RESEARCH STATEMENT

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I am interested in probability theory and stochastic processes, my current research focuses on random matrix theory.

A central problem of random matrix theory is to understand the structure of the *spectra of large random matrices*. Over the years, many tools have been developed from various branches of mathematics (including combinatorics, analysis, representation theory, etc.) to study the eigenvalue statistics of random matrix ensembles on different scales. To understand the global picture, one could study the empirical spectral measure, i.e. a random probability measure supported on the spectrum so that each eigenvalue has equal weight (up to multiplicity). Under the appropriate scaling, the empirical spectral measures of a wide class of ensembles converge in distribution to a deterministic limit as the size of the matrix grows to infinity. For example, the Wigner semicircle law arises for random symmetric matrices with independent entries, and the Marchenko–Pastur law for the sample covariance matrices (see e.g. [18],[1]). These classical results indicate certain universal asymptotic properties of the spectrum, and are essentially Law of Large Numbers type results.

Another natural object to investigate is the *local scaling limit*, which describes the limit of the spectrum in the scaling regime where the spacings between the eigenvalues remain of constant order. In contrary to the global picture, the local limit will describe the asymptotic behavior near a certain value (reference point), and the limit process could depend on whether the reference point is in the bulk or at the edge of the spectrum. For several classical ensembles, the bulk and edge scaling limits were derived by Dyson, Gaudin, and Mehta in the 1960s utilizing the algebraic structures present in the joint eigenvalue densities. They showed that these algebraic structures are preserved in the limit, and the joint intensity functions of the limit point processes can be fully described.

By generalizing the eigenvalue distributions of the classical ensembles of random matrix theory, we get the so-called β-ensembles, which can be viewed as one-parameter families of particle systems, see Section 1 below for more details. My research during my graduate studies has been mostly devoted to the study of the scaling limits of β-ensembles. I have one published [8] and one submitted paper [17] in this area.

In Section 1, I will give a brief introduction to β-ensembles and the random operator approach that I used in their study. Section 2 provides a short overview of my completed projects. In Section 3, I will present some ongoing projects and outline my research plan.

1. Introduction

Random matrices were first emerged from the study of sample covariance matrices in statistics in 1920s. The models considered by Wishart [29] are matrices of the form $MM^\dagger$ where M is an $n \times (n+a)$ matrix with i.i.d. standard real/complex/quaternion Gaussian entries. The models are called the Laguerre (or Wishart) ensembles. Noticing that the Laguerre ensembles are invariant under certain group conjugations, the joint eigenvalue densities can be computed explicitly. For a size n Laguerre ensemble (indexed by n, a) the eigenvalues have a joint density function given by

$$p_{n,\beta}(\lambda_1, \cdots, \lambda_n) = \frac{1}{Z_{n,\beta}} \prod_{j<k} |\lambda_k - \lambda_k|^\beta \prod_{k=1}^{n} e^{-V(\lambda_k)} \tag{1.1}$$

on $\mathbb{R}_+^n$, where the potential function $V(x) = \frac{\beta}{2}x - (\frac{\beta}{2}(a+1) - 1)\log x$, and the parameter $\beta = 1, 2, 4$ corresponds to the real/complex/quaternion entries, respectively. Here $Z_{n,\beta}$ is an explicitly computable normalizing constant.

In 1950s Wigner [28] used random matrices to model the energy levels of nuclei of heavy atoms in nuclear physics. The idea was to approximate a self-adjoint operator (Hamiltonians with certain symmetries) using a large symmetric or Hermitian matrix with i.i.d. real or complex standard normals. The resulting models are called the Gaussian orthogonal ensemble (GOE) or Gaussian unitary ensemble (GUE), which are classified by the group over which they are invariant. It turns out the joint eigenvalue densities of these Gaussian ensembles have the same structure as equation (1.1) with potential $V(x) = \frac{\beta}{4}x^2$, and $\beta = 1, 2$ for GOE or GUE, respectively. When $\beta = 4$ the density (1.1) is related to the Gaussian symplectic ensemble.

