Understanding Demand Estimation Methods

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Why Demand Estimation?

Structural demand estimation is the prerequisite for any structural analysis of market power and pricing strategies.

- Calculate precise price elasticities: addressing price endogeneity (p correlated with unobserved ξ). It is the starting point for any other analysis.
- Recover supply-side primitives: inverting the FOC allows us to recover marginal costs and markups without observing real supply-side data.
- Counterfactual analysis: merger simulation, introduction of new goods, or taxes.
- Basis for other topics: vertical relations, network effects, consumer search, and many others...

Why Mixed Logit & BLP?

Standard Logit is insufficient due to the IIA (Independence of Irrelevant Alternatives) property.

- **The IIA Restriction:**
  - Logit implies proportional substitution (Red Bus / Blue Bus).
  - Cross-price elasticities are restricted by market shares, not product similarity.
- **The Random Coefficients Solution:**
  - Allows for flexible substitution patterns: Consumers substitute towards products with similar characteristics.
  - Captures Unobserved Heterogeneity: $\beta_i = \bar{\beta} + \sigma \nu_i$. Different consumers value attributes differently.

Learning From Replication

Bridging the gap between textbook theory and computational practice:

- **Optimization:**
  - Understanding the "Inner Loop" (Contraction Mapping) vs. "Outer Loop" (GMM/MLE).
- **Computational Bottlenecks:**
  - The burden of high-dimensional integration (N consumers $\times$ T periods $\times$ R draws).
- **Modern Implementation:**
  - **Vectorization & Broadcasting:** Eliminating loops.
  - **JIT Compilation (`Numba`):** If loops are necessary.
  - **GPU Acceleration (`PyTorch`):** Parallelizing massive integral simulations.

The Foundation: Discrete Choice Framework

The Choice Setting

- Consumer i faces a set of J mutually exclusive alternatives (e.g., Coke, Pepsi, Sprite, Outside Option).
- Decision: Choose j if and only if $U_{ij} > U_{ik}, \forall k \neq j$.

We decompose utility into an observed part (V_{ij}) and an unobserved shock (ε_{ij}):

$$U_{ij} = \underbrace{V_{ij}}_{\text{Deterministic (e.g., } X_j\beta - \alpha p_j)} + \underbrace{\varepsilon_{ij}}_{\text{Random Shock}}$$

To make this tractable, we assume ε_{ij} follows an i.i.d. Type I Extreme Value (Gumbel) distribution.

- Why? It yields a closed-form solution for probabilities.

The Plain Logit Model (McFadden, 1974)

Under the Gumbel assumption, the probability of choosing j has a beautiful closed-form expression:

$$P_{ij} = \frac{\exp(V_{ij})}{\sum_{k=0}^{J} \exp(V_{ik})}$$

Key Features:

- 0 to 1 bounded: Automatically behaves like a probability.
- Analytically tractable: Market shares (s_j) are easy to compute.
- Global convexity: The likelihood function is globally concave (easy to estimate).

The Limitation: Independence of Irrelevant Alternatives (IIA)

The Definition: The ratio of probabilities between any two goods (A, B) depends only on A and B, not on the existence of a third good C.

$$\frac{P_A}{P_B} = \frac{e^{V_A}}{e^{V_B}} \quad \text{(Independent of } V_C\text{)}$$

The "Red Bus / Blue Bus" Paradox

- Scenario 1: Drive (50%) vs. Red Bus (50%). Ratio = 1:1.
- Scenario 2: Add a "Blue Bus" (identical to Red Bus).
- Logic Prediction (IIA): Ratio of Drive/Red Bus must stay 1:1.
- Result: Drive (33%), Red Bus (33%), Blue Bus (33%).
- Reality: Drive (50%), Red Bus (25%), Blue Bus (25%).

Consequence: Plain Logit forces proportional substitution. It cannot capture that Red Bus and Blue Bus are closer substitutes!

Breaking the IIA

Plain Logit is restricted by IIA. How do we relax it?

