recent works have proven it to have an incentive ratio of four [5] and improved to three subsequently [9].

B. Our Contribution

Our work studies the proportional response protocol on the ring network, and focuses on an individual agent's Sybil attack. Our work limits the level of deviation for an agent to perform a Sybil attack in this network. On the other direction, it was known that the lower bound for a Sybil attack is also two [5]. This work completes the analysis for the ring network to establish the tight bound of two in terms of the incentive ratio.

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Our work develops new techniques for the analysis of Sybil attack in the resource sharing protocol based on the BitTorrent protocol. In the past work to limit the incentive ratio for the proportional response protocol, the network structure without ring is the first important step. There is a fully analyzed case for the complete network. The ring makes it difficult to characterize the fixed point solution when each participator aims to achieve its own optimality. Our approach develops a new and successive structure to characterize how the bottleneck decomposition structure changes in the network. This makes the solution be well organized in the analysis. This new idea is the key for tightening up the incentive ratio. Based on the new structural understanding, we also develop new analysis methods. We identify vertices whose classes remain unchanged at different stages of the analysis. Then we develop an innovative counting method for vertices in the same bottleneck pair as the cheating agent. This way, we find a more accurate numeric formula for the final evaluation to tighten up the incentive ratio.

The new ingredients in this work shed new light in the incentive analysis on limiting the impact of Sybil attack against the BitTorrent resource sharing system. They may lead to a new direction to the full understanding for general networks in the future.

C. Paper Organization

In Section 2, we provide some notations, properties and a distributed protocol, called the *proportional response dynamics*, for the resource sharing problem on P2P networks, introduce a combinatorial structure of *bottleneck decomposition*, as well as the *BD Allocation Mechanism* to achieve the fixed point allocation of the proportional response dynamics. In Section 3, we propose the main result of tight incentive ratio of two in Theorem 8. Such a result is proved based on the conditions that the manipulative agent v is in C class or in B class on original ring, in which several properties and propositions in [7] are applied. The last section concludes this paper.

II. Preliminaries

We consider the resource sharing problem on a P2P network, which is modeled as an undirected graph $G = (V, E)$. Each vertex $v \in V$ represents an agent endowed with an amount of resource $w_v \geq 0$ available for exchanging with its neighbors. The neighborhood of v is denoted by $\Gamma(v) = \{u|(u, v) \in E\}$ and x_{vu} is the amount of resource that v allocates to its neighbor $u \in \Gamma(v)$. Then the total collection $X = \{x_{vu}\}_{(v,u)\in E}$ is called the *resource allocation* (or allocation for short) on G. Given an allocation X, each agent's utility is $U_v(X) = \sum_{u\in\Gamma(v)} x_{uv}$, i.e., the total resource obtained from all its neighbors.

A. Proportional Response Dynamics

From the perspective of algorithm design, Wu and Zhang formulated the idea in the *tit-for-tat* cooperative technique [10] of seeking pareto efficiency and achieving robustness in network resource utilization as *proportional response dynamics* [15].

Definition 1 (Proportional Response Dynamics)**.** *Let $x_{vu}(t)$ be the allocation at period t. Proportional response dynamics are defined as $x_{vu}(0) = w_v/d_v$, where d_v is the degree of vertex v, and*

$$x_{vu}(t+1) = \frac{x_{uv}(t)}{\sum_{k\in\Gamma(v)} x_{kv}(t)} \cdot w_v. \tag{1}$$

Wu and Zhang proved that the proportional response dynamics (1) converges to a fixed point allocation and then proposed an allocation mechanism with the help of a combinatorial structure, called *bottleneck decomposition*, to obtain the fixed point allocation of (1) directly.

B. Bottleneck Decomposition

For any vertex set $S \subseteq V$, let us define $w(S) = \sum_{v\in S} w_v$ and $\Gamma(S) = \cup_{v\in S}\Gamma(v)$. It follows that $S \cap \Gamma(S) \neq \emptyset$, if S is not independent. Denote $\alpha(S) = w(\Gamma(S))/w(S)$ to be the inclusive expansion ratio of S, or the *α-ratio* of S for short. A set $B \subseteq V$ is called a *bottleneck* of G if $\alpha(B) = \min_{S\subseteq V} \alpha(S)$. A bottleneck with the maximal size is called the *maximal bottleneck*.

Definition 2 (Bottleneck Decomposition [15])**.** *Given $G = (V, E; \mathbf{w})$. Start with $V_1 = V$, $G_1 = G$ and $i = 1$. Find the maximal bottleneck B_i of G_i and let G_{i+1} be the induced subgraph on the vertex set $V_{i+1} = V_i - (B_i \cup C_i)$, where $C_i = \Gamma(B_i) \cap V_i$, the neighbor set of B_i in G_i. Repeat if $G_{i+1} \neq \emptyset$ and set $k = i$ if $G_{i+1} = \emptyset$. Then we call $\mathcal{B} = \{(B_1, C_1), \cdots, (B_k, C_k)\}$ the bottleneck decomposition of G, (B_i, C_i) the i-th bottleneck pair and $\alpha_i = w(C_i)/w(B_i)$ the α-ratio of (B_i, C_i).*

Based on the definition of bottleneck decomposition, some important properties can be explored, which are crucial for the subsequent allocation mechanism design and the discussion in this paper.

Proposition 3 ([15])**.** *Given a graph G, the bottleneck decomposition of G is unique and*

1) $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_k \leqslant 1$.
2) *if $\alpha_i = 1$, then $i = k$ and $B_i = C_i$, otherwise B_i is an independent set and $B_i \cap C_i = \emptyset$.*
3) *there is no edge between B_i and B_j, $i \neq j \in \{1, \ldots, k\}$.*
4) *if there is an edge between B_i and C_j, then $j \leqslant i$.*

To give the readers a good understanding of the bottleneck decomposition and the properties in Proposition 3, we provide an example showen in Fig. 1, in which the first bottleneck pair is $(B_1, C_1) = (\{v_1, v_2\}, \{v_3\})$ with $\alpha_1 = 1/3$ and the second one is $(B_2, C_2) = (\{v_4, v_5, v_6\}, \{v_4, v_5, v_6\})$ with $\alpha_2 = 1$. It is not hard to see $\alpha_1 < \alpha_2 = 1$, B_1 is independent and $B_1 \cap C_1 = \emptyset$, and $B_2 = C_2$ with $\alpha_2 = 1$, verifying Proposition 3-(1) and (2). Proposition 3-(3) and (4) are right from the definition of bottleneck decomposition directly.

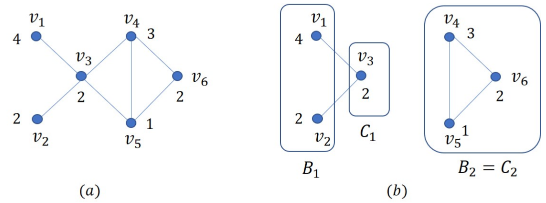

Figure 1. (a) The undirected graph G; (b) The bottleneck decomposition of G.

In addition, for the sake of the discussion, we partition the vertices in V into two classes as follows, based on whether the vertex is in B_i or C_i, $1 \le i \le k$.

Definition 4 (B class and C class)**.** *For pair (B_i, C_i) with $\alpha_i < 1$, each vertex in B_i (or C_i) is called a B class (or C class) vertex. If the last pair has the form as $B_k = C_k$ along with $\alpha_k = 1$, then all vertices in B_k are categorized as both B class and C class.*

It is worthy to note that a vertex in $B_k = C_k$ with $\alpha_k = 1$ is simultaneously a B class and C class vertex.

C. BD Allocation Mechanism

Given the bottleneck decomposition $\mathcal{B}$ of G, an allocation mechanism can be explored with the help of the maximum flow on a constructed network, which is named as the *BD Allocation Mechanism* for convenience.