Another well-studied ensemble in random matrix theory is the circular unitary ensemble (CUE), which describes the eigenvalue distribution of finite Haar unitary matrices. The model was introduced by Dyson [10] as a generalization of GUE on the unit circle. For a size n CUE, the eigenangles have a joint density proportional to $\prod_{j<k\le n} |e^{i\theta_j} - e^{i\theta_k}|^\beta$ with $\beta = 2$, which can be viewed as the density (1.1) evaluated on the unit circle with $V = 0$. The cases when $\beta = 1, 4$ correspond to symmetric/self-dual unitary matrices.

Note that the measure defined by (1.1) makes sense for all $\beta > 0$. Hence the classical Laguerre/Gaussian/circular ensembles (which correspond to Dyson's threefold way $\beta = 1, 2, 4$) can be naturally generalized to Laguerre β-ensemble (LβE), Gaussian β-ensemble (GβE), and circular β-ensemble (CβE) for all positive β. The size n versions of these β-ensembles are denoted by LβE$_n$, GβE$_n$, and CβE$_n$.

In the classical $\beta = 1, 2, 4$ cases, the ensembles are integrable (exactly solvable) in the sense that the local behavior (e.g. the joint intensity functions) of the eigenvalues can be explicitly described. However, the tools developed by Dyson, Gaudin and Mehta can not be adapted for general $\beta > 0$. This is due to the lack of the algebraic (determinantal/phaffian) structure. One particularly fruitful approach to study the asymptotic behavior of β-ensembles (especially on a microscopic level) is via random differential operators. This approach can be traced back to the work of Dumitriu and Edelman [9], where the authors constructed two families of tridiagonal matrix models whose eigenvalues obey GβE and LβE, respectively. See Killip and Nenciu [14] as well for matrix models for CβE. To study the local scaling limit of the spectrum, that is, to find the asymptotic limits of the rescaled (and possibly recentered) spectrum, Edelman and Sutton [11] presented heuristics that the scaled tridiagonal matrices should converge to certain random differential operators, from which the convergence of the eigenvalues would follow.

Centering near the edge of the spectrum, the arguments of [11] have been made rigorous by Ramírez, Rider, and Virág [21], and Ramírez and Rider [19] for two different types of limit behaviors. The limiting random differential operators constructed in [21] and [19] are second order Sturm-Liouville operators acting on certain subsets of L^2 functions, see Section 2.1 below for more details. The bulk scaling limit was derived by Valkó and Virág [24] for GβE, and Killip and Stoiciu [15] for CβE in different descriptions. Later it was proved in [25] that the two descriptions are equivalent, and that the bulk limit point process can be characterized as the spectrum of a first order Dirac operator acting on two-dimensional vector valued functions, see Section 2.2 below. The point process scaling limits for β-ensembles have been shown to be universal for a wide class of β-ensembles, see [4], [3], [16], [22]. Moreover, the last two results also show universality on the level of random operators near the edge.

To end the introductory part, we note that the random differential operator descriptions are novel even at the classical $\beta = 1, 2, 4$ cases. Using oscillation theory from differential equations, one can describe the limit point processes via their counting functions. These counting functions are related to coupled systems of stochastic differential equations. This representation provides access to various properties of the limit objects (e.g. large gap probability, tail asymptotics) simultaneously for all values of β using techniques from stochastic analysis.

2. Completed Works

My research is influenced by the above-mentioned random operator approach. My first result in this area is a joint work with Laure Dumaz and my advisor Benedek Valkó [8]. We proved the operator level hard to soft edge transition for β-ensembles, see Section 2.1 below for more details. My second paper is also joint with Valkó [17]. We studied the point process limit of the circular Jacobi β-ensemble (a family of measures on the unit circle), and proved the operator level convergence, see Section 2.2 below.