1. **Nested Logit**
   - Idea: Group similar alternatives into "nests" (e.g., Sedan, SUV, No Purchase).
   - Pros: Closed-form solution; easy to estimate.
   - Cons: You must pre-specify the nests. The substitution pattern is ad-hoc, not found by the data.
2. **Multinomial Probi**
   - Idea: Assume $\varepsilon \sim N(0, \Sigma)$. Allows flexible covariance.
   - Pros: Theoretically flexible.
   - Cons: Computationally intractable (requires high-dimensional integration).
3. **Mixed Logit / Random Coefficients**
   - Idea: Allow β to vary across individuals ($\beta_i = \bar{\beta} + \sigma\nu_i$).
   - Theorem: McFadden & Train (2000) proved that Mixed Logit can approximate any random utility model.

Mixed Logit

The utility of consumer i choosing product j is:

$$u_{ij} = x_j\beta_i + \varepsilon_{ij} = \underbrace{x_j\beta}_{\text{mean utility}} + \underbrace{\sum_k x_{jk}\,\sigma_k\,\nu_{ik}}_{\text{heterogeneity}} + \underbrace{\varepsilon_{ij}}_{\text{iid T1EV}} \quad (\text{Here, } \beta_i \sim N(\beta,\sigma^2))$$

Conditional on individual tastes ν_i, the probability is:

$$p_{ij}(\nu_i) = \frac{\exp(x_j\beta + x_j\,\sigma\,\nu_i)}{1 + \sum_{k=1}^{J}\exp(x_k\beta + x_k\,\sigma\,\nu_i)}$$

We observes choices y_{ij} ($=1$ means i chooses j). The log-likelihood function is

$$LL(\{\beta,\sigma\}) = \sum_{i=1}^{N}\log\int\left(\prod_{j=1}^{J} p_{ij}(\nu)^{y_{ij}}\right)\phi(\nu)d\nu$$

Numerical Integration

The goal is to maximize the likelihood, but it does not have a closed form since RCs follow a normal distribution.

How to do it?

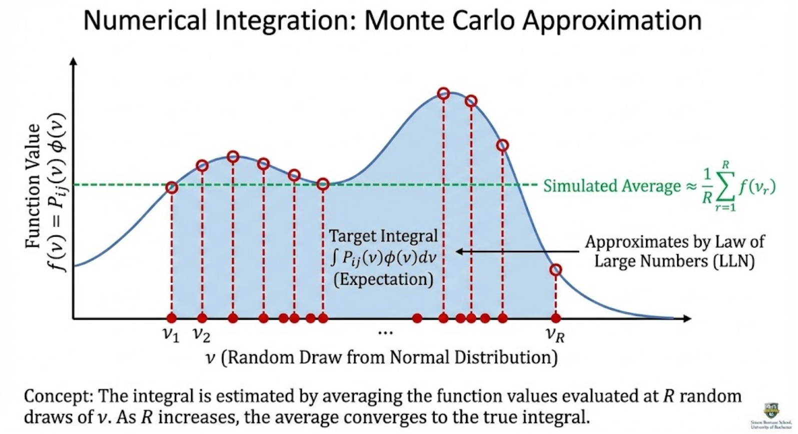
- Think of the integral as an expectation: $E_\nu[P_{ij}(\nu)]$.
- By the Law of Large Numbers, the sample mean converges to the expectation.

We simulate the integral:

1. Draw R random values: $\nu_1, \ldots, \nu_R \sim N(0,1)$.
2. Calculate probability for each draw: $P_{ij}(\nu_r)$.
3. Take the average:

$$SLL(\{\beta,\sigma\}) = \sum_{i=1}^{N} \log\left(\frac{1}{R}\sum_{r=1}^{R}\prod_{j=1}^{J} p_{ij}^{y_{ij}}(\nu_r)\right)$$

Numerical Integration


Numerical Integration: Monte Carlo Approximation
Function Value
$f(v) = P_{ij}(v)\,\phi(v)$
Simulated Average $\approx \frac{1}{R}\sum_{r=1}^{R} f(v_r)$
Target Integral
$\int P_{ij}(v)\phi(v)dv$
(Expectation)
Approximates by Law of Large Numbers (LLN)
v_1 v_2 … v_R
v (Random Draw from Normal Distribution)

Concept: The integral is estimated by averaging the function values evaluated at R random draws of v. As R increases, the average converges to the true integral.

To reduce the number of draws, we can use the Halton draw which covers space more uniformly. Halton$(R = 100) \approx$ Random$(R = 1000)$.

MLE Is Not Consistent

Let's distinguish between two types of convergence.

