SOLUTION OF SELECTED PROBLEMS FROM THE BOOK: FUNDAMENTALS OF ALGORITHMICS OF GUILLES BRASSARD & PAUL BRATLEY

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Download the `solution.pdf` file from:

https://github.com/lizard20/FundamentalsOfAlgorithmics

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INTRODUCTION

Problem 1.1. The word “algebra” is also connected with the mathematician al-Koârizmî, who gave his name to algorithms. What is the connection?
Solution:
To Muhammad ibn Musa al-Khwârizmî we owed the terms algebra and algorithm. Al-Khwarizmi’s popularizing treatise on algebra “The Compendious Book on Calculation by Completion and Balancing”, presented the first systematic solution of linear and quadratic equations. The term “algebra” is derived from the name of one of the basic operations with equations (al-jabr, meaning “restoration”, referring to adding a number to both sides of the equation to consolidate or cancel terms) described in this book.

The term “algorithm” is derived from the algorism, the technique of performing arithmetic with Hindu-Arabic numerals developed by al-Khwârizmî. Both “algorithm” and “algorism” are derived from the Latinized forms of al-Khwârizmî’s name, Algoritmi and Algorismi, respectively. [1]

Problem 1.5. Use multiplication *à la russe* multiply (a) 63 by 123, and (b) 64 by 123.
Solution:

	(a)	
63	123	123
31	246	246
15	492	492
7	984	984
3	1968	1968
1	3936	3936
		7749

	(b)	
64	123	~~123~~
32	246	~~246~~
16	492	~~492~~
8	984	~~984~~
4	1968	~~1968~~
2	3936	~~3936~~
1	7872	7872
		7872

1 Taken from Wikipedia, click here

Problem 1.6. Find a pocket calculator accurate to at least eight figures, that is, which can multiply a four-figure number by a four-figure number and get the correct the eight-figure answer. You are required to multiply 31415975 by 8182818. Show the divide-and-conquer multiplication algorithm of Section 1.2 can be used to reduce the problem to a small number of calculations that you can do on your calculator, follow by a simple paper-and-pencil addition. Carry out the calculation. *Hint:* Don't do it recursively!
Solution:

	Multiply		Shift	Result
i.)	0818	3141	8	2569338 . . . .
ii.)	0818	5975	4	4887550 . . . .
iii.)	2818	3141	4	8851338 . . . .
iv.)	2818	5975	0	16837550
				257071205717550

Problem 1.10. Are the two sets $X = \{1,2,3\}$ and $Y = \{2,1,3\}$ equal?
Solution:
Yes, two sets are equal if they have exactly the same elements. And it doesn't matter the order in which the items are placed.

Problem 1.11. Which of the following sets are finite:? $\varnothing$, $\{\varnothing\}$, $\mathbb{N}$, $\{\mathbb{N}\}$
What is the cardinatity of those among the above sets that are finite?
Solution:
$\varnothing$, the empty set is finite. And its cardinality is: $|\varnothing| = 0$.
$\{\varnothing\}$, this set has only one element; the empty set, therefore it is finite. Its cardinality is: $|\{\varnothing\}| = 1$
$\mathbb{N}$, it is an infinite and countable set. Its cardinality is infinite. It has a special symbol to denote its cardinality, $\aleph_0$. $|\mathbb{N}| = \aleph_0$
$\{\mathbb{N}\}$, this set has one element that is another set, it implies that the set is finite. The cardinality is: $|\{\mathbb{N}\}| = 1$

Problem 1.12. For which values of Boolean variable p, q and r is the Boolean formula $(p \wedge q) \vee (\neg q \wedge r)$ true?
Solution:

p	q	r	$\neg q$	$p \wedge q$	$\vee$	$\neg q \wedge r$
T	T	T	F	T	T	F
T	T	F	F	T	T	F
T	F	T	T	F	T	T
T	F	F	T	F	F	F
F	T	T	F	F	F	F
F	T	F	F	F	F	F
F	F	T	T	F	T	T
F	F	F	T	F	F	F

Problem 1.15. Prove that

$$\begin{aligned}
\log_a(xy) &= \log_a(x) + \log_a(y) \\
\log_a x^y &= y\log_a(x) \\
\log_a x &= \frac{\log_b x}{\log_z a} \\
x^{\log_b y} &= y^{\log_b x}
\end{aligned}$$

Solution:

$$\circ \log_a(xy) = \log_a(x) + \log_a(y)$$

Be

$$\begin{aligned}
\log_a x = m &\Rightarrow a^m = x \\
\log_a y = n &\Rightarrow a^n = y
\end{aligned}$$

then

$$\begin{aligned}
\log_a(xy) &= \log_a(a^m a^n) = \log_a(a^{m+n}) = m + n \\
\log_a(xy) &= m + n
\end{aligned}$$

then

$$\log_a(xy) = \log_a x + \log_a y \quad \blacksquare$$

$$\circ \log_a x^y = y\log_a(x)$$

Be

$$\log_a x = m \Rightarrow a^m = x$$

then

$$\begin{aligned}
\log_a(x^y) &= \log_a(a^m)^y = \log_a(a^{my}) = my = \log_a xy \\
\log_a(x^y) &= y\log_a x \quad \blacksquare
\end{aligned}$$

$$\circ \log_a x = \frac{\log_b x}{\log_b a}$$

Be

$$\begin{aligned}
\log_b x = m &\Rightarrow b^m = x \\
\log_b a = n &\Rightarrow b^n = a \\
\log_a x = s &\Rightarrow a^s = x
\end{aligned}$$

then

$$\begin{aligned}
a^s &= x \\
(b^n)^s &= x \\
b^{ns} &= x \\
x &= b^{ns} \\
b^m &= b^{ns} \Rightarrow m = ns \\
m &= ns \\
\log_b x &= \log_b a \log_a x \\
\log_a x &= \frac{\log_b x}{\log_b a} \quad \blacksquare
\end{aligned}$$

$$\circ x^{log_b y} = y^{log_b x}$$

We have

$$a^{\log_a x} = x$$

then

$$\begin{aligned}
x^{log_b y} &= (b^{log_b x})^{\log_b y} = (b^{\log_b y})^{\log_b x} \\
x^{log_b y} &= y^{\log_b x} \quad \blacksquare
\end{aligned}$$

Problem 1.16. Prove that $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$ for every real number x.
Solution:

$$\begin{aligned}
\lfloor x \rfloor &= n \ \text{ iff } \ n \leq x < n+1 \ \text{ for } \ x \in \mathbb{R}, n \in \mathbb{Z} \\
n \leq x & \ \text{ and } \ x < n+1 \\
x &< n+1 \\
x - 1 &< n \\
x - 1 &< n \leq x \\
x - 1 &< \lfloor x \rfloor \leq x \qquad (1) \\
\lceil x \rceil &= n \ \text{ iff } \ n-1 < x \leq n \ \text{ for } \ x \in \mathbb{R}, n \in \mathbb{Z} \\
n - 1 < x & \ \text{ and } \ x \leq n \\
n - 1 &< x \\
n &< x + 1 \\
x \leq n &< x + 1 \\
x \leq \lceil n \rceil &< x + 1 \qquad (2)
\end{aligned}$$

From (1) and (2), we have:

$$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1 \quad \blacksquare$$

Problem 1.17. An alternative proof fo Theorem 1.5.1, to the effect that there ar infinitely many primes, begins as follows. Assume for

contradiction that the set of prime numbers is finite. Let p be the largest prime. Consider $x = p!$ and $y = x + 1$. Your problem is to complete the proof from here and to distill from your proof an algorithm *Biggerprime*(p) that finds a prime larger than p. the proof of termination for your algorithm must be obvios, as well as the fact that it returns a value larger than p.
Solution:
Be all the prime numbers $p_1, p_2, \ldots, p$, where p is the largest prime number.
We have $x = p!$ and $y = x + 1$.
$y = 1 \times 2 \times 3 \times \ldots \times (p-1) \times p + 1$, is an integer and larger than all prime numbers.
Now, y is not a prime number, since p is the largest prime number, and $y > p$.