2.1. Operator level hard-to-soft transition for β-ensembles. Recall that the eigenvalues of LβE are a.s. positive. Let $\Lambda_{n,\beta,a} \sim \mathrm{L}\beta\mathrm{E}_n$ with parameter $a = a_n$. In the case when $a_n \equiv a > -1$, the smallest eigenvalue will converge in distribution to 0, and locally the spectrum will feel the hard constraint at the origin. It was proved in [19] that if $a_n \equiv a > -1$, then $n\Lambda_{n,\beta,a} \Rightarrow \mathrm{Bessel}_{\beta,a}$, where the $\mathrm{Bessel}_{\beta,a}$ process has the same distribution as the spectrum of the stochastic Bessel operator

$$\mathfrak{G}_{\beta,a} = -\frac{1}{m(x)}\frac{d}{dx}\left(\frac{1}{s(x)}\frac{d}{dx}\cdot\right), \qquad m(x) = e^{-(a+1)x-\frac{2}{\sqrt{\beta}}B(x)},\ s(x) = e^{ax+\frac{2}{\sqrt{\beta}}B(x)}. \tag{2.1}$$

Here B is a standard Brownian motion, and the operator $\mathfrak{G}_{\beta,a}$ is defined on a subset of $L^2(\mathbb{R}_+, m)$. This is called the hard edge scaling limit.

If $a_n \to \infty$ with at least a constant speed then one obtains a different scaling limit than the one seen in the hard edge case. Heuristically speaking, the tridiagonal matrix models for Λ_{n,β,a_n} become more diagonally dominated as $a_n \to \infty$, and the bottom of the spectrum is pushed away from the origin. Assuming $\liminf_{n\to\infty} a_n/n > 0$, it follows from the work of [21] that $c_n(\Lambda_{n,\beta,a_n} - d_n) \Rightarrow \mathrm{Airy}_\beta$, where $c_n = \frac{((n+a_n)n)^{1/6}}{(\sqrt{n+a_n}-\sqrt{n})^{4/3}}$, $d_n = (\sqrt{n+a_n} - \sqrt{n})^2$. The point process Airy_β has the same distribution as the spectrum of the stochastic Airy operator

$$\mathsf{A}_\beta = -\frac{d^2}{dx^2} + x + \frac{2}{\sqrt{\beta}}dB, \tag{2.2}$$

defined on a subset of $L^2(\mathbb{R}_+)$, and where B is a standard Brownian motion. This is called the soft edge scaling limit.

It is conjectured that the condition $\liminf_{n\to\infty} a_n/n > 0$ in the soft edge scaling limit could be relaxed to $\lim_{n\to\infty} a_n = \infty$. This conjecture, together with a diagonal argument, would imply the following point process level transition

$$a^{-4/3}(\mathrm{Bessel}_{\beta,2a} - a^2) \Rightarrow \mathrm{Airy}_\beta, \qquad \text{as } a \to \infty. \tag{2.3}$$

By analyzing the determinantal Airy and Bessel kernels, the transition (2.3) was proved in [2] for $\beta = 2$. For general $\beta > 0$, Ramírez and Rider [19] proved the scaling limit for the first point of the respective point processes. This result was extended to a full process level limit in [20] . With the operator descriptions, the transition (2.3) can be rewritten as $a^{-4/3}(\mathrm{spec}(\mathfrak{G}_{\beta,2a}) - a^2) \Rightarrow \mathrm{spec}(\mathsf{A}_\beta)$.

It is natural to ask whether it is possible to prove the corresponding limit on the level of the operators. In the joint work with Dumaz and Valkó [8], we obtained the following result.