1. The Role of Sample Size (N) As $N \to \infty$, the sample average converges to the expected value of the objective function (Law of Large Numbers):

$$\frac{1}{N}\sum_{i=1}^{N} \ln \hat{P}_i(\theta) \xrightarrow{p} E_{data}\left[\ln \hat{P}_i(\theta)\right]$$

2. The Role of Simulation Draws (R) However, because of Jensen's Inequality:

$$E\left[\ln \hat{P}_i\right] < \ln(E[\hat{P}_{ij}]) = \ln P_i$$

So, even with infinite data ($N \to \infty$), the estimator is not converging to the true value.

$R \to \infty$ Can Restore Consistency

Taylor Expansion: approximating $\ln(\hat{P})$ around the true probability P:

$$\ln(\hat{P}) \approx \ln(P) + \frac{1}{P}(\hat{P} - P) - \frac{1}{2P^2}(\hat{P} - P)^2$$

Taking expectations on both sides ($E[\hat{P}] = P$):

$$E[\ln(\hat{P})] \approx \ln(P) + 0 - \frac{1}{2P^2} \underbrace{Var(\hat{P})}_{\text{Simulation Variance}}$$

Since $\hat{P}$ is an average of R draws, $Var(\hat{P}) \propto \frac{1}{R}$.

$$\text{Bias} \approx -\frac{C}{R} \xrightarrow{R\to\infty} 0$$

Comparison: MLE vs. Method of Simulated Moments (MSM)

If MSLE is biased, why not use MSM?

Property	**MLE (Likelihood)**	**MSM (Moments)**
Objective	$\max \sum \ln(\hat{P})$	$\min \|\|\text{Data} - \text{Model}\|\|$
Linearity	Non-linear (ln)	Linear Difference
Bias (Fixed R)	Biased	Unbiased
Efficiency	High (Cramer-Rao)	Lower

Conclusion:

$$\mathrm{MSE}(\hat{\theta}) = \underbrace{\left(\mathbb{E}[\hat{\theta}] - \theta\right)^2}_{\text{Bias}^2} + \underbrace{\mathbb{E}\left[\left(\hat{\theta} - \mathbb{E}[\hat{\theta}]\right)^2\right]}_{\text{Variance}}$$

- MSM is consistent for fixed R, but less efficient (less information is used).
- In practice, we prefer MLE with large R (or Halton) because efficiency.

Specify Analytical Gradient

To maximize $f(x)$, we need to find a step Δx such that $f(x+\Delta x) > f(x)$.

$$f(x+\Delta x) \approx f(x) + \underbrace{\nabla f(x)^T \Delta x}_{\text{Must be negative}}$$

Without $\nabla f(x)$, the algorithm relies on slow finite-difference guesses.

Deriving Gradient

Consider the simplest case, no simulation. Let i choose product j. The Log-Likelihood is: Full derivation

$$LL(\beta) = \ln P_{ij} = \ln\left((\frac{\exp(x_{ij}\beta)}{\sum_{k=0}^{J} \exp(x_{ik}\beta)} \right) = x_{ij}\beta - \ln\left(\sum_{k=0}^{J} \exp(x_{ik}\beta)\right)$$

Take the derivative w.r.t β:

$$\begin{aligned}
\frac{\partial LL}{\partial \beta} &= x_{ij} - \frac{\partial}{\partial \beta} \ln\left(\sum_k e^{x_{ik}\beta}\right) \\
&= x_{ij} - \frac{1}{\sum_k e^{x_{ik}\beta}} \cdot \sum_k \left(e^{x_{ik}\beta} \cdot x_{ik}\right) \quad \text{(Chain Rule)} \\
&= x_{ij} - \sum_k \underbrace{\left(\frac{e^{x_{ik}\beta}}{\sum_m e^{x_{im}\beta}}\right)}_{P_{ik}(\beta)} \cdot x_{ik}
\end{aligned}$$

Deriving Gradient

The gradient has a beautiful, intuitive form:

$$\nabla_\beta LL = \underbrace{x_{ij}}_{\text{Observed Attribute}} - \underbrace{\sum_{k=1}^{J} P_{ik}(\beta) x_{ik}}_{\text{Expected Attribute } E[x]}$$

Intuition:

- Imagine x is "Horsepower". I chose a Ferrari (x_{high}).
- If the model predicts I like slow cars, then $E[x]$ is low.
- Gradient $= x_{high} - x_{low} > 0$.
- **Action:** Increase β_{HP}. The model stimulates the preference for horsepower up until Observed $\approx$ Expected.