Definition 5 (BD Allocation Mechanism)**.** *Given the bottleneck decomposition $\mathcal{B}$, an allocation X (named as* BD allocation*) is determined by distinguishing three cases.*

- *For (B_i, C_i) with $\alpha_i < 1$, consider the bipartite graph $\widehat{G} = (B_i, C_i; E_i)$ with $E_i = B_i \times C_i$. Construct a network $N = (V_N, E_N)$ with $V_N = \{s, t\} \cup B_i \cup C_i$ and directed edges (s, u) with capacity w_u for each $u \in B_i$, directed edges (v, t) with capacity w_v/α_i for each $v \in C_i$ and directed edges (u, v) with capacity $+\infty$ for each $(u, v) \in E_i$. The max-flow min-cut theorem ensures a maximal flow $\{f_{uv}\}$, $u \in B_i$ and $v \in C_i$, such that $\sum_{v \in \Gamma(u) \cap C_i} f_{uv} = w_u$ and $\sum_{u \in \Gamma(v) \cap B_i} f_{uv} = w_v/\alpha_i$. Let the allocation be $x_{uv} = f_{uv}$ and $x_{vu} = \alpha_i f_{uv}$ implying $\sum_{u \in \Gamma(v) \cap B_i} x_{vu} = \sum_{u \in \Gamma(v) \cap B_i} \alpha_i \cdot f_{vu} = w_v$.*
- *For the case that $\alpha_k = 1$ (i.e., $B_k = C_k$), construct a bipartite graph $\widehat{G} = (B_k, B'_k; E'_k)$ where B'_k is a copy of B_k, there is an edge $(u, v') \in E'_k$ if and only if $(u, v) \in E[B_k]$. Construct a network by the above method. For any edge $(u, v') \in E'_k$, there exists a flow $f_{uv'}$ such that $\sum_{v' \in \Gamma(u) \cap B'_k} f_{uv'} = w_u$. Let the allocation be $x_{uv} = f_{uv'}$.*
- *For any other edge $(u, v) \notin B_i \times C_i$, $i = 1, 2, \cdots, k$, define $x_{uv} = 0$.*

Proposition 6 ([15])**.** *The proportional response dynamics (1) converges to the BD allocation X, in which each agent v's utility from X is*

$$U_v = \begin{cases} w_v \cdot \alpha_i, & v \in B_i; \\ w_v/\alpha_i, & v \in C_i. \end{cases} \tag{2}$$

For the sake of convenience, we use α_v to denote the α-ratio of v, and define $\alpha_v = \alpha_i$ if $v \in B_i \cup C_i$.

D. Resource Sharing Game

From a system design point of view, a problem occurs that an agent may manipulate BD Allocation Mechanism to change the resulting allocation at the execution level. In particular, can agents make manipulative moves for gains in their utilities? A series of works [3]–[7], [9] studied this problem with incentive consideration, which is called the *resource sharing game*. Specifically, this work focuses on a kind of manipulative move, called the *Sybil attack*. An attacker can play a Sybil attack by creating more than one identity to use them for potentially more utility.

In this work, we study the impact of the Sybil attack on the BD Allocation Mechanism over a ring G. Sybil attack is formally modeled as follows: given the network G and weight profile $\mathbf{w}$, the manipulative agent v splits itself into m nodes $v^1, \cdots, v^m$, $1 \le m \le d_v$ (d_v is the degree of v) and assigns amount w_{v^i} of resource to each node v^i, satisfying $0 \le w_{v^i} \le w_v$ and $\sum_{i=1}^m w_{v^i} = w_v$. To disguise the identity of the fictitious nodes, the manipulative agent v imposes each of its neighbors to be connected with one of the fictitious nodes, but not vice versa. Let G' be the resulting network, and v's new utility, denoted by $U'_v(G'; w_{v^1}, \cdots, w_{v^m}, \mathbf{w}_{-v})$, is just defined as the sum of utilities from all copied nodes in G': $\sum_{i=1}^m U'_{v^i}(G'; w_{v^1}, \cdots, w_{v^m}, \mathbf{w}_{-v})$.

As the BD Allocation Mechanism is not truthful with respect to the Sybil attack [3]–[5], [9], we utilize the concept of *incentive ratio* [2] to characterize the rate of agent's improvement in its utility value by such an act.

Definition 7 (Incentive ratio)**.** *In a resource sharing game, given a network $G = (V, E)$ and a weight profile $\mathbf{w} = (w_1, \ldots, w_n)$, the incentive ratio of agent v under BD Allocation Mechanism against Sybil attack is*

$$\zeta_v = \max_{\substack{2 \le m \le d_v \\ w_{v^i} \in [0, w_v], i = 1, 2, \ldots, m \\ \sum_{i=1}^m w_{v^i} = w_v; G'}} \frac{U'_v(G'; w_{v^1}, \ldots, w_{v^m}, \mathbf{w}_{-v})}{U_v(G; \mathbf{w})}$$

The incentive ratio of BD Allocation Mechanism in a resource sharing game against Sybil attack is

$$\zeta = \max_{\substack{G=(V,E), v\in V \\ \mathbf{w}=(w_1,\ldots,w_n)}} \zeta_v$$

III. Tight Incentive Ratio of Two

In this section, we complete the study on the incentive ratio of BD Allocation Mechanism on rings to a tight bound of two.

Theorem 8. *If the network of the resource sharing system is a ring, then the incentive ratio of BD Allocation Mechanism against a Sybil attack is exactly two.*

A. Analysis and Discussion

Before providing the details of discussion, let us introduce some necessary notations, similar to those in [9], and some useful propositions. A manipulative agent v splits itself into two nodes v^1 and v^2, along with weight assignment w_{v^1} and w_{v^2}, to derive utilities from v^1 and v^2 on the resulting path $P_v(w_{v^1}, w_{v^2})$. For convenience, we use $U_{v^i}(w_{v^1}, w_{v^2})$, $\mathcal{B}(w_{v^1}, w_{v^2})$ and $\alpha_{v^i}(w_{v^1}, w_{v^2})$ to denote the utility of v^i, $i = 1, 2$, the bottleneck decomposition and the α-ratio of v^i, $i = 1, 2$, on path $P_v(w_{v^1}, w_{v^2})$, respectively.

Of course, the manipulative agent would like to maximize its utility by assigning its weight properly. Let w_1^* and w_2^* be the best decision to achieve the optimum, i.e.,

$$U_v(w_1^*, w_2^*) = \max_{\substack{0\le w_{v^i}\le w_v; \\ w_{v^1}+w_{v^2}=w_v}} U_v(w_{v^1}, w_{v^2}).$$

Thus, $P_v(w_1^*, w_2^*)$ is the resulting path when v plays the optimal strategy. Technically, the optimal weight assignment may not be unique. In this case, we arbitrarily pick one of them to achieve the maximal utility.

We are more concerned with a special path $P_v(w_1^0, w_2^0)$, in which w_1^0 and w_2^0 are the same amounts of the resource that v allocates to its two neighbors on the original ring under the BD Allocation Mechanism. Since the connections and the weights of other vertices remain unchanged, and v^1 and v^2 are leaves of $P_v(w_1^0, w_2^0)$, it is not hard to see the two neighbors of v^1 and v^2 receive the same amount of resource from v^1 and v^2 respectively on $P_v(w_1^0, w_2^0)$, and also return the same to v^1 and v^2 as before. So

Lemma 9. *Let w_1^0 and w_2^0 be the same amounts of the resource that manipulative agent v allocates to its two neighbors on the original ring under the BD Allocation Mechanism. Then $U_v(w_1^0, w_2^0) = U_{v^1}(w_1^0, w_2^0) + U_{v^2}(w_1^0, w_2^0) = U_v$.*

To compare the difference of U_v and $U_v(w_1^*, w_2^*)$, we shall study on the changes of utilities from the initial path $P_v(w_1^0, w_2^0)$ to the ultimate path $P_v(w_1^*, w_2^*)$. W.l.o.g., assume that $w_1^* > w_1^0$ and $w_2^* < w_2^0$. However, it is quite a challenge to compute the difference when the weights of v^1 and v^2 change simultaneously. We decompose the whole process from $P_v(w_1^0, w_2^0)$ to $P_v(w_1^*, w_2^*)$ into two stages according to whether v is in B class or in C class on original ring G. At each stage, only the weight of one of $\{v^1, v^2\}$ changes, and other vertices' weights are fixed.

Recently, Cheng et al. [7] studied the issue of agent deviation from the BD Allocation Mechanism by cheating on the resource amount it owns. In this issue, the weight of agent v is its private information and its reported resource amount is denoted by $x \in [0, w_v]$. Given other agents' weights, the utility and the α-ratio of v, and the bottleneck decomposition of the resulting network would change in accordance with the single parameter x. We can utilize the results in [7] to explore useful properties.