As y is not a prime number, then it is a composite number. So, exists a prime number, p_i, that is a factor of y, that is, p_i divides y: $\frac{y}{p_i} = \frac{x!}{p_i} + \frac{1}{p_i}$, p_i divides $x!$ but not divides 1, therefore y is not divisible; it is not an integer. This can be done with another prime number p_j and we'll get the same outcome that y is not divisible.
So, if there is not a prime number which divides y, then y is just divisible between 1 and itself, it means that y is a prime number.
But, we had established the y was a composite number. Here, there is a contradiction, a number cannot be a prime and a composite number at the same time. Then, the assumption that prime numbers are finite is false. Therefore, the conclusion is the opposite, there are infinite prime numbers. ∎

Problem 1.21. Prove by mathematical induction that the sums of the cubes of the first n positive integers is equal to the square of the sum of these integers.
Solution:

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2 \tag{1}$$

Base case: $n = 1$

$$\begin{aligned}\sum_{i=1}^{1} i^3 &= \left(\sum_{i=1}^{1} i\right)^2 \\ 1^3 &= (1)^2 \\ 1 &= 1\end{aligned}$$

Induction hypothesis: Assume that the equation is true for $n = k$, where $k \geq 1$

$$\sum_{i=1}^{k} i^3 = \left(\sum_{i=1}^{k} i\right)^2 \tag{2}$$

Induction step: We have to prove that it is true for $n = k+1$, where $k \geq 1$

$$\sum_{i=1}^{k+1} i^3 = \left(\sum_{i=1}^{k+1} i\right)^2 \tag{3}$$

$$\sum_{i=1}^{k} i^3 + (k+1)^3 = \left(\sum_{i=1}^{k} i + (k+1)\right)^2 \tag{4}$$

We know

$$\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4} \tag{5}$$

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2} \tag{6}$$

Substituting (5) and (6) in (4), we have

$$\frac{k^2(k+1)^2}{4} + (k+1)^3 = \left(\frac{k(k+1)}{2} + (k+1)\right)^2$$
$$\frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{k^2(k+1)^2}{4} + k(k+1)^2 + (k+1)^2$$

Dividing by $(k+1)^2$

$$\frac{k^2}{4} + k + 1 = \frac{k^2}{4} + k + 1 \quad \blacksquare$$

Problem 1.22. Following problem 1.21, prove that the sum of the cubes of the first n positive integers is equal to the square of the sum of these integers, but now use Proposition 1.7.13 rather than mathematical induction (Of course, Proposition 1.7.13 was proved by mathematical induction, too!)
Solution:

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2 \tag{1}$$

> Proposition 1.7.13 *For any integer* $k \geq 0$ *we have*
>
> $$\sum_{i=1}^{n} i(i+1)\ldots(i+k) = n(n+1)\ldots(n+k+1)/(k+2).$$

For $k = 0$:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\left(\sum_{i=1}^{n} i\right)^2 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\left(\sum_{i=1}^{n} i\right)^2 = \frac{n^4 + 2n^3 + n^2}{4} \tag{2}$$

For $k = 2$:

$$\sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

$$\sum_{i=1}^{n} i^3 + 3i^2 + 2i = \frac{n(n+1)(n+2)(n+3)}{4}$$

$$\sum_{i=1}^{n} i^3 = \frac{n(n+1)(n+2)(n+3)}{4} - 3i^2 - 2i$$

$$\sum_{i=1}^{n} i^3 = \frac{n^4 + 6n^3 + 11n^2 + 6n}{4} - 3\frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^4 + 6n^3 + 11n^2 + 6n}{4} - n^3 - \frac{5n^2}{2} - \frac{3n}{2}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^4 + 6n^3 + 11n^2 + 6n - 4n^3 - 10n^2 - 6n}{4}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^4 + 2n^3 + n^2}{4} \tag{3}$$

The second members of (2) and (3) are equal, therefore equality (1) has been proven to be true. ■

Problem 1.23. Determine by induction all the positive integers of n for which $n^3 > 2^n$. Prove your claim by mathematical induction.
Solution:
The table shows that from $n \geq 10$, $n^3 < 2^n$.

n	n^3	2^n
1	1	2
2	8	4
3	27	8
4	64	16
5	125	32
6	216	64
7	343	128
8	512	256
9	729	512
10	**1000**	**1024**
11	1331	2048
12	1728	4096

Now, we have to prove by mathematical induction that $2^n > n^3$ for $n \geq 10$

Base case: $n = 10$

$$2^n > n^3 \tag{1}$$
$$2^{10} > 10^3$$
$$1024 > 1000$$

Induction Hypothesis: Assume that the inequality is true for some $n = k$, where $k \geq 10$

$$2^k > k^3 \tag{2}$$

Induction step: We have to prove that it is true for $n = k + 1$

$$\begin{aligned} 2^{k+1} &> (k+1)^3 \qquad (3) \\ 2^{k+1} &> (k^3 + 3k^2 + 3k + 1) \end{aligned}$$

Now, we have

$$2^{k+1} = 2^k 2 \tag{4}$$

By Induction Hypothesis

$$\begin{aligned} 2^k 2 &> k^3 2 \qquad (5) \\ &> k^3 + k^3 \\ &> k^3 + kk^2 \end{aligned}$$

As $k \geq 10$, we have:

$$k^3 + kk^2 \geq k^3 + 10k^2$$

We have:

$$k^3 + 10k^2 = k^3 + 3k^2 + 3k^2 + 4k^2$$

For $k \geq 10$, we have:

$$3k^2 > 3k$$

and

$$4k^2 > 1$$

Then, we have

$$\begin{aligned} k^3 + 10k^2 &> k^3 + 3k^2 + 3k + 1 \\ k^3 + 10k^2 &> (k+1)^3 \qquad (6) \end{aligned}$$

From (4) and (6), we conclude that:

$$2^{k+1} > (k+1)^3 \quad \blacksquare$$

Problem 1.27. Recall that the *Fibonacci sequence* is defined as

$$\begin{cases} f_0 = 0; f_1 = 1 & \text{and} \\ f_n = f_{n-1} + f_{n-2} & \text{for } n \geq 2 \end{cases}$$

Prove by generalized mathematical induction that

$$f_n = \frac{1}{\sqrt{5}}[\phi^n - (-\phi)^{-n}],$$

where

$$\phi = \frac{1+\sqrt{5}}{2}$$

is the *golden ratio*. (This is known as the De Moivre's formula.)
Solution:
We have:

$$\begin{aligned}
\phi &= \phi^{-1} + 1 \\
\phi^{-1} &= \phi - 1 = \frac{1+\sqrt{5}}{2} - 1 = \frac{1+\sqrt{5}-2}{2} \\
\phi^{-1} &= -\frac{1-\sqrt{5}}{2} \qquad (1)
\end{aligned}$$

$$\begin{aligned}
f_n &= \frac{1}{\sqrt{5}}[\phi^n - (-\phi)^{-n}] \\
&= \frac{1}{\sqrt{5}}[\phi^n - (-\phi^{-1})^n] \qquad (2)
\end{aligned}$$

substituting ϕ and ϕ^{-1} into (2)

$$\begin{aligned}
f_n &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(-\left(-\frac{1-\sqrt{5}}{2}\right)\right)^n\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \qquad (3)
\end{aligned}$$

From (3) and for $n = 0$:

$$f_0 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^0 - \left(\frac{1-\sqrt{5}}{2}\right)^0\right] = \frac{1}{\sqrt{5}}(1-1) = 0$$

$n = 1$:

$$\begin{aligned}
f_1 &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right] = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}-1+\sqrt{5}}{2}\right] \\
&= \frac{1}{\sqrt{5}}\sqrt{5} = 1
\end{aligned}$$

$n = 2$:

$$\begin{aligned}
f_2 &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right] = \frac{1}{\sqrt{5}}\left[\frac{1+2\sqrt{5}+5-1+2\sqrt{5}-5}{4}\right] \\
&= \frac{1}{\sqrt{5}}\frac{4\sqrt{5}}{4} = 1
\end{aligned}$$