Theorem 2.1 (Operator level hard-to-soft transition, [8]). *There is a simple coordinate transformation of $\mathfrak{G}_{\beta,a}$, denoted by $\mathsf{G}_{\beta,a}$, and an explicit coupling of the Brownian motions in A_β and $\mathfrak{G}_{\beta,a}$ so that $a^{4/3}(\mathsf{G}_{\beta,2a}-a^2)^{-1} \to \mathsf{A}_\beta^{-1}$ a.s. in Hilbert-Schmidt norm as $a\to\infty$.*

The convergence in Theorem 2.1 is done in the strongest possible sense, and provides an alternative proof for the transition (2.3). Both the stochastic Airy and Bessel operators fit into the framework of generalized Sturm-Liouville operators. Thus, the resolvents of A_β and $\mathfrak{G}_{\beta,a}$ are Hilbert-Schmidt integral operators with explicit kernels depending on certain eigenfunctions of A_β and $\mathfrak{G}_{\beta,a}$. These eigenfunctions can be expressed in terms of certain diffusions driven by the Brownian motion appearing in the appropriate random operators.

Our proof relies on a careful study of the arising diffusions for the hard and soft edge operators. We show that in our coupling on a given compact interval the hard edge diffusions converge to the soft edge diffusion uniformly as the parameter a goes to ∞, which implies that the truncated versions of the hard edge operators converge to the truncated soft edge integral operator. We also show that the effect of the truncation disappears in the limit, this is the technically difficult part of the proof. A key step here is to understand the asymptotic behavior of the hard edge diffusion on various time scales. Up to a macroscopic time of order $a^{2/3}$, the hard edge diffusion still mimics the behavior of the soft edge diffusion. For larger times, the hard edge diffusion oscillates near a deterministic curve with possibly large excursions. Using tools from stochastic analysis (in particular coupling techniques), we establish a number of tight bounds on the large time behavior of the soft and hard edge diffusions. This allows us to derive the norm resolvent convergence stated in Theorem 2.1.

2.2. Operator level limits of the circular Jacobi β-ensemble. The circular Jacobi β-ensemble (CJβE) is a one-parameter generalization of CβE. Introduce $\delta\in\mathbb{C}$ with $\Re\delta>-1/2$, the size n CJβE (with parameters β,δ) is a distribution of n points on the unit circle, where the angles are distributed according to the density function

$$p_{n,\beta,\delta}(\theta_1,\dots,\theta_n)=\frac{1}{Z_{n,\beta,\delta}}\prod_{j<k\le n}|e^{i\theta_j}-e^{i\theta_k}|^\beta\prod_{1\le k\le n}(1-e^{-i\theta_k})^\delta(1-e^{i\theta_k})^{\bar\delta},\quad \theta_j\in[-\pi,\pi). \tag{2.4}$$

For $\beta=2$ the distribution is known as the Hua-Pickrell measure, and the local scaling limit of the angles was derived in [12] using the determinantal structure present in this case. For $\delta=0$ the distribution recovers CβE, where the scaling limit was first derived by Killip and Stoiciu [15], and then proved by Valkó and Virág [25] in the random operator framework.

[25] showed that a number of random matrix models (and their limits) can be represented using random Dirac differential operators. The ingredients to define a Dirac operator are a generating path $x+iy:[0,1)\mapsto\mathbb{H}=\{z\in\mathbb{C}:\Im z>0\}$, and two non-zero, non-parallel vectors $u_1,u_2\in\mathbb{R}^2$. Then we consider differential operators of the form

$$\tau: f\to R^{-1}(t)\begin{pmatrix}0&-1\\1&0\end{pmatrix}f',\qquad f:[0,1)\to\mathbb{R}^2,\qquad R=\tfrac{1}{2y}\begin{pmatrix}1&-x\\-x&x^2+y^2\end{pmatrix}. \tag{2.5}$$

Under some mild conditions on the triple $(x+iy,u_1,u_2)$, the associated Dirac operator is self-adjoint with pure point spectrum, and its inverse is a Hilbert-Schmidt integral operator with explicit kernel.