B. Changes of the utility, the α-ratio and the bottleneck decomposition with reported weight x

On network G in which other agents' weights are fixed except for agent v, we denote the utility and the α-ratio of v, and the bottleneck decomposition of the resulting network, by $U_v(x)$, $\alpha_v(x)$ and $\mathcal{B}(x)$, respectively, when v reports its resource amount $x \in [0, w_v]$. Though $\mathcal{B}(x)$ changes with the weight variable x, it is observed that $\mathcal{B}(x)$ could be the same when x falls in a segment of the same interval. Based on such an observation, we partition the interval $[0, w_v]$ into a number of disjoint subintervals $\{\langle a_i, b_i\rangle\}_i$. Here $\langle a_i, b_i\rangle$ represents a subinterval, which could be one of five forms $[a_i, b_i]$, $[a_i, b_i)$, $(a_i, b_i]$, (a_i, b_i) and $a_i = b_i$. W.l.o.g., assume $0 < a_i \le b_i = a_{i+1} \le b_{i+1}$. If $a_i = b_i$, then $\langle a_i, b_i\rangle$ contains exactly one point which can be viewed as a special closed interval. Therefore, a series of bottleneck decompositions $\{\mathcal{B}^i\}_i$ are constructed such that when $x \in \langle a_i, b_i\rangle$ is in the interior of each of such intervals, $\mathcal{B}(x) = \mathcal{B}^i = \{(B_1^i, C_1^i), \cdots, (B_{k^i}^i, C_{k^i}^i)\}$.

Theorem 10 ([7]). *In a resource sharing game, if an agent plays the misreporting strategy, then its utility $U_v(x)$ is continuous and monotonically non-decreasing on any of its reported weight $x \in [0, w_v]$.*

The changes of $\alpha_v(x)$ and the bottleneck pairs, containing the manipulative agent v, with respect to x follow these propositions.

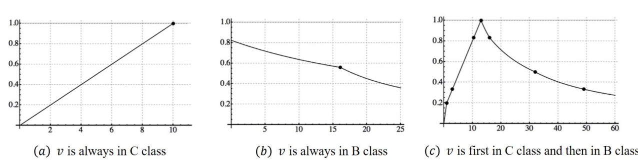

Figure 2. The changes of $\alpha_v(x)$.

Proposition 11 ([7]). *For any agent v and any other agents' weights, let $\alpha_v(x)$ be v's α-ratio function of the weight variable $x \in [0, w_v]$. Then, there exist three cases:*

- *Case B-1. $\alpha_v(x)$ is non-decreasing and v is in C class for all $x \in [0, w_v]$, shown in Fig. 2-(a);*
- *Case B-2. $\alpha_v(x)$ is non-increasing and v is in B class for all $x \in [0, w_v]$, shown in Fig.2-(b);*
- *Case B-3. there is a number $x^* \in (0, w_v]$ with $\alpha_v(x^*) = 1$, shown in Fig.2-(c), such that*
  - *$\alpha_v(x)$ is non-decreasing and v is in C class if $0 \le x \le x^*$;*
  - *$\alpha_v(x)$ is non-increasing and v is in B class if $x^* \le x \le w_v$.*

Given an interval $[a, b]$. Case B-1 in Proposition 11 indicates that if v is in C class when $x = b$, then v is always in C class when $x \in [a, b]$. Symmetrically, Case B-2 in Proposition 11 implies that if v is in B class when $x = a$, then v is always in B class when $x \in [a, b]$.

We mainly focus on the bottleneck pairs that v belongs to, in two adjacent bottleneck decompositions $\mathcal{B}^i$ and $\mathcal{B}^{i+1}$.

Proposition 12 ([7]). *Suppose v is in bottleneck pairs (B^i_j, C^i_j) and (B^{i+1}_l, C^{i+1}_l) in two adjacent bottleneck decompositions $\mathcal{B}^i$ and $\mathcal{B}^{i+1}$. Then,*

1) *$v \in B^i_j \cap B^{i+1}_l$ or $v \in C^i_j \cap C^{i+1}_l$;*
2) *if $v \in C^i_j \cap C^{i+1}_l$, then*
   a) *$v \in C^i_j \cap C^{i+1}_{j+1}$ ($l = j + 1$), $B^i_j = B^{i+1}_j \cup B^{i+1}_{j+1}$, $C^i_j = C^{i+1}_j \cup C^{i+1}_{j+1}$ and $\alpha^i_j(b_i) = \alpha^{i+1}_j(b_i) = \alpha^{i+1}_{j+1}(b_i)$, shown in Fig.3-(a), or*
   b) *$v \in C^i_j \cap C^{i+1}_j$ ($l = j$), $B^i_j \cup B^i_{j+1} = B^{i+1}_j$, $C^i_j \cup C^i_{j+1} = C^{i+1}_j$ and $\alpha^{i+1}_j(b_i) = \alpha^i_j(b_i) = \alpha^i_{j+1}(b_i)$, shown in Fig.3-(b);*
3) *if $v \in B^i_j \cap B^{i+1}_l$, then*
   a) *$v \in B^i_l \cap B^{i+1}_l$ ($j = l$), $B^i_l = B^{i+1}_l \cup B^{i+1}_{l+1}$, $C^i_l = C^{i+1}_l \cup C^{i+1}_{l+1}$ and $\alpha^i_l(b_i) = \alpha^{i+1}_l(b_i) = \alpha^{i+1}_{l+1}(b_i)$ or*
   b) *$v \in B^i_{l+1} \cap B^{i+1}_l$ ($j = l + 1$), $B^i_l \cup B^i_{l+1} = B^{i+1}_l$, $C^i_l \cup C^i_{l+1} = C^{i+1}_l$ and $\alpha^{i+1}_l(b_i) = \alpha^i_l(b_i) = \alpha^i_{l+1}(b_i)$.*

Proposition 12-(1) states that the manipulative agent v must be in the same class in adjacent two bottleneck decompositions. Moreover, if $v \in C^i_j \cap C^{i+1}_l$ and its reported weight x decreases from interval $\langle a_{i+1}, b_{i+1}\rangle$ to interval $\langle a_i, b_i\rangle$, then the pair containing it may combine another pair with a smaller α-ratio to form a new one or may be decomposed into two pairs, one containing v and the other has a larger index (implying it has a larger α-ratio). On the other hand, if x increases from interval $\langle a_i, b_i\rangle$ to interval $\langle a_{i+1}, b_{i+1}\rangle$, then the pair containing it may combine another pair with a larger α-ratio to form a new one or may be decomposed into two pairs, one containing v and the other has a smaller index (implying it has a smaller α-ratio). Fig.3 illustrates Proposition 12-(2) well.

Symmetrically, when $v \in B^i_j \cap B^{i+1}_l$ and x increases or decreases, the opposite situations happen. From Proposition 12, we can see if the class that v is in keeps the same when the weight variable x changes in some interval, then the classes that all other agents belong to also remain unchanged.

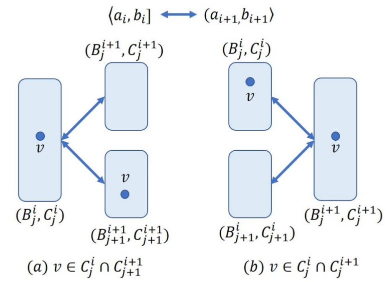

Figure 3. The changes of bottleneck pairs when v is in C class.

Lemma 13. *Suppose an agent v reports weight $x \in [0, w_v]$, and other agents' weights are fixed. Given an interval $[a, b] \subseteq [0, w_v]$, let $\mathcal{B}(a)$ and $\mathcal{B}(b)$ be the bottleneck decompositions of G when $x = a$ and b, respectively. If v is always in C class (or B class) when $x \in [a, b]$, then all other B class and C class vertices are always in B class and C class respectively. Furthermore,*

- *If v is a C class vertex in $[a, b]$, then*
  - *the bottleneck pairs in $\mathcal{B}(a)$, with α-ratio $< \alpha_v(a)$, are not impacted, when x increases from a to b,*
  - *the bottleneck pairs in $\mathcal{B}(b)$, with α-ratio $> \alpha_v(b)$, are not impacted, when x decreases from b to a.*
- *If v is a B class vertex in $[a, b]$, then*
  - *the bottleneck pairs in $\mathcal{B}(a)$, with α-ratio $> \alpha_v(a)$, are not impacted, when x increases from a to b,*
  - *the bottleneck pairs in $\mathcal{B}(b)$, with α-ratio $< \alpha_v(b)$, are not impacted, when x decreases from b to a.*

C. Proof for Theorem 8 when manipulative agent v is a C class vertex on original ring G

In this subsection, we will prove the tight incentive ratio of two, if v is a C class vertex on original ring G. W.l.o.g., assume that v is a C class vertex if $\alpha_v = 1$ on original ring G. Thus the process from $P_v(w^0_1, w^0_2)$ to $P_v(w^*_1, w^*_2)$ is decomposed into two stages as:

- Stage C-1: decrease v^2's weight from w^0_2 to w^*_2 and fix vertex v^1's weight as w^0_1;
- Stage C-2: increase v^1's weight from w^0_1 to w^*_1 and fix vertex v^2's weight as w^*_2.