This is essentially the Method of Moments condition!

Optimization Algorithms

Now we have the direction ∇LL (Gradient). How do we update β?

1. **Gradient Ascent:** Simple but slow. Just follow the steepest slope.

$$\beta_{new} = \beta_{old} + \alpha \cdot \nabla LL$$

2. **Newton's Method:** Fast but expsensive to get H

$$\beta_{new} = \beta_{old} - \underbrace{H^{-1}}_{\text{Curvature}} \nabla LL$$

3. **Quasi-Newton / BFGS**
   - **Idea:** Don't calculate H. Instead, approximate H^{-1} using the history of gradients.
   - **Result:** Speed of Newton + Low cost of Gradient Ascent.
   - This is what `pyblp` and most solvers use by default.

The Remaining Bottleneck

Even with analytical gradients, estimation is slow. Why?

- Heavy operations
  - We perform matrix multiplication ($X\beta$) and exponentiation (exp) for every consumer, every product, and every simulation draw.
  - Tensor Size: $N(10^4) \times J(10) \times R(500) = 50$ Million Operations.

- The CPU limitation (sequential computing)
  - `NumPy` runs primarily on the CPU.
  - CPUs have few cores (8-16). They process large matrices in "chunks" (sequentially).

Why PyTorch?

PyTorch is not just for neural networks. For structural estimation, it offers two advantages:

- Massive parallel computation (GPU Acceleration) (Later on this point!)
    - Move data from CPU to GPU.
    - GPUs have housands of small cores.
    - We can compute utility for all N consumers and R draws simultaneously.
- Automatic Differentiation (AutoGrad)
    - No need to manually derive the complex gradient.
    - `loss.backward()` computes exact gradients using the computational graph.

Backward Propagation

AutoGrad doesn't "know" the full formula. It just multiplies local gradients step-by-step.

$$\begin{tikzcd}
 & (X\beta + \dots) & \text{Softmax} & \text{Mean} & \\
\beta \arrow[r] & V_{ij} \arrow[r] \arrow[l, dashed, bend left, "\frac{\partial V}{\partial \beta} = X"] & P_{ij} \arrow[r] \arrow[l, dashed, bend left, "\frac{\partial P}{\partial V} \approx P(1-P)"] & \bar{P}_{ij} \arrow[r] \arrow[l, dashed, bend left, "\frac{\partial \bar{P}}{\partial P} = \frac{1}{R}"] & LL \arrow[l, dashed, bend left, "\frac{\partial LL}{\partial \bar{P}} = \frac{1}{\bar{P}}"]
\end{tikzcd}$$

The Logic:

$$\underbrace{\frac{\partial LL}{\partial \beta}}_{\text{Goal}} = \underbrace{\frac{\partial LL}{\partial \bar{P}}}_{\text{Step 1}} \cdot \underbrace{\frac{\partial \bar{P}}{\partial P}}_{\text{Step 2}} \cdot \underbrace{\frac{\partial P}{\partial V}}_{\text{Step 3}} \cdot \underbrace{\frac{\partial V}{\partial \beta}}_{\text{Step 4}}$$

Each node only needs to know its own derivative. The computer just multiplies them together!

Code Comparison

```
# Manual
def gradient(beta, sigma):
  # 1. Calc Shares (P)
  P = ...

  # 2. Manual Formula
  grad_beta = X - (P @ X)

  return grad_beta

# AutoGrad
# Define params
beta = torch.tensor([2.0], requires_grad=True)

# 1. Forward Pass
LL = compute_LL(beta, ...)

# 2. Backward Pass
LL.backward()
```

Application: Advertising and Demand for Addictive Goods (Tuchman, 2019)

Research Question: Do E-cigarette ads decrease demand for traditional cigarettes?

The Model: Mixed Logit with State Dependence (Addiction)
Consumer i chooses product j at week t. Utility depends on past choices:

$$U_{ijt} = \underbrace{\alpha p_{jt} + \phi A_{mt}}_{\text{Mean Utility}} + \underbrace{\gamma \mathbb{I}(y_{it-1} = j)}_{\text{Addiction (State Dep.)}} + \underbrace{\mu_{ij}}_{\text{Heterogeneity}} + \epsilon_{ijt}$$

- Data: Nielsen Household Panel.
- Key Challenge: We must estimate the random coefficients (σ_β) and addiction parameters (γ) by integrating over thousands of consumers.

