Discrete Mathematics Summary/Guide

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Discrete Maths

Graphs

(+ the Pigeonhole principle
- not on midterm
- review this lecture)

(+ Induction
- review Induction I lecture)

Set Theory

- sets = unordered structures of unique elements
  - ↳ cardinality = # elems
    - $|\emptyset|$ (empty set) = 0
    - $|\mathbb{N}|$ (or any other infinite set) = $\aleph_0$ (aleph-naught)
- $x \in A$ = x is an elem of A
  - $x \subseteq A$ = x is a subset of A (To prove $S \subseteq T$, pick arbitrary $x \in S$ and prove $x \in T$ + if $x \in T$ then $S \subseteq T$)
- set operations
  - union = $A \cup B$ — if $x \in S \cup T$, then $x \in S \vee x \in T$
  - intersection = $A \cap B$ — if $x \in S \cap T$, then $x \in S \wedge x \in T$
  - difference = $A - B$ — if $x \in S - T$, then $x \in S \wedge x \notin T$
  - symmetric difference = $A \Delta B$ — if $x \in S \Delta T$, then $(x \in S \wedge x \notin T) \vee (x \notin S \wedge x \in T)$
  - Powerset = $P(A)$ — $S = \{a, b\}$, $P(S) = \{\{a, b\}, \{a\}, \{b\}, \emptyset\}$
    - ↳ $|P(A)| > |A|$, always. To prove $T \in P(S)$, prove $T \subseteq S$

set equality = if $S = T$, then $S \subseteq T$ & $T \subseteq S$

Mathematical Logic

- propositions = a statement that is either T / F
  - ↳ prop. variables represent ^ like a boolean
- (material) logical implication = statement of the form "If P, then Q" — $P \to Q$ — implications are connections of propositions
  - $P \to Q$, $P \subseteq Q$
  - $!Q \to !P$ (contrapositive) — not P, not Q, $!Q \subseteq !P$
  - ↳ biconditionals = "if & only if", two-way implications — $P \leftrightarrow Q$
  - ↳ vacuous truth = when the antecedent is false and no contradiction can be found to prove an implication is false

P	Q	P→Q
F	F	T
F	T	T
T	F	F
T	T	T

Propositional logic

- propositional connectives = any operation w/ propositions to produce a statement whose bool value depends on each prop. var involved (in order ↓)
  - $\neg$ = "not" / ! / negation
  - $\wedge$ = "and" / && / conjunction
  - $\vee$ = "or" / || / disjunction
  - $\to$ = "implies" / implication
  - $\leftrightarrow$ = "iff" / biconditional
  - $\top$ = Truth
  - $\bot$ = Falsity

*Guide to negations

$\dfrac{\neg(A \vee B)}{\neg A \wedge \neg B}$ $\dfrac{\neg(A \wedge B)}{\neg A \vee \neg B}$ (deMorgan's Laws) or use $\dfrac{\neg(A \wedge B)}{A \to \neg B}$

$\dfrac{\neg(A \to B)}{A \wedge \neg B}$ $\dfrac{\neg(A \leftrightarrow B)}{\neg A \leftrightarrow B}$ or use $\dfrac{\neg(A \leftrightarrow B)}{A \leftrightarrow \neg B}$

$\dfrac{\neg \forall x. A}{\exists x. \neg A}$ $\dfrac{\neg \exists x. A}{\forall x. \neg A}$ $\dfrac{\neg \forall x \in S. A}{\exists x \in S. \neg A}$ $\dfrac{\neg \exists x \in S. A}{\forall x \in S. \neg A}$

- You always just keep on "pushing in" the logic negation in the logic statement as far as you can go

Functions

$\forall a \in A. \exists b \in B. f(a) = b$
$\forall a_1 \in A\ \forall a_2 \in A. (a_1 = a_2 \to f(a_1) = f(a_2))$

- take in one input and output one value = deterministic — $f: A \to B$ / $B f(A x)$ (domain, codomain) (image & range are subsets of domain & codomain, respectively) (the preimage ($f^{-1}[Y]$) is the set of inputs that output the elements in specified set Y)
- you can take composites of functions ($g(f(x)) = (g \circ f)(x)$. if $g: B \to C$ & $f: A \to B$, then $g \circ f: A \to C$)
- injectives = every output has no more than one input — $\forall a_1 \in A. \forall a_2 \in A\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$; $\forall a_1 \in A. \forall a_2 \in A\ (f(a_1) = f(a_2) \to a_1 = a_2)$ — $|A| \leq |B|$ (composites of injectives are injective)
- surjectives = every output has at least one input — $\forall b \in B. \exists a \in A\ f(a) = b$ — $|A| \geq |B|$ (composites of surjectives are surjectives)
- bijectives = every output has exactly one input (injective && surjective) — $|A| = |B|$ → more in case reader abt relationship between cardinality & functions
- true inverse of a function = left & right inverse, only one per invertible function
  - ↳ left inverse ($\forall a \in A. g(f(a)) = a$), f must be injective — f(x), g(x), h(x)
  - ↳ right inverse ($\forall b \in B. f(g(b)) = b$), f must be surjective — A, B, f(x), h(x), g(x)
- involutions: $f(f(n)) = n$

Functions are a kind of binary relation

Relations

aRb if (some property) for a & b holds (→ in A)
uses "if" but definitely an equivalence / ↔ statement so you CAN use negation of this!

- Binary Relation = a comparison between an ordered pair from a set A
- Equivalence relations: (prove all three:)
  - ↳ reflexive $\forall x \in A. xRx$
  - ↳ symmetric $\forall x \in A. \forall y \in A\ (xRy \to yRx)$
  - ↳ transitive $\forall x \in A. \forall y \in A. \forall z \in A. (xRy \wedge yRz \to xRz)$

partitions
equivalence classes
system of Representatives

- strict order relations: (prove all three:)
  - ↳ irreflexive $\forall x \in A. x \not R x$
  - ↳ asymmetric $\forall x \in A. \forall y \in A. (xRy \to y \not R x)$ (irreflexive & transitive implies asymmetry)
  - ↳ transitive $\forall x \in A. \forall y \in A. \forall z \in A\ (xRy \wedge yRz \to xRz)$

Prerequisites