We have

$$\begin{aligned}
& U_v(w^*_1, w^*_2) - U_v \\
= \;& \big(U_v(w^0_1, w^*_2) - U_v(w^0_1, w^0_2)\big) + \big(U_v(w^*_1, w^*_2) - U_v(w^0_1, w^*_2)\big) \\
= \;& \sum_{i=1}^{2} \big(U_{v^i}(w^0_1, w^*_2) - U_{v^i}(w^0_1, w^0_2)\big) + \sum_{i=1}^{2} \big(U_{v^i}(w^*_1, w^*_2) - U_{v^i}(w^0_1, w^*_2)\big) \\
= \;& \left[\delta^{(1)}_{v^1} + \delta^{(1)}_{v^2}\right] + \left[\delta^{(2)}_{v^1} + \delta^{(2)}_{v^2}\right],
\end{aligned}$$

where $\delta_{v^1}^{(1)}$ (or $\delta_{v^2}^{(1)}$) denotes the difference of v^1's (or v^2's) utility at Stage C-1, and $\delta_{v^1}^{(2)}$ (or $\delta_{v^2}^{(2)}$) denotes the difference of v^1's (or v^2's) utility at Stage C-2.

The proof of Theorem 8, when v is a C class vertex on original ring, includes three parts: to prove $\delta_{v^1}^{(1)} \leq 0$, $\delta_{v^2}^{(1)} \leq 0$ at Stage C-1 (Lemma 16); $\delta_{v^1}^{(2)} \leq U_v$, $\delta_{v^2}^{(2)} \leq 0$ at Stage C-2, if v^1 is a C class vertex on $P_v(w_1^*, w_2^*)$ (Lemma 18); and to prove $U_v(w_1^*, w_2^*) \leq 2U_v$ directly if v^1 is a B class vertex on $P_v(w_1^*, w_2^*)$ (Lemma 19).

Lemma 14. *If v is a C class vertex on original ring G, then the bottleneck decomposition $\mathcal{B}(w_1^0, w_2^0)$ of $P_v(w_1^0, w_2^0)$ have one of the following forms:*

- *Case C-1. $\mathcal{B}(w_1^0, w_2^0) = \{(B_1(w_1^0, w_2^0), C_1(w_1^0, w_2^0))\}$ with $v^1 \in B_1(w_1^0, w_2^0)$, $v^2 \in C_1(w_1^0, w_2^0)$ and $\alpha_1(w_1^0, w_2^0) = \alpha_v$, there are even number of vertices on $P_v(w_1^0, w_2^0)$, and the B class and C class vertices appear alternatively, shown in Fig. 4-(a);*
- *Case C-2. $v^1 \in B_j(w_1^0, w_2^0)$ with $w_1^0 = 0$ and $v^2 \in C_i(w_1^0, w_2^0)$ with $w_2^0 = w_v$ in $\mathcal{B}(w_1^0, w_2^0)$, shown in Fig. 4-(b);*
- *Case C-3. $v^1 \in C_j(w_1^0, w_2^0)$ with $w_1^0 \geq 0$, $v^2 \in C_i(w_1^0, w_2^0)$ with $w_2^0 \geq 0$ in $\mathcal{B}(w_1^0, w_2^0)$, and $j \geq i$ along with $\alpha_j(w_1^0, w_2^0) \geq \alpha_i(w_1^0, w_2^0) = \alpha_v$, shown in Fig. 4-(c).*

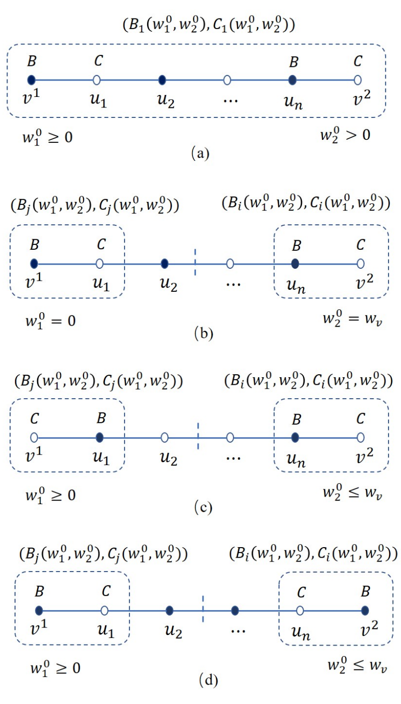

Figure 4. The possible forms of $\mathcal{B}(w_1^0, w_2^0)$.

Given the bottleneck decomposition $\mathcal{B}$ of a graph G, all vertices are categorized in the B-class or the C-class but not both if $\alpha_i < 1$. We must deal with the case of $\alpha_v = 1$ more carefully.

If the original ring G is an odd ring and only has one bottleneck pair (B_1, C_1) with $\alpha_1 = 1$, then $\mathcal{B}(w_1^0, w_2^0)$ has the form in Case C-1. Other cases with $\alpha_v = 1$ may be G is an even ring and only has one bottleneck pair (B_1, C_1) with $\alpha_1 = 1$, or the $\mathcal{B}$ of G has more than one bottleneck pair, the last pair $B_k = C_k$ contains v and the induced subgraph of B_k is a path. Then we let v be in C class, and the neighbor(s) in induced subgraph be in B class, and so on, such that the vertices on induced subgraph are B and C class alternately. If v is a C class vertex on G, and its two neighbors u_1 and u_n are in C class and B class, respectively, then there is no resource exchanging between v and u_1, and thus $\mathcal{B}(w_1^0, w_2^0)$ has the form in Case C-2. If v is in C class on G and its two neighbors u_1 and u_n are both in B class, then $\mathcal{B}(w_1^0, w_2^0)$ has the form in Case C-3. The detailed proof for Lemma 14 is provided in the full version.

The analysis for Case C-3 is a little more complicated. If $\mathcal{B}(w_1^0, w_2^0)$ has the form in Case C-3, it is possible that v^1 and v^2 are both in $C_j(w_1^0, w_2^0)$, indicating $\alpha_{v^1}(w_1^0, w_2^0) = \alpha_{v^2}(w_1^0, w_2^0) = \alpha_v$, and $\sum_{i=1}^2 U_{v^i}(w_1^0, w_2^0) = \sum_{i=1}^2 \frac{w_i^0}{\alpha_j(w_1^0, w_2^0)} = \sum_{i=1}^2 \frac{w_i^0}{\alpha_v} = U_v$. Let us change the v^1 and v^2's weights simultaneously. If path $P_v(w_{v^1}, w_{v^2})$ has the same bottleneck decomposition as the one of $P_v(w_1^0, w_2^0)$, during the whole process that w_{v^1} increases from w_1^0 to w_1^* and w_{v^2} decreases from w_2^0 to w_2^*, then $(B_j(w_1^0, w_2^0), C_j(w_1^0, w_2^0))$ is not impacted, and then

$$(B_j(w_1^*, w_2^*), C_j(w_1^*, w_2^*)) = (B_j(w_1^0, w_2^0), C_j(w_1^0, w_2^0)).$$

So $\alpha_j(w_1^*, w_2^*) = \alpha_j(w_1^0, w_2^0) = \alpha_v$, and $U_v(w_1^*, w_2^*) = U_{v^1}(w_1^*, w_2^*) + U_{v^2}(w_1^*, w_2^*) = \frac{w_1^* + w_2^*}{\alpha_v} = \frac{w_1^0 + w_2^0}{\alpha_v} = U_{v^1}(w_1^0, w_2^0) + U_{v^2}(w_1^0, w_2^0) = U_v$. If such a situation happens, then agent v cannot improve its utility by Sybil attack directly.