Implementation: Manual Derivation vs. PyTorch AutoGrad

The Task: Estimate parameters (θ) using simulated maximum likelihood.

- Numpy (Manual gradient)
  - Complex math: Must derive $\frac{\partial LL}{\partial \gamma}$ manually.
  - Runtime: approximately 1 hour 22 minutes on BlueHive.
- PyTorch (AutoGrad)
  - Zero math: Just write the utility function U_{ijt}.
  - Runtime: approximately 16 minutes on an RTX 4060 Ti

Transition: From Micro-Data to Market-Level Data

We have seen how Mixed Logit handles heterogeneity using household panel data.

But what if we only have Aggregate Data?

- Data: Market shares (s_{jt}), Prices (p_{jt}), Characteristics (x_{jt}).
- No individual choices (y_{ijt}) are observed.

The "Unobserved Quality" Problem (ξ_{jt})

$$u_{ijt} = \alpha_i p_{jt} + x_{jt}\beta_i + \underbrace{\xi_{jt}}_{\text{Unobserved Quality}} + \epsilon_{ijt}$$

- ξ_{jt}: Characteristics observed by consumers and firms (e.g., style, prestige), but *unobserved* by the econometrician.
- Endogeneity: Firms set prices based on ξ_{jt} (High quality $\rightarrow$ High price).
- Result: $Corr(p_{jt}, \xi_{jt}) > 0$, which yields biased price elasticities.

The BLP Solution: GMM with Instrumental Variables

Berry, Levinsohn, and Pakes (1995) propose a solution to estimate mixed logit with aggregate data and endogenous prices.

Key Idea: Inversion

- We cannot use MLE because ξ_{jt} is not random noise; it's correlated with price.
- Instead, we invert the market share function to recover the "Mean Utility" (δ_{jt}):

$$s_{jt}(\delta, \theta_2) = S_{jt}^{Data} \implies \delta_{jt}(\theta_2) = s^{-1}(S^{Data}, \theta_2)$$

Once we recover δ_{jt}, we have a linear equation:

$$\delta_{jt} = x_{jt}\beta + \alpha p_{jt} + \xi_{jt}$$

Now we can use IVs to handle the correlation between p_{jt} and ξ_{jt}.

Berry's Inversion (1994)

The market share (there is no RC):

$$s_{jt} = \frac{\exp(x_{jt}\beta - \alpha p_{jt} + \xi_{jt})}{1 + \sum_k \exp(x_{kt}\beta - \alpha p_{kt} + \xi_{kt})}$$

Normalize the outside utility to zero:

$$s_{0t} = \frac{1}{1 + \sum_k \exp(\delta_{kt})}$$

This implies

$$\ln \frac{s_{jt}}{s_{0t}} = x_{jt}\beta - \alpha p_{jt} + \xi_{jt}$$

Implication: We can now estimate this using IV Regression (2SLS/GMM) to solve the correlation between p_{jt} and ξ_{jt}.

When There are Random Coefficients

The market share is the integral of individual probabilities over the population distribution $F(\nu)$:

$$s_{jt}(\delta, \theta_2) = \int \frac{\exp(\delta_{jt} + \mu_{ijt})}{1 + \sum_k \exp(\delta_{kt} + \mu_{ikt})} dF(\nu)$$

Here, $\delta_{jt} = x_{jt}\beta - \alpha p_{jt} + \xi_{jt}$, and $\mu_{ijt} = \sum_k x_{jtk}(\sigma_k \nu_{ik}) - p_{jt}(\sigma_p \nu_{ip})$

Since the integral has no closed form, we approximate it by simulation (averaging over R draws):

$$s_{jt} \approx \frac{1}{R} \sum_{r=1}^{R} \frac{\exp(\delta_{jt} + \mu_{rjt})}{1 + \sum_k \exp(\delta_{kt} + \mu_{rkt})}$$

Inner Loop: Contraction Mapping

Since we cannot analytically invert the integral $s_{jt}(\delta, \theta_2)$, we solve for δ numerically.

