Numerical modeling of spiral resonators

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I. SPIRAL GEOMETRY

A spiral is specified by three parameters: its external radius R_e, internal radius R_i, and pitch p. The shape of the spiral is given in polar coordinates by

$$\rho(\phi) = R_e(1 - \alpha\phi) \tag{1}$$

where

$$\alpha = \frac{p}{2\pi R_e}.$$

Then the maximum value of ϕ is given by setting $\rho = R_i$:

$$\phi_{\max} = \frac{1}{\alpha}\left(1 - \frac{R_i}{R_e}\right).$$

We specify position along the length of the spiral in terms of a coordinate s, which has its origin at the outside of the spiral. It will be useful later to be able to express the angular position ϕ as a function of s, which we will do now. Consider a differential element of length along the spiral ds, of radial distance from the center of the spiral $d\rho$, and of angle $d\phi$. These are related by

$$ds^2 = d\rho^2 + \rho^2 d\phi^2,$$

or using Eq (1) to express $d\rho$ in terms of $d\phi$,

$$\left(\frac{ds}{d\phi}\right)^2 = \alpha^2 R_e^2 + \rho^2. \tag{2}$$

It is not straightforward to solve this exactly, and numerical integration would add computational cost to the model; however, we can approximate the solution using a geometric argument. Consider the area occupied by a length s of a given spiral. This can be approximated as the area between concentric circles,

$$\text{Area} = \pi(R_e^2 - \rho(s)^2),$$

but we can also approximate it as the length of the spiral multiplied by the width of each turn,

$$\text{Area} = ps;$$

setting these two equal to each other yields an approximation for $\phi(s)$:

$$\phi(s) = \frac{1}{\alpha}\left[1 - \sqrt{1 - \frac{2\alpha s}{R_e}}\right].$$

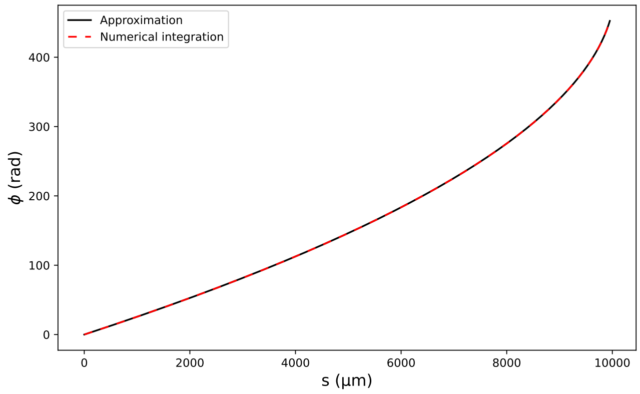

Figure 1: Comparison between approximate $\phi(s)$ and the value computed using numerical integration.

We can check that this approximation is valid by computing $ds/d\phi$ and comparing it to Eq. (2). Doing so we find

$$\frac{ds}{d\phi} = \rho^2;$$

the approximation is therefore valid when the difference between these is negligible, i.e.

$$\rho \gg \alpha R_e = \frac{p}{2\pi};$$

that is, the radial distance we are considering must be larger than the pitch, which is generally satisfied for the spirals we consider. Figure 1 shows a representative example which demonstrates the very good agreement between this approximation and the result of a numerical integration.

II. RESONANT FREQUENCY CALCULATION

Following [1, 2], we express the vector potential due to the spiral at a point (r, θ) in the plane of the spiral as

$$\vec{A}(r,\theta) = \frac{\mu_0 I}{4\pi}\int \frac{e^{-ikR}}{R}\psi(s)d\vec{s} \tag{3}$$

where $k = \omega/c_eff$, $\psi(s)$ is the current distribution along the spiral, and I is the amplitude of the current. The distance R is given by

$$R = \sqrt{r^2 + R_e^2(1-\alpha\phi)^2 - 2rR_e(1-\alpha\phi)\cos(\phi-\theta)};$$

note that the ϕ in this equation is the angle corresponding to position s along the spiral, hence the need for our approximation.

The electric field components in the plane of the spiral are

$$E_r = \frac{1}{i\omega\epsilon_0\mu_0}\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rA_r)\right]$$

and

$$E_\theta = -i\omega A_\theta;$$

the boundary condition that must be satisfied is for the electric field along the direction of the spiral to be zero,

$$R_e\alpha E_r + rE_\theta = 0, \tag{4}$$

and it must also be the case that the current at both ends of the spiral is zero, i.e.

$$\psi(0) = \psi(s_{\max}) = 0. \tag{5}$$

In the cases where $R_e - R_i \ll R_e$ [1] or $R_i = 0$ [2], Maleeva et al. apply a set of approximations that allow for an analytic solution for the frequency ω as well as $\psi(s)$. In our case, however, we wish to model the intermediate region, where these approximations are no longer valid. We therefore make no approximations and instead solve for ω and $\psi(s)$ numerically. The current distribution $\psi(s)$ is parameterized as a sum of sine functions:

$$\psi(s) = \sum_n c_n \sin\left(\frac{n\pi s}{s_{\max}}\right)$$

up to some finite order; note that this parameterization automatically enforces boundary condition (4). The process for finding a numerical solution is:

1. Given values for ω and each coefficient c_n, compute the vector potential and the electric field due to the spiral via numerical integration of Eq. (**??**) and numerical differentiation of the result, respectively. This is done for a set of points drawn from the area of the spiral, $\{(r_i, \theta_i)\}$.

2. At each of these points compute $E_i^{||} = R_e\alpha E_{r_i} + r_i E_{\theta_i}$, i.e. the electric field in the direction along the wire

[1] N. Maleeva, M. V. Fistul, A. Karpov, A. P. Zhuravel, A. Averkin, P. Jung, and A. V. Ustinov, Journal of Applied Physics **115**, 064910 (2014).
[2] N. Maleeva, A. Averkin, N. N. Abramov, M. V. Fistul, A. Karpov, A. P. Zhuravel, and A. V. Ustinov, Journal of Applied Physics **118**, 033902 (2015).