For Case C-3, let us continue to consider the situation that v^1 and v^2 are both in $C_j(w_1^0, w_2^0)$ and there is a critical point $z \in [0, w_2^0 - w_2^*)$, such that on $P_v(w_1^0 + z, w_2^0 - z)$,

$$\mathcal{B}(w_1^0 + z, w_2^0 - z) = \mathcal{B}(w_1^0, w_2^0),$$

implying $B_j(w_1^0 + z, w_2^0 - z) = B_j(w_1^0, w_2^0)$ and $C_j(w_1^0 + z, w_2^0 - z) = C_j(w_1^0, w_2^0)$, but on $P_v(w_1^0 + (z+\epsilon), w_2^0 - (z+\epsilon))$ for any sufficient $\epsilon > 0$, $(B_j(w_1^0 + z, w_2^0 - z), C_j(w_1^0 + z, w_2^0 - z))$ is decomposed into two pairs, one containing v^1 and the other containing v^2. At this critical point z, v^1 and v^2 are both in $C_j(w_1^0 + z, w_2^0 - z)$, and

$$\begin{aligned} & \alpha_j(w_1^0 + z, w_2^0 - z) \\ = \quad & \frac{w(C_j(w_1^0 + z, w_2^0 - z) - \{v^1, v^2\}) + (w_1^0 + z) + (w_2^0 - z)}{w(B_j(w_1^0 + z, w_2^0 - z))} \\ = \quad & \frac{w(C_j(w_1^0, w_2^0))}{w(B_j(w_1^0, w_2^0))} = \alpha_j(w_1^0, w_2^0). \end{aligned}$$

So on $P_v(w_1^0+z, w_2^0-z)$,

$$\begin{aligned}&\sum_{i=1}^{2} U_{v^i}(w_1^0+z, w_2^0-z)\\ =\ & \frac{w_1^0+z}{\alpha_j(w_1^0+z, w_2^0-z)}+\frac{w_2^0-z}{\alpha_j(w_1^0+z, w_2^0-z)}\\ =\ & \frac{w_1^0+w_2^0}{\alpha_j(w_1^0, w_2^0)}=U_v.\end{aligned}$$

Based on this observation, we use $P_v(w_1^0+z, w_2^0-z)$ to replace $P_v(w_1^0, w_2^0)$, as the initial path. Let $w_1^0 = w_1^0+z$ and $w_2^0 = w_2^0-z$. For convenience, we name such a technique as *Adjusting Technique*.

When w_{v^2} decreases from w_2^0 to w_2^*, we are also concerned with the changes of the bottleneck pair that v^1 is in. Our purpose is to prove $\delta_{v^1}^{(1)} \le 0$ by showing the bottleneck pair that v^1 is in is not impacted at Stage C-1. We observe that v^2 is in C class on initial path $P_v(w_1^0, w_2^0)$ and its weight decreases from w_2^0 to w_2^*, indicating v^2 is always in C class when $w_{v^2} \in [w_2^*, w_2^0]$ by Proposition 11. Such an observation inspires us to prove the bottleneck pair that v^1 is in is not impacted by Lemma 13. However, there is a difficulty that v^1 and v^2 are both in $(B_j(w_1^0, w_2^0), C_j(w_1^0, w_2^0))$ on $P_v(w_1^0, w_2^0)$ (after Adjusting Technique if necessary) with $\alpha_{v^1}(w_1^0, w_2^0) = \alpha_{v^2}(w_1^0, w_2^0)$. It leads to the changes of $(B_j(w_1^0, w_2^0), C_j(w_1^0, w_2^0))$ that v^1 is in with the decreasing of w_{v^2}. To overcome it, we make a skill to consider path $P_v(w_1^0, w_2^0-\epsilon)$ for any sufficient $\epsilon > 0$, on which $(B_j(w_1^0, w_2^0), C_j(w_1^0, w_2^0))$, that v^1 and v^2 are both in, splits into two pairs, one containing v^1 and the other containing v^2 with $\alpha_{v^1}(w_1^0, w_2^0) = \alpha_{v^1}(w_1^0, w_2^0-\epsilon) > \alpha_{v^2}(w_1^0, w_2^0-\epsilon)$. Thus we have the following lemma.

Lemma 15. *For Case C-3, if v^1 and v^2 are in the same pair on $P_v(w_1^0, w_2^0)$, then on $P_v(w_1^0, w_2^0-\epsilon)$ for any sufficient $\epsilon > 0$, such a pair splits into two bottleneck pairs with $\alpha_{v^2}(w_1^0, w_2^0-\epsilon) < \alpha_{v^1}(w_1^0, w_2^0-\epsilon) = \alpha_{v^1}(w_1^0, w_2^0)$.*

With the help of Lemma 15, the following lemma is derived.

Lemma 16. *If v is a C class vertex on original ring G, then $\delta_{v^1}^{(1)} \le 0$ and $\delta_{v^2}^{(1)} \le 0$.*

Proof: At Stage C-1, v^2's weight w_{v^2} decreases from w_2^0 to w_2^*, and other vertices' weights are fixed. By Theorem 10, we have $\delta_{v^2}^{(1)} \le 0$. Now let us analyze $\delta_{v^1}^{(1)}$ based on the three cases in Lemma 14. As v^2 is in C class in all cases on initial path $P_v(w_1^0, w_2^0)$, and its weight decreases from w_2^0 to w_2^*, it must be always in C class and its α-ratio non-increases along with w_{v^2} decreasing by Proposition 11. Thus the classes of all other vertices remain unchanged at Stage C-1 by Lemma 13.

For Case C-1, $\mathcal{B}(w_1^0, w_2^0)$ only has one bottleneck pair $\{(B_1(w_1^0, w_2^0), C_1(w_1^0, w_2^0))\}$,

$$\alpha_1(w_1^0, w_2^0) = \alpha_{v^2}(w_1^0, w_2^0) = \alpha_{v^1}(w_1^0, w_2^0),$$

and v^1 is always in B class at Stage C-1. There exits a break point $y \in [w_2^*, w_2^0]$, such that when $w_{v^2} = y$,

$$(B_1(w_1^0, y), C_1(w_1^0, y)) = (B_1(w_1^0, w_2^0), C_1(w_1^0, w_2^0)),$$

and when $w_{v^2} = y-\epsilon$, $(B_1(w_1^0, y-\epsilon), C_1(w_1^0, y-\epsilon))$ is decomposed into two pairs, one containing v^2 and the other containing v^1. In addition, at this break point y, the pair that v^1 is in has α-ratio $= \alpha_{v^2}(w_1^0, y) > \alpha_{v^2}(w_1^0, y-\epsilon)$, and then this pair is not impacted when w_{v^2} decreases from y to w_2^* by Lemma 13. Therefore,

$$\begin{aligned}&U_{v^1}(w_1^0, w_2^*) = w_1^0 \cdot \alpha_{v^2}(w_1^0, y) \le w_1^0 \cdot \alpha_{v^2}(w_1^0, w_2^0)\\ =\ & w_1^0 \cdot \alpha_{v^1}(w_1^0, w_2^0) = U_{v^1}(w_1^0, w_2^0).\end{aligned}$$

It implies $\delta_{v^1}^{(1)} \le 0$.

For Case C-2, since $w_{v^1} = 0$, we have $U_{v^1}(w_1^0, w_2^*) = U_{v^1}(w_1^0, w_2^0) = 0$, showing $\delta_{v^1}^{(1)} = 0$.

For Case C-3, if v^1 and v^2 are in different bottleneck pairs on $P_v(w_1^0, w_2^0)$, we have $\alpha_{v^2}(w_1^0, w_2^0) < \alpha_{v^1}(w_1^0, w_2^0)$ by Lemma 14. Since v^2's α-ratio non-increases at Stage C-1, the bottleneck of v^1 is not impacted by Lemma 13, which means $\delta_{v^1}^{(1)} = 0$. If v^1 and v^2 are in the same bottleneck pair on $P_v(w_1^0, w_2^0)$, we let v^2 decrease a sufficient $\epsilon > 0$ to make sure v^1 and v^2 are not in the same pair and $\alpha_{v^2}(w_1^0, w_2^0 - \epsilon) < \alpha_{v^1}(w_1^0, w_2^0-\epsilon) = \alpha_{v^1}(w_1^0, w_2^0)$ by Lemma 15. Since v^2's α-ratio non-increases at Stage C-1, the bottleneck of v^1 is not impacted by Lemma 13. Hence,

$$\begin{aligned}&U_{v^1}(w_1^0, w_2^*) = U_{v^1}(w_1^0, w_2^0-\epsilon)\\ =\ & \frac{w_1^0}{\alpha_{v^1}(w_1^0, w_2^0-\epsilon)} = \frac{w_1^0}{\alpha_{v^1}(w_1^0, w_2^0)} = U_{v^1}(w_1^0, w_2^0),\end{aligned}$$

implying $\delta_{v^1}^{(1)} = 0$. ■

From Lemma 16, we can obtain the following corollary.