From Berry (1994), we use the following iteration:

$$\delta_{jt}^{h+1} = \delta_{jt}^{h} + \ln(S_{jt}^{Data}) - \ln(s_{jt}(\delta^h, \theta_2))$$

- **Intuition:** If Model Share $<$ Data Share, increase δ.
- **Convergence:** $T(\delta) = \delta + \ln(s) - \ln(s(\delta))$ is a contraction mapping (Berry, 1994).

Connection to Rust's NFXP: BLP is structurally identical to John Rust's Nested Fixed Point algorithm.

$$T(W(x)) = \max_d \left[u(x, d; \theta) + \beta\, E[W(x') \mid x, d; \theta]\right]$$

The Outer Loop: Estimating Parameters via GMM

Once the contraction mapping converges ($|\delta_{jt}^{h+1} - \delta_{jt}^{h}| <$ threshold) for a given guess of non-linear parameters θ_2, we obtain the mean utilities $\delta_{jt}(\theta_2)$.

1. Recover linear parameters $(\hat{\beta}, \hat{\alpha})$ using 2SLS and the structural error (ξ_{jt}):

$$\xi_{jt}(\theta) = \delta_{jt}(\theta_2) - (x_{jt}\hat{\beta} - \hat{\alpha}p_{jt})$$

2. Create moment conditions: since price is endogenous $(E[\xi p] \neq 0)$, we use Instruments Z_{jt}:

$$\bar{g}(\theta) = \frac{1}{J \times T} \sum_{j,t} Z_{jt}' \xi_{jt}(\theta)$$

3. GMM: we search for parameters $\hat{\theta}_2$ that minimize the distance of moments to zero:

$$\min_{\theta_2} J(\theta_2) = \bar{g}(\theta_2)' W \bar{g}(\theta_2) = \xi(\theta_2)' Z W Z' \xi(\theta_2)$$

Identification

Recall the moment conditions we used:

$$E(\xi_{jt} X_{jt}) = 0, \qquad E(\xi_{jt} Z_{jt}) = 0.$$

$E(\xi_{jt} X_{jt}) = 0$ are used to identify β. One of the $E(\xi_{jt} Z_{jt}) = 0$ conditions (where Z_{jt} is an IV for price) is used to identify α. The identification of σ comes from the rest of the $E(\xi_{jt} Z_{jt}) = 0$ conditions.

Types of instruments: (Assumptions?)

- Cost shifters
- Hausman IV: price of product j in other markets $p_{j't}$
- BLP IV: characteristics of rival products $x_{-j,t}$, and other variants:
  - $\sum_{j' \neq j} \left(X_{jtm} - X_{j',t,m}\right)^2$
  - $\sum_{j' \neq j} \left(X_{jtm} - X_{j',t,m}\right)\left(X_{jtm'} - X_{j',t,m'}\right)$

Identificaiton

Question: What variations identify random coefficients?
Answer: Substitution patterns - with random coefficients, consumers substitute more to a similar product.

Variations in substitution patterns:

- Substitution based on characteristic distance.
- Choice set variations.

Sometimes, there is not enough variation to identify the unobserved heterogeneity in the data. Instead, we can identify the heterogeneity correlated to demographics. It is easier to use observables to explain substitution patterns.

$$\beta_i = \bar{\beta} + \underbrace{\Sigma \nu_i}_{\text{unobserved}} + \underbrace{\Pi D_i}_{\text{demographics}}$$

Application: Inference and Impact of Category Captaincy (Zhu, 2024)

This paper focuses on category captaincy, which is a vertical arrangement where retailers delegate shelf placement and pricing to a leading manufacturer. However, the contract is confidential. In the data, we can just see Dannon leads in some chains, Yoplait in others. Is this due to consumer preference or shelf advantage (better display, space)?