Now, we prove for n assuming that (3) is true for $n-1$ and $n-2$

$$
\begin{aligned}
f_n &= f_{n-1} + f_{n-2} \\
f_n &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \right. \\
&\left.\left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1} + \left(\frac{1+\sqrt{5}}{2}\right)^{n-2} - \right. \\
&\left.\left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left(1+\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\left(1+\frac{1-\sqrt{5}}{2}\right)\right]
\end{aligned} \tag{4}
$$

on the other hand we have:

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} = \frac{3}{2}+\frac{\sqrt{5}}{2} = 1+\frac{1}{2}+\frac{\sqrt{5}}{2}$$

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = 1+\frac{1+\sqrt{5}}{2} \tag{5}$$

$$\left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{1-2\sqrt{5}+5}{4} = \frac{6-2\sqrt{5}}{4} = \frac{3}{2}-\frac{\sqrt{5}}{2} = 1+\frac{1}{2}-\frac{\sqrt{5}}{2}$$

$$\left(\frac{1-\sqrt{5}}{2}\right)^2 == 1+\frac{1-\sqrt{5}}{2} \tag{6}$$

substituting (1) and (2) into (1)

$$
\begin{aligned}
f_n &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^{n-2}\left(\frac{1-\sqrt{5}}{2}\right)^2\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \left(\frac{1-\sqrt{5}}{2}\right)^{n}\right] \quad \blacksquare
\end{aligned}
$$

Problem 1.28. Following Problem 1.27, prove by mathematical induction that $f_n > (\frac{3}{2})^n$ for all sufficiently large integer n. How large does n have to be? Do not use de Moivre's formula.
Solution:

n	f_n	$(\frac{3}{2})^n$	n	f_n	$(\frac{3}{2})^n$
0	0	1.0	7	13	17.09
1	1	1.5	8	21	25.63
2	1	2.25	9	34	38.44
3	2	3.38	10	55	57.67
4	3	5.06	**11**	**89**	**86.5**
5	5	7.59	12	144	129.75
6	8	11.39	13	233	194.62

From the table, we can see that from $n \geq 11$:

$$f_n > \left(\frac{3}{2}\right)^n \tag{1}$$

The Finonacci sequence is defined as:

$$\begin{cases} f_0 = 0; f_1 = 1 & \text{and} \\ f_n = f_{n-1} + f_{n-2} & \text{for } n \geq 2 \end{cases} \tag{2}$$

Let's prove by mathematical induction the (1) inequality

Base case: $n = 11$:

From (2) and the lhs of (1), we have

$$\begin{aligned} f_{11} &= f_{10} + f_9 \\ f_{11} &= 55 + 34 \\ f_{11} &= 89 \end{aligned} \tag{3}$$

On the rhs of (1), we have:

$$\left(\frac{3}{2}\right)^{11} = 86.497 \tag{4}$$

From (3) and (4), we have:

$$\begin{aligned} f_{11} &> \left(\frac{3}{2}\right)^{11} \\ 89 &> 86.497 \end{aligned}$$

Induction Hypothesis: Assume that the inequality is true for some $n = k$, where $k \geq 11$

$$f_k > \left(\frac{3}{2}\right)^k \tag{5}$$

Induction step: we have to prove that it is true for $n = k+1$, where $k \geq 11$

$$f_{k+1} > \left(\frac{3}{2}\right)^{k+1} \tag{6}$$

From (2), we have

$$f_{k+1} = f_k + f_{k-1} \tag{7}$$

From (5), we have

$$\begin{aligned} f_{k+1} &> \left(\frac{3}{2}\right)^{k} + \left(\frac{3}{2}\right)^{k-1} \\ &> \left(\frac{3}{2}\right)^{k-1}\left(\frac{3}{2}+1\right) \end{aligned} \tag{8}$$

Likewise, we have

$$\begin{aligned} \frac{3}{2}+1 &= \frac{5}{2} \\ &= \frac{9}{4}+\frac{1}{4} \\ &> \frac{9}{4} \\ &> \left(\frac{3}{2}\right)^{2} \\ \frac{3}{2}+1 &> \left(\frac{3}{2}\right)^{2} \end{aligned} \tag{9}$$

From (8) and (9), we have

$$\begin{aligned} \left(\frac{3}{2}\right)^{k-1}\left(\frac{3}{2}+1\right) &> \left(\frac{3}{2}\right)^{k-1}\left(\frac{3}{2}\right)^{2} \\ &> \left(\frac{3}{2}\right)^{k+1} \end{aligned} \tag{10}$$

Finally, from (8) and (10), we conclude that

$$f_{k+1} > \left(\frac{3}{2}\right)^{k+1} \quad \blacksquare$$

Problem 1.36. Use the l'Hôpital's rule to find the limits as n tends to infinity of $(\log\log n)^a / \log n$, where $a > 0$ is an arbitrary positive constant.
Solution:
We apply the l'Hôpital's rule successively until the indeterminacy

disappear.
We take the first derivative:

$$\frac{d}{dx}\frac{(\log(\log x))^a}{\log x} = \frac{a(\log(\log x))^{a-1}\dfrac{1}{\log x}\dfrac{1}{x}}{\dfrac{1}{x}}$$
$$= \frac{a(\log(\log x))^{a-1}}{\log x}$$

the second derivative:

$$\frac{d^2}{dx^2}\frac{a(\log(\log x))^{a-1}}{\log x} = \frac{a(a-1)(\log(\log x))^{a-2}\dfrac{1}{\log x}\dfrac{1}{x}}{\dfrac{1}{x}}$$
$$= \frac{a(a-1)(\log(\log x))^{a-2}}{\log x}$$

the $(a-1)th$ derivative

$$\frac{d^{(a-1)}}{dx^{(a-1)}}\frac{(\log(\log x))^a}{\log x} = \frac{a(a-1)\ldots 2\ \log(\log x)}{\log x}$$

and finally the *ath* derivative:

$$\frac{d^{(a)}}{dx^{(a)}}\frac{(\log(\log x))^a}{\log x} = \frac{a(a-1)\ldots 2\ \dfrac{1}{\log x}\dfrac{1}{x}}{\dfrac{1}{x}}$$
$$= \frac{a(a-1)\ldots 2}{\log x} \tag{1}$$

the indeterminacy disappear after the *ath* derivative.

According to l'Hôpital's rule:

$$\lim_{x\to\infty}\frac{(\log(\log x))^a}{\log x} = \lim_{x\to\infty}\frac{\dfrac{d^{(a)}}{dx^{(a)}}(\log(\log x))^a}{\dfrac{d^{(a)}}{dx^{(a)}}\log x}$$

from (1)

$$\lim_{x\to\infty}\frac{(\log(\log x))^a}{\log x} = \lim_{x\to\infty}\frac{a(a-1)\ldots 2}{\log x} = 0$$
$$\lim_{n\to\infty}\frac{(\log(\log n))^a}{\log n} = 0$$

Problem 1.38. Give a simple proof, not using integrals, that the harmonic series diverges.
Solution:

$$H_n = \sum_{i=1}^{\infty} \frac{1}{n}$$

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{1} + \frac{1}{12} + \dots \tag{1}$$
$$H_n = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \dots + \frac{1}{16}\right) + \dots$$

Building G_n as follows:

$$G_n = 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \dots \frac{1}{16}\right) + \dots \tag{2}$$

the terms in parentheses of G_n are less than respective terms in H_n, therefore:

$$H_n > G_n \tag{3}$$

$$\begin{aligned} H_n &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \dots \frac{1}{16}\right) + \dots \\ &= 1 + \frac{1}{2} + \left(\frac{2}{4}\right) + \left(\frac{4}{8}\right) + \left(\frac{8}{16}\right) + \dots \\ &= 1 + \frac{1}{2} + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \dots \end{aligned}$$

We see that the sum of terms in parentheses is $1/2$ and the sum of these terms growths indefinetely, therefore G_n diverges. As $H_n > G_n$ then H_n also diverges. ■