Corollary 17. *If $\mathcal{B}(w_1^0, w_2^0)$ has the form in Case C-3, then $\alpha_{v^1}(w_1^0, w_2^*) > \alpha_{v^2}(w_1^0, w_2^*)$ at the end of Stage C-1, and v^1 and v^2 are not in the same pair on $P_v(w_1^0, w_2^*)$.*

Lemma 18. *If v is a C class vertex on original ring G and v^1 is also in C class on ultimate path $P_v(w_1^*, w_2^*)$, then $\delta_{v^1}^{(2)} \le U_v$ and $\delta_{v^2}^{(2)} = 0$.*

Proof: At Stage C-2, v^1's weight w_{v^1} increases from w_1^0 to w_1^* and other vertices' weights are fixed. By the condition that v^1 is a C class vertex on the ultimate path $P_v(w_1^*, w_2^*)$, we can see v^1 is always in C class at Stage C-2, implying v^1 is a C class vertex on $P_v(w_1^0, w_2^*)$. By the proof of Lemma 16, the classes of all vertices remain unchanged at Stage C-1, indicating v^1 is always in C class at Stage C-1. However, v^1 is a B class vertex in Case C-1 and Case C-2 in Lemma 14. Such a contradiction shows Case C-1 and Case C-2 in Lemma 14 cannot happen.

For Case C-3, we know v^1 and v^2 are in the different pairs on $P_v(w_1^0, w_2^*)$ and $\alpha_{v^1}(w_1^0, w_2^*) > \alpha_{v^2}(w_1^0, w_2^*)$ by Corollary 17. Additionally combining the conditions that v^1 is in C class, w_{v^1} increases from w_1^0 to w_1^*, we confirm that the bottleneck pair that v^2 is in is not impacted at Stage C-2 by Lemma 13. It follows $U_{v^2}(w_1^*, w_2^*) = U_{v^2}(w_1^0, w_2^*)$, implying $\delta_{v^2}^{(2)} = 0$.

On the other hand, as w_{v^1} increases and other vertices' weights are fixed, $\alpha_{v^1}(w_1^*, w_2^*) \geq \alpha_{v^1}(w_1^0, w_2^*)$. So

$$\begin{aligned}\delta_{v^1}^{(2)} &= U_{v^1}(w_1^*, w_2^*) - U_{v^1}(w_1^0, w_2^*) \leq U_{v^1}(w_1^*, w_2^*) \\ &= \frac{w_1^*}{\alpha_{v^1}(w_1^*, w_2^*)} \leq \frac{w_1^*}{\alpha_{v^1}(w_1^0, w_2^*)} \\ &= \frac{w_1^*}{\alpha_{v^1}(w_1^0, w_2^0)} \leq \frac{w_v}{\alpha_v} = U_v.\end{aligned}$$

■

From above discussion, we have completely proved $U_v(w_1^*, w_2^*) \leq 2U_v$ under the conditions that v is a C class vertex on G and v^1 is a C class vertex on $P_v(w_1^*, w_2^*)$, by showing $\delta_{v^1}^{(1)} \leq 0$, $\delta_{v^2}^{(1)} \leq 0$ in Lemma 16 and $\delta_{v^1}^{(2)} \leq U_v$, $\delta_{v^2}^{(2)} = 0$ in Lemma 18. Next we continue to study on the situation that v^1 is a B class vertex on $P_v(w_1^*, w_2^*)$ and prove Theorem 8 directly.

Lemma 19. *If v is a C class vertex on the original ring G, and v^1 is a B class vertex on the ultimate path $P_v(w_1^*, w_2^*)$, then $U_v(w_1^*, w_2^*) \leq 2U_v$.*

Proof: During the process that w_{v^1} increases from w_1^0 to w_1^*, the bottleneck pairs containing v^1 and v^2 may combine or decompose, and the classes of vertices may change, as it is possible that v^1 changes to be a B class vertex, while it is in C class at the beginning of Stage C-2. Thus we partition $[w_1^0, w_1^*]$ into four disjoint subintervals: $[w_1^0, a\rangle$, $\langle a, b\rangle$, $\langle b, c\rangle$ and $\langle c, w_1^*]$ (some of them may be empty).

In $[w_1^0, a\rangle$, v^1 is in C class (indicating $\mathcal{B}(w_1^0, w_2^0)$ has the form of Case C-3), v^1 and v^2 are in different pairs with $\alpha_{v^1}(w_1^0, w_2^*) > \alpha_{v^2}(w_1^0, w_2^*)$ (by Corollary 17) and $\alpha_{v^1}(a, w_2^*) \geq \alpha_{v^1}(w_1^0, w_2^*)$ (by monotone property in Proposition 11). Furthermore, the pair containing v^2 is not impacted by Lemma 13, indicating $\alpha_{v^2}(a, w_2^*) = \alpha_{v^2}(w_1^0, w_2^*)$ and $U_{v^2}(a, w_2^*) = U_{v^2}(w_1^0, w_2^*)$. Since v^1 is in B class on $P_v(w_1^*, w_2^*)$, the final state cannot be reached in $[w_1^0, a\rangle$.

In $\langle a, b\rangle$, v^1 changes to be a B class vertex at critical point a, v^1 is in B class in this subinterval, and v^1 and v^2 are in different pairs. It follows the pair containing v^2 is still not impacted. So $\alpha_{v^2}(b, w_2^*) = \alpha_{v^2}(a, w_2^*) = \alpha_{v^2}(w_1^0, w_2^*)$, and $U_{v^2}(b, w_2^*) = U_{v^2}(w_1^0, w_2^*)$. It is possible that the final state may be reached in $\langle a, b\rangle$, as v^1 is a B class vertex in $\langle a, b\rangle$. If this situation happens, then $b = w_1^*$ and $U_{v^1}(w_1^*, w_2^*) \leq w_1^* \leq w_v \leq U_v$, in which the last inequality is from the condition that v is in C class on original ring. At the same time,

$$U_{v^2}(w_1^*, w_2^*) = U_{v^2}(w_1^0, w_2^*) = U_{v^2}(w_1^0, w_2^0) + \delta_{v^2}^{(1)} \leq U_v,$$

as $U_{v^2}(w_1^0, w_2^0) \leq U_v$ and $\delta_{v^2}^{(1)} \leq 0$. Hence,

$$U_v(w_1^*, w_2^*) = U_{v^1}(w_1^*, w_2^*) + U_{v^2}(w_1^*, w_2^*) \leq 2U_v.$$

In $\langle b, c\rangle$, v^1 is in B class, and the pairs containing v^1 and v^2, respectively, combine together at the break point b. Such a pair that contains v^1 and v^2 is denoted by $(\widehat{B}, \widehat{C})$, with $v^1 \in \widehat{B}$ and $v^2 \in \widehat{C}$. Additionally,

$$\frac{w(\widehat{C})}{w(\widehat{B})} = \alpha_{v^2}(b, w_2^*) = \alpha_{v^2}(w_1^0, w_2^*).$$

When w_{v^1} increases from b to c, $(\widehat{B}, \widehat{C})$ keeps the same. Thus

$$\begin{aligned}\delta_{v^2}^{(2)} &= w_2^* \left(\frac{1}{\alpha_{v^2}(c, w_2^*)} - \frac{1}{\alpha_{v^2}(b, w_2^*)}\right) \\ &= w_2^* \left(\frac{w(\widehat{B} - \{v^1\}) + c}{w(\widehat{C})} - \frac{w(\widehat{B} - \{v^1\}) + b}{w(\widehat{C})}\right) \\ &= w_2^* \frac{c - b}{w(\widehat{C})} \leq c - b \leq w_1^*, \qquad (3)\end{aligned}$$

where the first inequality is right since $v^2 \in \widehat{C}$ and thus $w_2^* \leq w(\widehat{C})$.

The final state may be reached in $\langle b, c\rangle$ or $\langle c, w_1^*]$. Since the proof of $U_v(w_1^*, w_2^*) \leq 2U_v$ for $\langle c, w_1^*]$ is an extension of the proof for $\langle b, c\rangle$, we would propose the proof for $\langle c, w_1^*]$ in detail.