Utility of consumer i for product j at retailer r:

$$u_{ijrmt} = \bar{u}(x_{jrmt}, \beta, \eta_{irmt}) + \xi_{jrmt} + \varepsilon_{ijrmt}$$

Key Innovation: Decompose unobserved quality ξ_{jrmt}:

$$\xi_{jrmt} = \xi_{jt} + \underbrace{\xi_{brt}}_{\text{Captaincy Effect}} + \Delta\xi_{jrmt}$$

Application: Inference and Impact of Category Captaincy

Identification: To solve price endogeneity, the paper uses two sets of IVs (selected using the specification test (Gandhi and Houde 2019) and Lasso (Belloni et al. 2012)):

- **Cost Shifters**: cost or markup shifters, which includes input/transportation costs interacted with product characteristics, assortment variables, and demographic characteristics of a brand's main market within retailer interacted with that brand's product characteristics
- **Differentiation IVs**: characterizes the degree of differentiation of each product in a retailer market and exploits variation in household demographics across retailer markets (Gandhi & Houde, 2019).

PyBLP

The burdens of hand-coding BLP: updating the optimal weighting matrix, deriving the analytical Jacobian, and managing numerical stability with high-precision tolerances. PyBLP handles these burdensome steps. That's why I recommend it.

Aside from the estimation of BLP, it can also do post-estimation practices: price elasticities, consumer surplus calculation, markup and margin calculation, and merger simulation.

The estimation in `pyblp` is organized around three core objects: Formulation, Integration, and Problem.

The Core Workflow in PyBLP

1. Model Configuration: separate linear parameters (β, α) from non-linear random coefficients (Σ).

```
X1_formulation = pyblp.Formulation(
    '0 + prices + sugar + sodium + fat + size + # flavor + calorie +
        organic + # sales',
    absorb='C(fe_product_year) + C(fe_brand_retailer_year)'
)

X2_formulation = pyblp.Formulation(
    '1 + prices + # flavor + size + sodium + sugar + fat'
)

agent_formulation = pyblp.Formulation('0 + income + kids + edu')
```

The Core Workflow in PyBLP

2. Integration and Initialization:

```
problem = pyblp.Problem(
    product_formulations=(X1_formulation, X2_formulation),
    product_data=product_data,
    agent_data=agent_data,
    agent_formulation=agent_formulation
)

results = problem.solve(
    sigma=initial_sigma,
    sigma_bounds=(np.zeros((7, 7)), np.zeros((7, 7))),
    pi=initial_pi, pi_bounds=(pi_lower, pi_upper),
    optimization=pyblp.Optimization('l-bfgs-b', {'gtol': 1e-4}),
    method='2s', se_type='clustered'
)
```

Revisit Tuchman (2019)

Problem:

- Micro data alone: Sparse. Hard to estimate precise brand-market-time fixed effects (δ_{jmt}) needed to control for price endogeneity.
- Macro data alone: Hard to identify complex heterogeneity distributions (Σ) and state dependence parameters (γ).

Solution:

- Use aggregate data to pin down mean utilities (δ_{jmt}).
- Use household data to identify non-linear parameters (Σ, γ).
- Consistency Constraint: The δ_{jmt} that rationalizes aggregate shares must be the same δ_{jmt} entering the household likelihood.

The Estimation Algorithm

We iterate between two steps until convergence of non-linear parameters $\theta_2 = \{\Sigma, \gamma\}$:

Step 1: The Macro Step (Contraction Mapping)

- Given θ_2, solve for δ_{jmt} such that $s^{pred}(\delta, \theta_2) = s^{obs}$.
- Challenge: Standard BLP is parallel over markets (t is independent). Tuchman's model is recursive: Market shares at t depend on market shares $t-1$.

Step 2: The Micro Step (MLE)

- Treat δ_{jmt} as data (fixed).
- Maximize $LL(\theta_2 | \delta, \text{Household Data})$.

Step 3: Linear Parameters (Post-Convergence)

- Regress converged δ_{jmt} on Prices/Ads (Border strategy).

Step 1: Handling Recursion with Numba

$$s_{jmt} = \int \sum_k \pi_{jmt}(k) \cdot \underbrace{\Pr(y_{it-1} = k)}_{\text{Evolves over time!}} dF(\beta)$$

Implementation Detail:

- Cannot vectorize over T. We must loop $t = 1 \dots T$.
- Pure python loops are too slow for the contraction mapping.
- **Solution:** `@jit(nopython=True)` compiles the recursion to machine code.

```
# Core logic inside Numba
for t in range(T):
    if t > 0:
        delta_t = delta_sol[t-1, :, :].copy() # History dependence

    # Contraction Mapping Loop for current t
    while error > 1e-9:
        # ... solve for delta_t ...

    # Update state distribution for t+1
    prob_state_lag = update_probs(probs)
```