Problem 1.41. Prove that $\binom{n}{r} \leq 2^{n-1}$ for $n > 0$ and $0 \leq r \leq n$.
Solution:

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n$$

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n-2} + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

the number of binomial coefficients is $n+1$. When this number is even, every coefficient is repeated twice. When this number is odd, there is one coefficient that does not repeat, when $r = n/2$, where n is even. Then, we can always choose two repeated binomial coefficients, and in the case that the number of binomial coefficients is odd avoid the coefficient when $r = n/2$. The repeating coefficients are:

$$\binom{n}{r} \text{ and } \binom{n}{n-r} \tag{1}$$

we can show that these two coefficients are equal

$$\begin{aligned}\binom{n}{r} &= \binom{n}{n-r}\\ &= \frac{n!}{(n-r)!(n-(n-r))!}\\ &= \frac{n!}{(n-r)!r!}\\ \binom{n}{r} &= \binom{n}{r}\end{aligned}$$

then, if we pick out two repeated binomial coefficients, we have

$$\begin{aligned}\binom{n}{r} + \binom{n}{n-r} &\leq \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n}\\ \binom{n}{r} + \binom{n}{r} &\leq 2^n\\ 2\binom{n}{r} &\leq 2^n\\ \binom{n}{r} &\leq 2^{n-1} \quad \blacksquare\end{aligned}$$

Problem 1.43. Show that $\sum_{r=1}^{n} r\binom{n}{r} = n2^{n-1}$ for $n > 0$.
Hint: Differentiate both sides of Theorem 1.7.20.

Solution:

> Proposition 1.7.20 (Newton). *Let n be a positive integer. Then*
>
> $$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n-1}x^{n-1} + x^n.$$

We have:

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n-1}x^{n-1} + x^n$$

differentiating

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + \ldots + (n-1)\binom{n}{n-1}x^{n-2} + nx^{n-1}$$

Be $x = 1$

$$n2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + \ldots + (n-1)\binom{n}{n-1} + n \qquad (1)$$

on the other hand we have

$$\begin{aligned}\sum_{r=1}^{n} r\binom{n}{r} &= \binom{n}{1} + 2\binom{n}{2} + \ldots + (n-1)\binom{n}{n-1} + n\binom{n}{n} \\ &= \binom{n}{1} + 2\binom{n}{2} + \ldots + (n-1)\binom{n}{n-1} + n \qquad (2)\end{aligned}$$

From (1) and (2) we have

$$n2^{n-1} = \sum_{r=1}^{n} r\binom{n}{r}$$

$$\sum_{r=1}^{n} r\binom{n}{r} = n2^{n-1} \quad \blacksquare$$

Problem 1.44. Show that $\dfrac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 + \ldots$ for $-1 < x < 1$.
Hint. Use Proposition 1.7.12
Solution:

Proposition 1.7.12. *The infinite series* $r + 2r^2 + 3r^3 + \ldots$ *converges when* $-1 < r < 1$, *and in this case its sum is* $r/(1-r)^2$

We have:

$$\frac{r}{(1-r)^2} = r + 2r^2 + 3r^3 + 4r^4 + 5r^5 \ldots \qquad (1)$$

dividing by $r,\ r \neq 0$ (2)

$$\frac{1}{(1-r)^2} = 1 + 2r + 3r^2 + 4r^3 + 5r^4 \ldots \qquad (3)$$

Substituting $-x$ for r in (2)

$$\frac{1}{(1-(-x))^2} = 1 + 2(-x) + 3(-x)^2 + 4(-x)^3 + 5(-x)^4 \ldots$$

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 + \ldots \quad \blacksquare$$

Problem 1.45. Show that two mutually exclusive events are *not* independent except in the trivial case that at least one of them has probability zero.
Solution:
If two events A and B are mutually exclusive, then

$$A \cap B = \varnothing \qquad (1)$$

On the other hand, two events A and B are independent if:

$$Pr(A \cap B) = Pr(A)Pr(B) \tag{2}$$

substituying (1) in (2)

$$\begin{aligned} Pr(\varnothing) &= Pr(A)Pr(B) \\ 0 &= Pr(A)Pr(B) \end{aligned}$$

this equality is satisfied if at least $Pr(A)$ or $Pr(B)$ is zero. But in the general case neither $Pr(A)$ nor $Pr(B)$ is zero, then

$$Pr(A)Pr(B) \neq 0$$

therefore events A and B are not independent. In conclusion, if A and B are mutually exclusive events, they are not independent except in the trivial case, when at least one of them has a probability zero. ■

Problem 1.46. Consider a random experiment whose sample space S is $\{1, 2, 3, 4, 5\}$. Let A and B the events "the outcome is divisible by 2" and "the outcome is divisible by 3", respectively. What are $Pr[A]$, $Pr[B]$ and $Pr[A \cap B]$? Are the events A and B independent?
Solution:
The events $A = \{2, 4\}$, $B = \{3\}$ and $A \cap B = \varnothing$
The probability of these events are:

$$\begin{aligned} Pr[A] &= \frac{2}{5} \\ Pr[B] &= \frac{1}{5} \\ Pr[A \cap B] &= 0 \end{aligned}$$

Two events are independent if:

$$Pr[A \cap B] = Pr[A]Pr[B]$$

checking:

$$\begin{aligned} 0 &\neq \frac{2}{5} \times \frac{1}{5} \\ 0 &\neq \frac{2}{25} \end{aligned}$$

Then, the two events, A and B, are not independent.

Problem 1.47. Show that for the horse-race of Section 1.7.4, 99% of the time our expected winnings $\overline{w}$ averaged over 50 races lie between -24.3 and $+2.3$.
Solution:
If the distribution is aproximately normal, the 99% of time the expected winning averaged over 50 races lie between $\overline{w} - 2.576 \times \sigma$ and $\overline{w} + 2.576 \times \sigma$. Where $\overline{w} = -11$ and $\sigma = 5.15$

$$\begin{aligned} -11 - 2.576 \times 5.15 \ \text{ and } \ -11 + 2.576 \times 5.15 \\ -24.3 \ \text{ and } \ 2.3 \end{aligned}$$

2

ELEMENTARY ALGORITHMICS

Problem 2.1. Suppose you measure the performance of a program, perhaps using some kind of run-time trace, and then you optimize the heavily-used parts of the code. However, you are careful no to change to the underlying algorithm. Would you expect to obtain (a) a gain in efficiency by a constant factor, whatever the problem being solved, or (b) a gain in efficiency that gets proportionally greater as the problem size increases? Justify your answer.
Solution:
The correct answer is (a). It is because the invariance principle states that the efficiency of two implementations of the same algorithm do not differ by more than a constant. This principle applies to any computer used to implement the algorithm, regardless of the programming language and compiler used. It is even independent of the skill of the programmer, while he (or she) does not modify the algorithm. Simply by changing the algorithm we can improve the efficiency as the size of the problem increases.