In $\langle c, w_1^*]$, $(\widehat{B}, \widehat{C})$ is decomposed into two pairs at break point c, one containing v^1 and the other containing v^2. In this subinterval, v^1 is in B class, v^2 is in C class, and the pair v^2 is in remains unchanged. Therefore,

$$\begin{aligned}&\alpha_{v^1}(c, w_2^*) = \alpha_{v^2}(c, w_2^*) = \alpha_{v^2}(w_1^*, w_2^*) \\ &U_{v^2}(w_1^*, w_2^*) = \frac{w_2^*}{\alpha_{v^2}(w_1^*, w_2^*)} = \frac{w_2^*}{\alpha_{v^2}(c, w_2^*)}. \qquad (4)\end{aligned}$$

We have $U_{v^1}(c, w_2^*) \leq U_{v^1}(w_1^*, w_2^*)$ by the monotone property of utility function in Theorem 10. Since v^1 is in B class, we have $\alpha_{v^1}(c, w_2^*) \geq \alpha_{v^1}(w_1^*, w_2^*)$ by the monotone property of α-ratio function in Proposition 11. So,

$$U_{v^1}(c, w_2^*) \leq U_{v^1}(w_1^*, w_2^*) = w_1^* \alpha_{v^1}(w_1^*, w_2^*) \leq w_1^* \alpha_{v^1}(c, w_2^*). \qquad (5)$$

By (4) and (5), we have

$$\begin{aligned}U_v(w_1^*, w_2^*) &= \sum_{i=1}^{2} U_{v^i}(w_1^*, w_2^*) \leq w_1^* \alpha_{v^1}(c, w_2^*) + \frac{w_2^*}{\alpha_{v^2}(w_1^*, w_2^*)} \\ &= w_1^* \alpha_{v^1}(c, w_2^*) + \frac{w_2^*}{\alpha_{v^2}(c, w_2^*)} \qquad (6)\end{aligned}$$

Furthermore, $\frac{w_2^*}{\alpha_{v^2}(w_1^*, w_2^*)} = \frac{w_2^*}{\alpha_{v^2}(w_1^0, w_2^*)} + \delta_{v^2}^{(2)}$, and by (3)

$$\begin{aligned}U_v(w_1^*, w_2^*) &\leq w_1^* \alpha_{v^1}(c, w_2^*) + \frac{w_2^*}{\alpha_{v^2}(w_1^0, w_2^*)} + \delta_{v^2}^{(2)}. \\ &\leq w_1^* \alpha_{v^1}(c, w_2^*) + \frac{w_2^*}{\alpha_{v^2}(w_1^0, w_2^*)} + w_1^*. \qquad (7)\end{aligned}$$

If $w_1^* \alpha_{v^2}(c, w_2^*) > w_2^*$, then $w_2^*/\alpha_{v^2}(c, w_2^*) < w_1^*$, and from (6)

$$U_v(w_1^*, w_2^*) \leq w_1^* \alpha_{v^1}(c, w_2^*) + w_1^* \leq 2w_1^* \leq 2w_v \leq 2U_v.$$

If $w_1^* \alpha_{v^1}(c, w_2^*) = w_1^* \alpha_{v^2}(c, w_2^*) \leq w_2^*$, then from (7)

$$\begin{aligned}U_v(w_1^*, w_2^*) &\leq w_2^* + w_1^* + \frac{w_2^*}{\alpha_{v^2}(w_1^0, w_2^*)} \leq U_v + \frac{w_2^*}{\alpha_{v^2}(w_1^0, w_2^*)} \\ &= U_v + U_{v^2}(w_1^0, w_2^*) = U_v + U_{v^2}(w_1^0, w_2^0) + \delta_{v^2}^{(1)} \\ &\leq U_v + U_{v^2}(w_1^0, w_2^0) \leq 2U_v,\end{aligned}$$

where the penultimate inequality is right as $\delta^{(1)}_{v^2} \le 0$ in Lemma 16. ■

D. Proof for Theorem 8 when manipulative agent v is a B class vertex on original ring G

In this subsection, we will prove the tight incentive ratio of two, if v is a B class vertex on original ring G (whose α-ratio < 1). The process from $P_v(w_1^0, w_2^0)$ to $P_v(w_1^*, w_2^*)$ is decomposed into two stages as:

- Stage D-1: increase v^1's weight from w_1^0 to w_1^* and fix vertex v^2's weight as w_2^0;
- Stage D-2: decrease v^2's weight from w_2^0 to w_2^* and fix vertex v^1's weight as w_1^*.

Let us denote $\Delta^{(1)}_{v^i} = U_{v^i}(w_1^*, w_2^0) - U_{v^i}(w_1^0, w_2^0)$, and $\Delta^{(2)}_{v^i} = U_{v^i}(w_1^*, w_2^*) - U_{v^i}(w_1^*, w_2^0)$, $i = 1, 2$. Then

$$\begin{aligned} & U_v(w_1^*, w_2^*) - U_v \\ = \ & \big(U_v(w_1^*, w_2^0) - U_v(w_1^0, w_2^0)\big) + \big(U_v(w_1^*, w_2^*) - U_v(w_1^*, w_2^0)\big) \\ = \ & \Big[\Delta^{(1)}_{v^1} + \Delta^{(1)}_{v^2}\Big] + \Big[\Delta^{(2)}_{v^1} + \Delta^{(2)}_{v^2}\Big]. \end{aligned}$$

The proof of Theorem 8, when v is a B class vertex on original ring, includes two parts: to prove $\Delta^{(1)}_{v^1} \le U_v$, $\Delta^{(1)}_{v^2} \le 0$ at Stage D-1 (Lemma 22); and to prove $\Delta^{(2)}_{v^1} \le 0$, $\Delta^{(2)}_{v^2} \le 0$ at Stage D-2 (Lemma 24).

Lemma 20. *If v is a B class vertex on original ring G, then the bottleneck decomposition $\mathcal{B}(w_1^0, w_2^0)$ of $P_v(w_1^0, w_2^0)$ has the following form:*

- *Case D-1. $v^1 \in B_j(w_1^0, w_2^0)$ with $w_1^0 \ge 0$, $v^2 \in B_i(w_1^0, w_2^0)$ with $w_2^0 \le w_v$, and $j \le i$ implying $\alpha_j(w_1^0, w_2^0) \le \alpha_i(w_1^0, w_2^0) = \alpha_v$ (shown in Fig. 4-(d)).*

As v is a B class vertex on G, it must be adjacent to two C class vertices. Thus the bottleneck decomposition $\mathcal{B}(w_1^0, w_2^0)$ of $P_v(w_1^0, w_2^0)$ has the form in Case D-1, whose detailed proof is provided in the full version.

It is possible that v^1 and v^2 are in the same pair $(B_j(w_1^0, w_2^0), C_j(w_1^0, w_2^0))$. For this special case, we shall do the same operation as before to increase w_{v^1} from w_1^0 to $w_1^0 + z$ and to decrease w_{v^2} from w_2^0 to $w_2^0 - z$, $z \in [0, w_2^0 - w_2^*]$, simultaneously. If $\mathcal{B}(w_1^*, w_2^*) = \mathcal{B}(w_1^0, w_2^0)$ when $z = w_2^0 - w_2^* = w_1^* - w_1^0$, then v^1 and v^2 are also both in $(B_j(w_1^*, w_2^*), C_j(w_1^*, w_2^*))$, which is equal to $(B_j(w_1^0, w_2^0), C_j(w_1^0, w_2^0))$. Therefore, on path $P_v(w_1^*, w_2^*)$, $\alpha_j(w_1^*, w_2^*) = \alpha_j(w_1^0, w_2^0) = \alpha_v$, and

$$\begin{aligned} & U_v(w_1^*, w_2^*) = U_{v^1}(w_1^*, w_2^*) + U_{v^2}(w_1^*, w_2^*) = (w_1^* + w_2^*)\alpha_v \\ = \ & (w_1^0 + w_2^0)\alpha_v = U_{v^1}(w_1^0, w_2^0) + U_{v^2}(w_1^0, w_2^0) = U_v. \end{aligned}$$

When this situation happens, we conclude that agent v cannot improve its utility by Sybil attack directly.

In the following, for this special case, we only consider the situation that there exists a critical point $z \in [0, w_2^0 - w_2^*)$, such that $(B_j(w_1^0 + z, w_2^0 - z), C_j(w_1^0 + z, w_2^0 - z)) = (B_j(w_1^0, w_2^0), C_j(w_1^0, w_2^0))$, while on path $P_v(w_1^0 + (z + \epsilon), w_2^0 - (z + \epsilon))$, $(B_j(w_1^0 + z, w_2^0 - z), C_j(w_1^0 + z, w_2^0 - z))$ is decomposed into two pairs, one containing v^1 and the other containing v^2. Thus we use $P_v(w_1^0 + z, w_2^0 - z)$ to replace $P_v(w_1^0, w_2^0)$, and let $w_1^0 = w_1^0 + z$, $w_2^0 = w_2^0 - z$ by applying Adjusting Technique. Similar to Lemma 15, we have the following lemma.