Comparison

```python
import time
import numpy as np
from numba import jit

data = np.arange(1_000_000)
# Loop
result = []
for x in data:
    result.append(x**2) # 0.0493s

# Vectorization
result = data ** 2 # 0.0091s

# Numba
@jit(nopython=True, cache=True)
def square_loop(data):
    result = np.empty(len(data))
    for i in range(len(data)):
        result[i] = data[i] ** 2
    return result

_ = square_loop(data) # First time: 0.0377s
_ = square_loop(data) # After compilation: 0.0008s
```

Step 2: GPU Acceleration with PyTorch

$$LL = \sum_i \ln \int \prod_t P_{ijt}(\delta_{jmt}, \gamma, \Sigma) dF(\nu)$$

Implementation Detail:

- We need to broadcast the δ_{jmt} (solved in Step 1) to thousands of households.
- This involves massive matrix operations: $(N_{HH} \times T \times R_{draws})$.
- **Solution:** Move tensors to GPU.

```
# Broadcasting Macro delta to Micro observations
t_delta_i = t_delta_est[self.t_h_week, self.t_h_mkt, :].unsqueeze(2)

# Adding Heterogeneity (Sigma) and Addiction (Gamma)
t_U = t_delta_i + t_mu_matrix + t_lag_boost

# Choice Probabilities computed in parallel on GPU
t_exp_U = torch.exp(t_U)
```

Handling Initial Conditions: The Burn-in Period

Issue:

- At $t = 0$, we don't know the distribution of addicted users.
- Assuming an arbitrary starting distribution (e.g., all non-smokers) biases the early estimates.

Solution in Code:

- Simulate the first 24 weeks without using them in the likelihood.
- Start from a steady state:

$$LL(\theta) = \sum_{t=\text{burn\_in}}^{T} \ln P(y_{it} | y_{it-1}, \dots)$$

Conclusion

Summary of Methods:

- Mixed Logit: Best for micro-data, handles heterogeneity via simulation, but computationally heavy.
- BLP: Best for aggregate-data, handles price endogeneity via inversion (δ) and IVs.
- Integrated Model: Combines the identification power of micro-data (Σ, γ) with the endogeneity control of aggregate-data (δ).

The Takeaway:

- Tools like `PyTorch` and `Numba` allow us to estimate complex dynamics (e.g., addiction, state dependence) that were previously intractable.
- Final Thought: A good model is not just theoretically sound, but computationally feasible.

Deriving Gradient

Let $P_{ij}(\nu_r)$ be the probability of the chosen alternative j for consumer i in simulation draw r.

The simulated probability is the average over draws:

$$\hat{P}_{ij} = \frac{1}{R}\sum_{r=1}^{R} P_{ij}(\nu_r)$$

The gradient of the Log-Likelihood with respect to β:

$$\frac{\partial SLL}{\partial \beta} = \sum_{i=1}^{N} \frac{\partial \log \hat{P}_{ij}}{\partial \beta} = \sum_{i=1}^{N} \frac{1}{\hat{P}_{ij}} \cdot \frac{\partial \hat{P}_{ij}}{\partial \beta}$$

Substitute the derivative of the simulated probability:

$$\frac{\partial \hat{P}_{ij}}{\partial \beta} = \frac{1}{R}\sum_{r=1}^{R} P_{ij}(\nu_r)(x_{ij} - \sum_{k} P_{ik}(\nu_r)x_{ik})$$

Deriving Gradient

Combining terms, the gradient has a beautiful structure:

$$\nabla_\beta SLL = \sum_{i=1}^{N} \sum_{r=1}^{R} \underbrace{\left[\frac{P_{ij}(\nu_r)}{\sum_{m=1}^{R} P_{ij}(\nu_m)} \right]}_{\text{Weight } w_{ir}} \cdot \underbrace{(x_{ij} - \bar{x}_i(\nu_r))}_{\text{Deviation}}$$

Interpretation:

- Deviation: $(x_{ij} - \bar{x}_i(\nu_r))$ is the difference between the observed attribute and the predicted attribute.
- Weights w_{ir}: These are posterior probabilities of the draws (there exists baysesian updating within MLE).
- **Mechanism:** The gradient gives more weight to simulation draws (ν_r) that make the observed choice more likely.