Problem 2.2. A sorting algorithm takes one second to sort 1000 items on your local machine. How long would you expect it to take to sort 10000 items. (a) If you believe that the algorithm takes a time roughly proportional to n^2, and (b) if you believe that the algorithm takes a time roughly proportional to $n \log n$?
Solution:
(a) The algorithm time is proportional to n^2, $t \propto n^2$, then $t = cn^2$. Let's calculate the constant c using the number of items and the time:

$$\begin{aligned} 1 &= c\ 1000^2 \\ c &= \frac{1}{1000^2} \\ c &= 10^{-6} \end{aligned}$$

now, calculate the time that the algorithm takes when $n = 10000$:

$$\begin{aligned} t &= 10^{-6}(10000)^2 \\ t &= 10^{-6}10^8 \\ t &= 10^2 \\ t &= 100 \end{aligned}$$

It means that the algorithm takes 100 seconds when $n = 10000$.
(b) If the algorithm is proportional to $n \log n$ then $t = cn \log n$. Let's calculate the constant c:

$$\begin{aligned}
1 &= cn \log n \\
1 &= c(1000) \log 1000 \\
c &= \frac{1}{1000 \log 1000} \\
c &= 1.448 \times 10^{-4}
\end{aligned}$$

Now, calculate the time for $n = 10000$:

$$\begin{aligned}
t &= 1.448 \times 10^{-4}(10000) \log 10000 \\
t &= 13.34
\end{aligned}$$

The algorithm takes 13.34 seconds when $n = 10000$

Problem 2.3. Two algorithms take n^2 days and n^3 seconds respectively to solve an instance of size n. Show that it is only on instances requiring more than 20 million years to solve that the quadratic algorithm outperforms the cubic algorithm.
Solution:
Be $t_1 = n^2$ and $t_2 = n^3$
transforming units:
$s - seconds,\ h - hours,\ d - days,\ y - years$

$$\begin{aligned}
t_1 &= n^2\ d \times \frac{1\ y}{365\ d} \\
t_1 &= 2.74 \times 10^{-3} n^2 \text{ y} \\
t_2 &= n^3\ s \times \frac{1\ h}{3600\ s} \times \frac{1\ d}{24\ h} \times \frac{1\ y}{365\ d} \\
t_2 &= 3.17 \times 10^{-8} n^3 \text{ y}
\end{aligned}$$

although the performance of the quadratic algorithm is better than the cubic algorithm, the constants affect this performance. In this case, there is an instance, n, for which the quadratic algorithm outperforms the cubic algorithm.
Let's find the n value at which t_1 and t_2 are equal:

$$\begin{aligned}
t_1(n) &= t_2(n) \\
2.74 \times 10^{-3} n^2 &= 3.17 \times 10^{-8} n^3 \\
2.74 \times 10^{-3} &= 3.17 \times 10^{-8} n \\
n &= \frac{2.74 \times 10^{-3}}{3.17 \times 10^{-8}} \\
n &= 86435.3
\end{aligned}$$

then, from the instance size $n = 86436$, the quadratic algorithm outperforms the cubic.
Now, let's calculate the time to compute n.

$$\begin{aligned}
t_1(86436) &= 2.74 \times 10^{-3} \times (86436)^2\ y \\
t_1 &= 20471038.9 \text{ y}
\end{aligned}$$

if the calculation is done using t_2, we have:

$$t_2(86436) = 3.17 \times 10^{-8} * (86436)^3$$
$$t_2 = 20471197.3 \text{ y}$$

in both results, the time is around to 20 millons years.

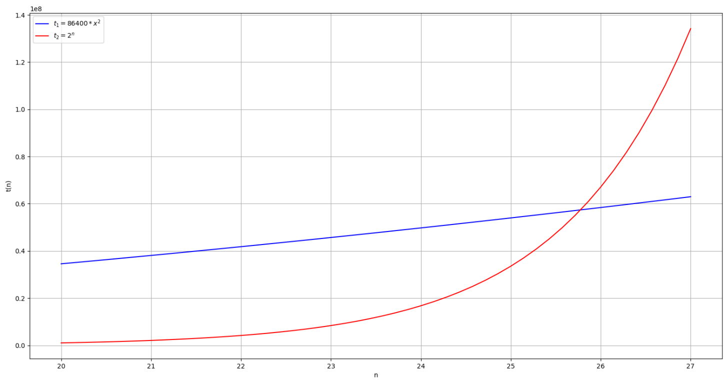
Problem 2.8 Two algorithms take n^2 days and 2^n seconds respectively to solve an instance of size n. What is the size of the smallest instance on which the former algorithm outperforms the latter? Approximately how long does such an instance takes to solve?
Solution:
Be $t_1 = n^2$ days and $t_2 = 2^n$ seconds
transforming units:

$$t_1 = n^2\ d \times \frac{24\ h}{1\ d} \times \frac{3600\ s}{1\ h}$$
$$t_1 = 86400\ n^2\ s$$

let's find the value at which t_1 and t_2 are equal:

$$t_1 = t_2$$
$$86400\ n^2 = 2^n$$

we'll use a graphical method to solve the former equation:



the two curves intersect at n value between 25 and 26.
The answer to the first question about the smallest instance is: $n = 26$
The time required to solve this instance is:

$$t_1 = 84600 \times 26^2$$
$$t_1 = 58406400\ s$$
$$t_1 \approx 1.85 \text{ y}$$

if we calculate the time with t_2:

$$t_2 = 2^{26}$$
$$t_2 = 67108864\ s$$
$$t_2 \approx 2.13 \text{ y}$$

the time taken to solve the instance $n = 26$, depends on the algorithm we used. If we choose the quadratic algorithm it takes 1.85 years to solve the instance. And if we use the exponential algorithm it takes 2.13 years.

Problem 2.9. Simulate both the insertion sort and the selection sort algorithms of Section 2.4 on the following two arrays: $U = [1, 2, 3, 4, 5, 6]$ and $V = [6, 5, 4, 3, 2, 1]$. Does insertion sorting run faster on the array U or the array V? And selection sorting? Justify your answer.
Solution:
Insertion sort algorithm

```
1: procedure insert(T[1..n])
2:     for i ← 2 to n do
3:         x ← T[i]
4:         j ← i − 1
5:         while j > 0 and x < T[j] do
6:             T[j + 1] ← T[j]
7:             j ← j − 1
8:         end while
9:         T[j + 1] ← x
10:    end for
11: end procedure
```

Simulation:
U= [1,2,3,4,5,6]

i	**op**	**array**	***i***	**op**	**array**	***i***	**op**	**array**
2	$i \leftarrow 2$		4	$i \leftarrow 4$		6	$i \leftarrow 6$	
	$x \leftarrow 2$			$x \leftarrow 4$			$x \leftarrow 6$	
	$j \leftarrow 1$			$j \leftarrow 3$			$j \leftarrow 5$	
	$cmp(1,2)$[1]			$cmp(3,4)$			$cmp(5,6)$	
	$U[2] \leftarrow 2$	[1,2,3,4,5,6]		$U[4] \leftarrow 4$	[1,2,3,4,5,6]		$U[6] \leftarrow 6$	[1,2,3,4,5,6]
3	$i \leftarrow 3$		5	$i \leftarrow 5$				
	$x \leftarrow 3$			$x \leftarrow 5$				
	$j \leftarrow 2$			$j \leftarrow 4$				
	$cmp(2,3)$			$cmp(4,5)$				
	$U[3] \leftarrow 3$	[1,2,3,4,5,6]		$U[5] \leftarrow 5$	[1,2,3,4,5,6]			

1 $cmp(j,x)$, refers to line 5 of the insertion algorithm, *while* $(j < 0)$ *and* $(x < T[j])$. It includes three elemental operations, two comparisons and a boolean operation.

V=[6,5,4,3,2,1]