In [25], it was showed that in the case when the generating path is a time-changed hyperbolic Brownian motion in $\mathbb{H}$, and one chooses the appropriate boundary conditions, then the spectrum of the corresponding Dirac operator (denoted by $\mathtt{Sine}_\beta$) has the same distribution as the universal bulk limit point process (denoted by Sine_β). Using the theory of orthogonal polynomials on unit circle (see e.g. [23]), the authors in [25] also provided Dirac operator representations for finite unitary matrices. The idea was that for finitely supported probability measures on the unit circle, the Szegő recursion of the normalized orthogonal polynomials (the analogue of the three-term recursion for real orthogonal polynomials) can be translated into the eigenvalue equation for a Dirac operator with piecewise constant path. The path can be constructed from the so-called modified Verblunsky coefficients appearing in the Szegő recursion.

The main goal of [17] (joint with Valkó) is to understand the local scaling limit of CJβE, using the Dirac operator theoretic framework. In [5] the authors constructed matrix models for CJβE and described explicitly the distribution of the modified Verblunsky coefficients. These constructions lead to the random Dirac operators $\mathtt{CJ}_{n,\beta,\delta}$, whose spectrum is the periodic extension of CJβE$_n$ with an extra magnification by n. Under the appropriate scaling, the piecewise constant paths associated to the random operators $\mathtt{CJ}_{n,\beta,\delta}$ converge to the time-changed hyperbolic Brownian motion with drift. As shown in [25], one can construct random differential operators in terms of the limiting diffusion. Denote the limiting operator and its spectrum by $\mathtt{HP}_{\beta,\delta}$ and $\mathrm{HP}_{\beta,\delta}$, respectively (note that $\mathtt{HP}_{\beta,0} = \mathtt{Sine}_\beta$ and $\mathrm{HP}_{\beta,0} = \mathrm{Sine}_\beta$). Our main result in [17] is the following operator level convergence.

Theorem 2.2 (Operator limit of $\mathtt{CJ}_{n,\beta,\delta}$,[17])**.** *Fix $\beta > 0$ and $\Re\delta > -1/2$. Then there is a coupling of the random operators $\mathtt{CJ}_{n,\beta,\delta}, n \geq 1$ and $\mathtt{HP}_{\beta,\delta}$ so that $\mathtt{CJ}_{n,\beta,\delta}$ converges to $\mathtt{HP}_{\beta,\delta}$ a.s. in norm resolvent sense as $n \to \infty$. In particular, if $\Lambda_{n,\beta,\delta} \sim \mathrm{CJ}\beta\mathrm{E}_n$ with parameter δ then $n\Lambda_{n,\beta,\delta} \Rightarrow \mathrm{HP}_{\beta,\delta}$. In this coupling the normalized characteristic polynomial of $\Lambda_{n,\beta,\delta}$ converges a.s. uniformly on compacts to a random analytic function constructed from the $\mathtt{HP}_{\beta,\delta}$ operator.*

The proof of Theorem 2.2 relies on a uniform integrable upper bound on the kernels of the inverses of the $\mathtt{CJ}_{n,\beta,\delta}$ operators. The key is to understand how the two coordinates of the random walks depend on each other. Along the way, we established tight bounds on the piecewise constant paths which were sufficient to show the desired operator level convergence. Our path bounds also imply the convergence of the normalized characteristic polynomials of CJβE to the random analytic function constructed from the $\mathtt{HP}_{\beta,\delta}$ operator (see [27], [17] for more details about the constructions).

The limit objects are interesting on their own. In [17], we also characterized the $\mathrm{HP}_{\beta,\delta}$ process via its counting function. This description allows us the derive the asymptotics of the large gap probability (probability of having no eigenvalues in a large interval) of the $\mathrm{HP}_{\beta,\delta}$ process. We also derived various characterizations of the limiting random analytic function, by describing the joint distribution of the Taylor coefficients at zero, and also by using an entire function valued stochastic differential equation.