Lemma 21. *For Case D-1, if v^1 and v^2 are in the same pair on $P_v(w_1^0, w_2^0)$, then on $P_v(w_1^0 + \epsilon, w_2^0)$ for any sufficient $\epsilon > 0$, such a pair splits into two bottleneck pairs with $\alpha_{v^1}(w_1^0 + \epsilon, w_2^0) < \alpha_{v^2}(w_1^0 + \epsilon, w_2^0) = \alpha_{v^2}(w_1^0, w_2^0)$.*

Lemma 22. *If v is a B class vertex on original ring G, then $\Delta^{(1)}_{v^1} \le U_v$ and $\Delta^{(1)}_{v^2} = 0$.*

Proof: At Stage D-1, v^1's weight w_{v^1} increases from w_1^0 to w_1^*, and other vertices' weights are fixed. Since v^1 is a B class vertex on $P_v(w_1^0, w_2^0)$, we can see v^1 is always in B class at Stage D-1 and v^1's α-ratio is non-increasing by Proposition 11. So we have $\alpha_{v^1}(w_1^*, w_2^0) \le \alpha_{v^1}(w_1^0, w_2^0)$,

$$\begin{aligned} & \Delta^{(1)}_{v^1} = U_{v^1}(w_1^*, w_2^0) - U_{v^1}(w_1^0, w_2^0) \le U_{v^1}(w_1^*, w_2^0) \\ = \ & w_1^*\alpha_{v^1}(w_1^*, w_2^0) \le w_1^*\alpha_{v^1}(w_1^0, w_2^0) \le w_v\alpha_v = U_v. \end{aligned}$$

If v^1 and v^2 are in different bottleneck pairs on $P_v(w_1^0, w_2^0)$, we have $\alpha_{v^2}(w_1^0, w_2^0) < \alpha_{v^1}(w_1^0, w_2^0)$ by Lemma 20. Since v^1's α-ratio non-increases when w_{v^1} increases from w_1^0 to w_1^* at Stage D-1, the bottleneck pair containing v^2 is not impacted by Lemma 13, indicating $\Delta^{(1)}_{v^2} = 0$.

If v^1 and v^2 are in the same bottleneck pair on $P_v(w_1^0, w_2^0)$, we can let v^1 increase a sufficient $\epsilon > 0$ to make sure v^1 and v^2 are not in the same pair and $\alpha_{v^1}(w_1^0 + \epsilon, w_2^0) < \alpha_{v^2}(w_1^0 + \epsilon, w_2^0) = \alpha_{v^2}(w_1^0, w_2^0)$ by Lemma 21. Since v^1's α-ratio non-increases during Stage D-1, the bottleneck of v^2 is not impacted by Lemma 13. Hence,

$$\begin{aligned} & U_{v^2}(w_1^*, w_2^0) = U_{v^2}(w_1^0 + \epsilon, w_2^0) \\ = \ & w_2^0\alpha_{v^2}(w_1^0 + \epsilon, w_2^0) = w_2^0\alpha_{v^2}(w_1^0, w_2^0) = U_{v^2}(w_1^0, w_2^0), \end{aligned}$$

implying $\Delta^{(1)}_{v^2} = 0$. ■

Corollary 23. *In Case D-1, $\alpha_{v^1}(w_1^*, w_2^0) < \alpha_{v^2}(w_1^*, w_2^0)$, and v^1 and v^2 are not in the same pair on $P_v(w_1^*, w_2^0)$.*

Proof: By the proof of Lemma 22, we know $\alpha_{v^1}(w_1^0 + \epsilon, w_2^0) < \alpha_{v^2}(w_1^0, w_2^0)$, for any sufficient $\epsilon > 0$, and the pair v^2 is in is not impacted when w_{v^1} increases from $w_1^0 + \epsilon$ to w_1^*. It follows v^1 and v^2 are not in the same bottleneck pair on $P_v(w_1^*, w_2^0)$ and by the monotone property of α_{v^1},

$$\alpha_{v^1}(w_1^*, w_2^0) \le \alpha_{v^1}(w_1^0 + \epsilon, w_2^0) < \alpha_{v^2}(w_1^0, w_2^0) = \alpha_{v^2}(w_1^*, w_2^0).$$

■

Lemma 24. *If v is a B class vertex on original ring G, then $\Delta^{(2)}_{v^1} \le 0$ and $\Delta^{(2)}_{v^2} \le 0$.*

Proof: At Stage D-2, v^2's weight w_{v^2} decreases from w_2^0 to w_2^*, and other vertices' weights are fixed. By Theorem 10, we have $\Delta^{(2)}_{v^2} \le 0$. We will discuss $\Delta^{(2)}_{v^1}$ separately, depending on whether v^2 is in B class or C class on the ultimate path $P_v(w_1^*, w_2^*)$.

If v^2 is in B class on the ultimate path $P_v(w_1^*, w_2^*)$, we can see v^2 is always in B class and its α-ratio is non-decreasing at Stage D-2 by Proposition 11. So $\alpha_{v^2}(w_1^*, w_2^*) \geq \alpha_{v^2}(w_1^*, w_2^0)$. Since v^1 and v^2 are in the different pairs on $P_v(w_1^*, w_2^0)$ and $\alpha_{v^1}(w_1^*, w_2^0) < \alpha_{v^2}(w_1^*, w_2^0)$ by Corollary 21, we confirm that the bottleneck pair that v^1 is in is not impacted at Stage D-2 by Lemma 13. It follows $U_{v^1}(w_1^*, w_2^*) = U_{v^1}(w_1^*, w_2^0)$, implying $\Delta_{v^1}^{(2)} = 0$.

If v^2 is in C class on the ultimate $P_v(w_1^*, w_2^*)$, then v^2's α-ratio is non-increasing after it changes to be a C class vertex. If v^1 and v^2 have never been in the same pair, meaning the pair v^1 is in is not impacted, then we have $\Delta_{v^1}^{(2)} = 0$. But during the process in which w_{v^2} decreases, there might be an interval, denoted by $\langle x, y \rangle$, such that the pairs, containing v^1 and v^2 respectively, combine together when $w_{v^2} \in \langle x, y \rangle$. Since v^1 and v^2 are in the same pair, we have $\alpha_{v^1}(w_1^*, w_{v^2}) = \alpha_{v^2}(w_1^*, w_{v^2})$, $w_{v^2} \in \langle x, y \rangle$. In addition, because v^2 is in C class and its weight w_{v^2} is decreasing, we have $\alpha_{v^2}(w_1^*, w_{v^2})$ non-increases, implying $\alpha_{v^2}(w_1^*, x) \geq \alpha_{v^2}(w_1^*, y)$. So

$$\begin{aligned}
\Delta_{v^1}^{(2)} &= U_{v^1}(w_1^*, y) - U_{v^1}(w_1^*, x) \\
&= (\alpha_{v^1}(w_1^*, y) - \alpha_{v^1}(w_1^*, x))w_1^* \\
&= (\alpha_{v^2}(w_1^*, y) - \alpha_{v^2}(w_1^*, x))w_1^* \leq 0
\end{aligned}$$

In summary, we have

$$U_v(w_1^*, w_2^*) - U_v = \left[\Delta_{v^1}^{(1)} + \Delta_{v^2}^{(1)}\right] + \left[\Delta_{v^1}^{(2)} + \Delta_{v^2}^{(2)}\right] \leq U_v$$

which means $U_v(w_1^*, w_2^*) \leq 2U_v$. ■

IV. Conclusion

In this paper we discuss the influence from the Sybil attack against the proportional response protocol for the P2P resource sharing problem over networks. Since the manipulative agent can obtain more utility by playing the Sybil attack, we characterize the improvement in utility, measured by the incentive ratio. Previous works have proved the incentive ratio is lower bounded by two and upper bounded by four. The current best bound is three [9]. How to tighten the incentive ratio is an open problem listed in [5] and [9]. In this paper, we completely resolve this open problem to tighten up the incentive ratio of two. The Adjusting Technique provides a new approach toward the problem on general P2P networks, for which we also conjecture to demand an incentive ratio of two.

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