i	op	array	i	op	array	i	op	array
2	$i \leftarrow 2$		4	$j \leftarrow 2$		5	$j \leftarrow 0$	
	$x \leftarrow 5$			$V[3] \leftarrow 5$	[4,5,5,6,2,1]		$cmp(0,2)$	
	$j \leftarrow 1$			$cmp(2,3)$			$V[1] \leftarrow 2$	[2,3,4,5,6,1]
	$cmp(1,5)$			$V[2] \leftarrow 4$	[4,4,5,6,2,1]	6	$i \leftarrow 6$	
	$V[2] \leftarrow 6$	[6,6,4,3,2,1]		$j \leftarrow 1$			$x \leftarrow 1$	
	$j \leftarrow 0$			$cmp(1,3)$			$j \leftarrow 5$	
	$cmp(0,5)$			$V[2] \leftarrow 4$	[4,4,5,6,2,1]		$cmp(5,1)$	
	$V[1] \leftarrow 5$	[5,6,4,3,2,1]		$j \leftarrow 0$			$V[6] \leftarrow 6$	[2,3,4,5,6,6]
3	$i \leftarrow 3$			$cmp(0,3)$			$j \leftarrow 4$	
	$x \leftarrow 4$			$V[1] \leftarrow 3$	[3,4,5,6,2,1]		$cmp(4,1)$	
	$j \leftarrow 2$		5	$i \leftarrow 5$			$V[5] \leftarrow 5$	[2,3,4,5,5,6]
	$cmp(2,4)$			$x \leftarrow 2$			$j \leftarrow 3$	
	$V[3] \leftarrow 6$	[5,6,6,3,2,1]		$j \leftarrow 4$			$cpm(3,1)$	
	$j \leftarrow 1$			$cmp(4,2)$			$V[4] \leftarrow 4$	[2,3,4,4,5,6]
	$cmp(1,4)$			$V[5] \leftarrow 6$	[3,4,5,6,6,1]		$j \leftarrow 2$	
	$V[2] \leftarrow 5$	[5,5,6,3,2,1]		$j \leftarrow 3$			$cmp(2,1)$	
	$j \leftarrow 0$			$cmp(3,2)$			$V[3] \leftarrow 3$	[2,3,3,4,5,6]
	$cmp(0,4)$			$V[4] \leftarrow 5$	[3,4,5,5,6,1]		$j \leftarrow 1$	
	$V[1] \leftarrow 4$	[4,5,6,3,2,1]		$j \leftarrow 2$			$cmp(1,1)$	
4	$i \leftarrow 4$			$cmp(2,2)$			$V[2] \leftarrow 2$	[2,2,3,4,5,6]
	$x \leftarrow 3$			$V[3] \leftarrow 4$	[3,4,4,5,6,1]		$j \leftarrow 0$	
	$j \leftarrow 3$			$j \leftarrow 1$			$cmp(0,1)$	
	$cmp(3,3)$			$cmp(1,2)$			$V[1] \leftarrow 1$	[1,2,3,4,5,6]
	$V[4] \leftarrow 6$	[4,5,6,6,2,1]		$V[2] \leftarrow 3$	[3,3,4,5,6,1]			

Selection sort algorithm

```
procedure select(T[1..n])
    for i ← 1 to n-1 do
        minj ← i
        minx ← T[i]
        for j ← i + 1 to n do
            if T[j] < minx then
                minj ← j
                minx ← T[j]
            end if
            T[minj] ← T[i]
            T[i] ← minx
        end for
    end for
end procedure
```

Simulation:
U=[1,2,3,4,5,6]

i	op	array	i	op	array	i	op	array
1	$i \leftarrow 1$		2	$minx \leftarrow 2$		3	$U[3] \leftarrow 3$	[1,2,3,4,5,6
	$minj \leftarrow 1$			$j \leftarrow 3$			$U[3] \leftarrow 3$	[1,2,3,4,5,6]
	$minx \leftarrow 1$			$U[2] \leftarrow 2$	[1,2,3,4,5,6]		$j \leftarrow 6$	
	$j \leftarrow 2$			$U[2] \leftarrow 2$	[1,2,3,4,5,6]		$U[3] \leftarrow 3$	[1,2,3,4,5,6]
	$U[1] \leftarrow 1$	[1,2,3,4,5,6]		$j \leftarrow 4$			$U[3] \leftarrow 5$	[1,2,3,4,5,6]
	$U[1] \leftarrow 1$	[1,2,3,4,5,6]		$U[2] \leftarrow 2$	[1,2,3,4,5,6]	4	$i \leftarrow 4$	
	$j \leftarrow 3$			$U[2] \leftarrow 2$	[1,2,3,4,5,6]		$minj \leftarrow 4$	
	$U[1] \leftarrow 1$	[1,2,3,4,5,6]		$j \leftarrow 5$			$minx \leftarrow 4$	
	$U[1] \leftarrow 1$	[1,2,3,4,5,6]		$U[2] \leftarrow 2$	[1,2,3,4,5,6]		$j \leftarrow 5$	
	$j \leftarrow 4$			$U[2] \leftarrow 2$	[1,2,3,4,5,6]		$U[4] \leftarrow 4$	[1,2,3,4,5,6]
	$U[1] \leftarrow 1$	[1,2,3,4,5,6]		$j \leftarrow 6$	[3,4,5,6,6,1]		$U[4] \leftarrow 4$	[1,2,3,4,5,6]
	$U[1] \leftarrow 1$	[1,2,3,4,5,6]		$U[2] \leftarrow 2$	[1,2,3,4,5,6]		$j \leftarrow 6$	[2,3,3,4,5,6]
	$j \leftarrow 5$			$U[2] \leftarrow 2$	[1,2,3,4,5,6]		$U[4] \leftarrow 4$	[1,2,3,4,5,6]
	$U[1] \leftarrow 1$	[1,2,3,4,5,6]	3	$i \leftarrow 3$			$U[4] \leftarrow 4$	[1,2,3,4,5,6]
	$U[1] \leftarrow 1$	[1,2,3,4,5,6]		$minj \leftarrow 3$		5	$i \leftarrow 5$	
	$j \leftarrow 6$			$minx \leftarrow 3$			$minj \leftarrow 5$	
	$U[1] \leftarrow 1$	[1,2,3,4,5,6]		$j \leftarrow 4$			$minx \leftarrow 5$	
	$U[1] \leftarrow 1$	[1,2,3,4,5,6]		$U[3] \leftarrow 3$	[1,2,3,4,5,6]		$j \leftarrow 6$	
2	$i \leftarrow 2$			$U[3] \leftarrow 3$	[1,2,3,4,5,6]		$U[5] \leftarrow 5$	[1,2,3,4,5,6]
	$minj \leftarrow 2$			$j \leftarrow 5$			$U[5] \leftarrow 6$	[1,2,3,4,5,6]

V=[6,5,4,3,2,1]

i	op	array	*i*	op	array	*i*	op	array
1	*i*←1		2	*minx*←6		3	*minj*←5	
	minj←1			*j*←3			*minx*←4	
	minx←6			*minj*←3			*V*[5]←5	[1,2,5,6,5,3]
	j←2			*minx*←5			*V*[3]←4	[1,2,4,6,5,3]
	minj←2			*V*[3]←6	[1,6,6,4,3,2,]		*j*←6	
	minx←6			*V*[2]←6	[1,5,6,4,3,2]		*minj*←6	
	V[2]←6	[6,6,4,3,2,1]		*j*←4			*minx*←3	
	V[1]←5	[5,6,4,3,2,1]		*minj*←4			*V*[6]←4	[1,2,4,6,5,4]
	j←3			*minx*←4			*V*[3]←3	[1,2,3,6,5,4]
	minj←3			*V*[4]←4	[1,5,6,5,3,2]	4	*i*←4	
	minx←4			*V*[2]←4	[1,4,6,5,3,2]		*minj*←4	
	V[3]←5	[5,6,5,3,2,1]		*j*←5			*minx*←6	
	V[1]←4	[4,6,5,3,2,1]		*minj*←5			*j*←5	
	j←4			*minx*←3			*minj*←5	
	minj←4			*V*[5]←4	[1,4,6,5,4,2]		*minx*←5	
	minx←3			*V*[2]←3	[1,3,6,5,4,2]		*V*[5]←6	[1,2,3,5,6,4]
	V[4]←4	[4,6,5,4,2,1]		*j*←6			*V*[4]←5	[1,2,3,5,6,4]
	V[1]←3	[3,6,5,4,2,1]		*minj*←6			*j*←6	
	j←5			*minx*←2			*minj*←4	
	minj←5			*V*[6]←3	[1,3,6,5,4,3]		*minx*←4	
	minx←2			*V*[2]←2	[1,2,6,5,4,3]		*V*[6]←5	[1,2,3,5,6,5]
	V[5]←3	[3,6,5,4,3,1]	3	*i*←3			*V*[4]←4	[1,2,3,4,6,5]
	V[1]←2	[2,6,5,4,3,1]		*minj*←3		5	*i*←5	
	j←6			*minx*←6			*minj*←5	
	minj←6			*j*←4			*minx*←6	
	minx←1			*minj*←4			*j*←6	
	V[6]←2	[3,6,5,4,3,2]		*minx*←5			*minj*←6	
	V[1]←1	[1,6,5,4,3,2]		*V*[4]←6	[1,2,6,6,4,3]		*minx*←5	
2	*i*←2			*V*[3]←5	[1,2,5,6,4,3]		*V*[6]←6	[1,2,3,4,6,6]
	minj←2			*j*←5			*V*[5]←5	[1,2,3,4,5,6]

From the simulations we can see that the *insertion algorithm* run faster thatn *selection algorithm*. Althoug both of them have nested loops.
For example, in the best case the insertion algorithm take n steps for sorting, while the selection algorithm takes n^2 steps.