3. Ongoing projects and Future directions

In this section, I describe some projects that I am currently exploring, or planning to investigate in the near future.

3.1. Soft edge limit for the Laguerre β-ensemble "near" the hard edge. Let $\Lambda_{n,\beta,a} \sim \mathrm{L}\beta\mathrm{E}_n$ with parameter $a = a_n$. As briefly discussed in Section 2.1 earlier, it is conjectured that the convergence

$$\frac{((n+a_n)n)^{1/6}}{(\sqrt{n+a_n}-\sqrt{n})^{4/3}}(\Lambda_{n,\beta,a_n} - (\sqrt{n+a_n}-\sqrt{n})^2) \Rightarrow \mathrm{Airy}_\beta$$

holds for all sequences of a_n such that $\lim_{n\to\infty} a_n = \infty$. The case when $a_n \to \infty$ subject to $\lim_{n\to\infty} a_n/n = 0$ remains open. (See however [7] for the proof of the case $\beta = 2$, $a_n = c\sqrt{n}$ using the determinantal structure present at $\beta = 2$.)

Jointly with Valkó, we are investigating this remaining case by adapting the arguments in [8] in a discrete setting. The key step is to control the inverses of the tridiagonal matrix models for Λ_{n,β,a_n} (viewed as discrete integral operators).

3.2. Limits of the truncated circular β-ensemble. This is also a joint work with Valkó. In [30], Życzkowski and Sommers derived the joint eigenvalue distribution of a truncated Haar unitary matrix. Killip and Kozhan [13] extended this density for general $\beta > 0$, which we call the truncated CβE. Using the Szegő recursion and Verblunsky coefficients, they also provided sparse matrix models whose eigenvalues are distributed as the truncated CβE.

In this project we aim to find the point process scaling limit of the truncated CβE. Since the truncated models are sub-unitary, we cannot expect to find a self-adjoint operator in the limit. However, by analyzing the Szegő recursion directly, we can find the limits of the appropriate orthogonal polynomials, and the point process limit.

3.3. Future directions. In the following, I list some problems that I would like to explore further.

(1) Note that the coupling in Theorem 2.2 is not explicit. It would be interesting to construct a strong coupling of the random operators $\mathtt{CJ}_{n,\beta,\delta}, n \geq 1$ and $\mathtt{HP}_{\beta,\delta}$ so that the random walks are embedded into the limit diffusion. See [26] for the construction when $\delta = 0$ using the rotational invariance of the models. Also, a lot more can be analyzed explicitly when $\delta = 0$; for example, Valkó and Virág [27] computed the expectations of certain functionals of the random entire function associated to the $\mathtt{Sine}_\beta$ operator. It is then natural to ask if the moment formulas in [27] can be generalized for all $\Re\delta > -1/2$.

(2) For CJβE, it was proved in [5] that if $\delta = \delta_n = \frac{\beta}{2}nz$ with $\Re z \geq 0$ then the corresponding empirical spectral measure converges weakly to some nontrivial limit as $n \to \infty$. It is already interesting to derive the scaling limit in this case. Inspired by the transition (2.3), one can also ask if any process level transition can be observed for the $\mathrm{HP}_{\beta,\delta}$ process in the $\delta \to \infty$ limit.

(3) It would be interesting to construct matrix models for the truncated versions of the CJβE using (modified) Verblunsky coefficients. This should be parallel to the construction in [13]. If such constructions are possible, then a natural further step is to study the point process scaling limit for truncations of CJβE.

Although I am interested in exploring these open questions related to my thesis work, I would also like to expand my research areas and learn new tools and techniques. For example, I would be interested in projects related to interacting particle systems, integrable probability, and stochastic PDEs. During my graduate studies I have taken various graduate and summer courses in the topic of integrable probability, in particular about random growth models belonging to the KPZ universality class. I also participated in a semester-long reading seminar on the topic of the directed landscape [6], a central object in the field. I would be very interested in working on problems related to any of these areas.

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