Problem 2.10. Suppose you try to "sort" an array $W = [1, 1, 1, 1, 1, 1]$ all of whose elements are equal using (a) insertion sorting and (b) selection sorting. How does this compare to sorting the arrays U and V of the previous problem?
Solution:
(a) In the case of the *insertion algorithm*, the code line number 5

```
5     while j > 0 and x < W[j]
```

proves if `x < W[j]`, this condition never will be true, because all numbers are equal to 1. Therefore sorting array W will take the same time as sorting the array U.

Problem 2.15. A certain algorithm takes $10^{-4} \times 2^n$ seconds to solve an instance of size n. Show that in a year it could just solve an instance of size 38. What size of instance could be solved in a year on a machine one hundred times as fast?
A second algorithm takes $10^{-2} \times n^3$ seconds to solve an instance of size n. What size instance can it solve in a year? What size instance could be solved in a year on a machine one hundred times as fast.
Show that the second algorithm is nevertheless slower than the first for instances of size less than 20.
Solution:
Be

$$\begin{aligned} t(n) &= 10^{-4} \times 2^n \text{ s} \\ 1 \text{ y} &= 31536000 \text{ s} \end{aligned}$$

if $t(n) = 1\text{y}= 31536000$ s

$$\begin{aligned} 31536000 \text{ s} &= 10^{-4} \times 2^n \text{ s} \\ 2^n &= 31536000 \times 10^4 \end{aligned}$$

taking the $\log_2$

$$\begin{aligned} n &= \log_2 (31536000 \times 10^4) \\ \log_a x &= \frac{\log_b x}{\log_b a} \\ n &= \frac{log_{10}(31536000 \times 10^4)}{\log_{10} 2} \\ n &= 38.2 \end{aligned}$$

Then, in a year the algorithm solves an instance of size 38. Now, if we have a machine one hundred times faster:

$$\begin{aligned} t(n) &= \frac{10^{-4} 2^n}{100} \\ t(n) &= 10^{-6} 2^n \end{aligned}$$

if $t(n) = 1 \text{ y} = 31536000$ s

$$\begin{aligned} 31536000 &= 10^{-6} 2^n \\ n &= \log_2 31536000 \times 10^6 \\ n &= \frac{log_{10}(31536000 \times 10^6)}{\log_{10} 2} \\ n &= 44.8 \end{aligned}$$

the one hundred faster machine takes one year to solve an instance of size 44.
In the case of the second algorithm, we have:

$$t(n) = 10^{-2}n^3 \text{ s}$$

when $t(n) = 1$ y= 31536000 s

$$\begin{aligned} 31536000 \text{ s} &= 10^{-2}n^3 \text{ s} \\ n &= (31536000 \times 10^2)^{1/3} \\ n &= 1466.45 \end{aligned}$$

then, the algorithm takes one year to solve an instance of size 1466.
Now, if we have a machine one hudred times faster:

$$t(n) = \frac{10^{-2}n^3}{100} \text{ s}$$

when $t(n) = 1$ y= 31536000 s

$$\begin{aligned} 31536000 \text{ s} &= 10^{-4}n^3 \text{ s} \\ n &= (31536000 \times 10^4)^{1/3} \\ n &= 15260.06 \end{aligned}$$

the one hundred faster machine takes one year to solve an instance of size 15260.

n	$10^{-4}2^n$	$10^{-2}n^3$	n	$10^{-4}2^n$	$10^{-2}n^3$
1	0.0002	0.01	12	0.4096	17.28
2	0.0004	0.08	13	0.8192	21.97
3	0.0008	0.27	14	1.6384	27.44
4	0.0016	0.64	15	3.2768	33.75
5	0.0032	1.25	16	6.5536	40.96
6	0.0064	2.16	17	13.1072	49.13
7	0.0128	3.43	18	26.2144	58.32
8	0.0256	5.12	19	52.4288	68.59
9	0.0512	7.29	**20**	**104.8576**	**80.0**
10	0.1024	10.0	21	209.7152	92.61
11	0.2048	13.31	22	419.4304	106.48

The table shows that for $n < 20$, cubic algorithm is slower than exponential.

3

ASYMPTOTIC NOTATION

Problem 3.1. Consider an implementation of an algorithm that takes a time that is bounded above by the unlikely function

$$t(n) = 3 \text{ seconds} - 18n \text{ milliseconds} + 27n^2 \text{ microseconds}$$

to solve an instance of size n. Find the simplest possible function $f : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$ such that the algorithm takes time in the order of $f(n)$.
Solution:
Be

$$f : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0},\ f(n) = n^2$$

$$\begin{aligned}
t(n) &= 3 \text{ seconds} - 18n \text{ milliseconds} + 27n^2 \text{ microseconds} \\
t(n) &= 3 - 18 \times 10^{-3} n + 27 \times 10^{-6} n^2 \leq 3n^2 + 27 \times 10^{-6} n^2 \\
t(n) &= 3.0000027n^2 \approx 3n^2 \\
t(n) &= 3f(n)
\end{aligned}$$

Problem 3.4. What does $O(1)$ mean? $\Theta(1)$
Solution:
O, expresses the worst case complexity scenario, it is an asymptotic upper bound.

$$O(f(n)) = \{t : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0} | (\exists c \in \mathbb{R}^{+})\ (\overset{\infty}{\forall} n \in \mathbb{N} | [t(n) \leq cf(n)]\}$$

when $f(n) = 1,$ we have $t(n) \leq c(1) \Rightarrow t(n) \leq c.$
Then, we have found that $\exists c \in \mathbb{R}^{+}, \forall n \in \mathbb{N},$ such that $t(n) \leq c.$
The function $t(n)$ is upperly bounded by a constant. Then, $O(1)$ means that $t(n)$ runs in constant time regardless of the size of the input.

Θ, expresses the average case complexity scenario.

$$\Theta(f(n)) = \{t : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0} | (\exists c, d \in \mathbb{R}^{+})\ (\overset{\infty}{\forall} n \in \mathbb{N} | [df(n) \leq t(n) \leq cf(n)]\}$$

when $f(n) = 1,$ we have $d(1) \leq t(n) \leq c(1) \Rightarrow d \leq t(n) \leq c.$
Then, we have found that $\exists c \in \mathbb{R}^{+}$ and $\exists d \in \mathbb{R}^{+}, \forall n \in \mathbb{N},$ such that $d \leq t(n) \leq c.$
The function $t(n)$ is upper bounded by c and lower bounded by d. Then, $\Theta(1)$ means that on average $t(n)$ runs in constant time regardless of the size of the input.

Problem 3.5. Which of the following statements are true? Prove your answers.

1. $n^2 \in O(n^3)$

2. $n^2 \in \Omega(n^3)$

3. $2^n \in \Theta(2^{n+1})$

4. $n! \in \Theta((n+1)!)$

Solution:
1. If $t(n) \in O(f(n))$

$$\exists c \in \mathbb{R}^+ \text{ and } n_0 \in \mathbb{N} : \forall n > n_0, \text{ such that } 0 \leq t(n) \leq cf(n)$$

then, we have to find a $c > 0$ such that

$$n^2 \leq cn^3 \tag{1}$$

We know that

$$n^2 \leq n^3,\ \forall n \geq 1 \tag{2}$$

If we make $c = 1$ in (1), we get (2), $n^2 \leq n^3$. Then, exists c, in particular $c = 1$, such that $\forall n > n_0$, in this case $n_0 = 1$, it is true that $n^2 \leq cn^3$. Therefore, the statement that $n^2 \in O(n^3)$ is true.
2. The definition of $\Omega(f(n))$

$$\Omega(f(n)) = \{t : \rightarrow \mathbb{R}^+ | (\exists d \in \mathbb{R}^+)(\overset{\infty}{\forall} n \in \mathbb{N})[t(n) \geq df(n)]\}$$

that is

$$\exists d > 0 \in \mathbb{R}^+, n_0 \in \mathbb{N} : \forall n > n_o, \text{ it's satisfies } t(n) \geq df(n)$$

Substituting, we have

$$\begin{gathered} n^2 \geq (d)n^3 \\ \text{be } n \geq 1 \text{ then} \\ 1 \geq (d)n \end{gathered} \tag{3}$$

as $d > 0$, then $(d)n$ tends to infinity as n tends to infinity, then the inequality (3) is false. Therefore the statement that $n^2 \in \Omega(n^3)$ is false.
3. Applying the limit rule for Θ.
If there are two arbitrary functions f and $g : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$

$$\text{if } \lim_{n\to\infty} \frac{f(n)}{g(n)} \in \mathbb{R}^+ \text{ then } f(n) \in \Theta(g(n))$$

$f(n) = 2^n$ and $g(n) = 2^{n+1}$ substituting and taking the limit

$$\lim_{n\to\infty} \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \frac{2^n}{2^n 2} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$$

As the limit $\frac{1}{2} \in \mathbb{R}^+$, then the statement $2^n \in \Theta(2^{n+1})$ is true.
4. As in the previous case, apply the limit rule for Θ.
$f(n) = n!$ and $g(n) = (n+1)!$

$$\lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{n!}{(n+1)n!} = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

according to the limit rule, if the limit is 0, then the statement $n! \in \Theta((n+1)!)$ is false.

Problem 3.6. Prove that if $f(n) \in O(n)$ then $[f(n)]^2 \in O(n^2)$.
Solution:
if $f(n) \in O(n)$, then

$$\exists c \in \mathbb{R}^+ \text{ and } n_0 \in \mathbb{N} : \forall n > n_0, \text{ such that } 0 \leq f(n) \leq cn$$

then

$$f(n) \leq cn \tag{1}$$

Square both members of (1)

$$\begin{aligned} [t(n)]^2 &\leq (cn)^2 \\ [t(n)]^2 &\leq c^2 n^2 \\ \text{be } c' &= c^2 \\ [t(n)]^2 &\leq c' n^2 \end{aligned} \tag{2}$$

(2) implies that $\exists c' > 0, n'_0 \in \mathbb{N} : \forall n > n'_0$, it's satisfies $[f(n)]^2 \in O(n^2)$. ■

Problem 3.9. Prove that the O notation is reflexive: $f(n) \in O(f(n))$ for any function $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$
Solution:
The reflexive property sets:

$$f(n) \in O(f(n)) \tag{1}$$

then $\exists c \in \mathbb{R}^+$ and $n_0 \in \mathbb{N} : \forall n > n_0$, such that

$$f(n) \leq cf(n) \tag{2}$$

inequality (2) is true for $c \geq 1$. Then, we can choose any $c \geq 1$ and n_o, $\forall n > n_0$, (2) is true. Therefore $f(n) \in O(f(n))$ ■

Problem 3.10. Prove that the O notation is transitive: it follows from

$$f(n) \in O(g(n)) \text{ and } g(n) \in O(h(n))$$

that $f(n) \in O(h(n))$ for any functions $f, g, h : \mathbb{N} \to \mathbb{R}^{\geq 0}$
Solution:
if $f(n) \in O(g(n))$, then $\exists c_1 \in \mathbb{R}^+ : \forall n \in \mathbb{N}, n > n_1$, it is true that

$$f(n) \leq c_1 g(n) \tag{1}$$

if $g(n) \in O(h(n))$, then $\exists c_2 \in \mathbb{R}^+ : \forall n \in \mathbb{N}, n > n_2$, it is true that

$$g(n) \leq c_2 h(n) \tag{2}$$

from (1) we have:

$$\frac{f(n)}{c_1} \leq g(n) \tag{3}$$

then, from (2) and (3) we have

$$\begin{aligned} \frac{f(n)}{c_1} \leq & g(n) \leq c_2 h(n) \\ \frac{f(n)}{c_1} & \leq c_2 h(n) \\ f(n) & \leq c_1 c_2 h(n) \\ \text{be } c & = c_1 c_2 \\ f(n) & \leq c h(n) \end{aligned} \tag{4}$$

from (4) we concluded that $\exists c \in \mathbb{R}^+$, $\forall n \in \mathbb{N}$, $n > n_o$, such that $f(n) \in O(h(n))$ ∎

Problem 3.12. Prove that the Ω notation is reflexive and transitive: for any functions $f, g, h : \mathbb{N} \to \mathbb{R}^{\geq 0}$,

1. $f(n) \in \Omega(f(n))$
2. if $f(n) \in \Omega(g(n))$ and $g(n) \in \Omega(h(n))$ then $f(n) \in \Omega(h(n))$.

Rather than proving this directly (which would be easier!), use the duality rule and the results of Problems 3.9 and 3.10.
Solution:
The duality rule states that let there be two functions: $f, t : \mathbb{N} \to \mathbb{R}^{\geq 0}$ then

$$t(n) \in \Omega(f(n)) \iff f(n) \in O(t(n))$$

1. We have that $f(n) = g(n)$, then by the duality rule, we have:

$$f(n) \in \Omega(f(n)) \iff f(n) \in O(f(n))$$

In Problem 3.9, we proved that O notation is reflexive, then:

$$f(n) \in \Omega(f(n))$$

is true. Therefore Ω notation is also reflexive. ∎

2. We have

$$f(n) \in \Omega(g(n)) \text{ and } g(n) \in \Omega(h(n))$$

By duality, we have

$$g(n) \in O(f(n)) \text{ and } h(n) \in O(g(n))$$

Likewise

$$h(n) \in O(g(n)) \text{ and } g(n) \in O(f(n))$$

By Problem 3.10, we have that O notation is transitive

$$h(n) \in O(f(n))$$

By duality rule, we conclude that

$$f(n) \in \Omega(h(n)) \quad \blacksquare$$

Problem 3.27. Consider any b-smooth function $f : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$. Let c and n_0 be constants such that $f(bn) \leq cf(n)$ for all $n > n_0$. Consider any positive integer i. Prove by mathematical induction that $f(b^i n) \leq c^i f(n)$ for all $n \leq n_0$
Solution:
We know that:

$$f(bn) \leq cf(n) \quad \forall n \geq n_0$$

$i.$)For $i = 1$

$$\begin{aligned} f(b^1 n) &\leq c^1 f(n) \\ f(bn) &\leq cf(n) \end{aligned}$$

$ii.$)For $i = k + 1$

$$\begin{aligned} f(b^{k+1}) &\leq c^{k+1} f(n) \\ f(b^k bn) &\leq c^{k+1} f(n) \\ \text{be } n' &= bn \end{aligned}$$

supposse that the inequality is valid for $i = k$

$$\begin{aligned} f(b^k n') &\leq c^k f(n') = c^k f(bn) \\ f(b^k n') &\leq c^k f(bn) = c^k c f(n) \\ f(b^k n') &\leq c^{k+1} f(n) \\ f(b^k bn) &\leq c^{k+1} f(n) \\ f(b^{k+1} n) &\leq c^{k+1} f(n) \quad \blacksquare \end{aligned}$$

4

ANALYSIS OF ALGORITHMS

Problem 4.10. Prove by mathematical induction that if $d_0 = n$ and $d_l \leq d_{l-1}/2$ for all $l \geq 1$, then $d_l \leq n/2^l$ for all $l \geq 0$, (This is relevant to the analysis of the time taken by binary search: see Section 4.2.4.)
Solution:
We know that:

$$d_0 = n \tag{1}$$
$$d_l \leq \frac{d_{l-1}}{2} \quad \forall l \geq 1 \tag{2}$$

For $l = 1$:

from (2) we have:

$$d_1 \leq \frac{d_0}{2}$$

from (1) we have:

$$d_1 \leq \frac{n}{2}$$

For $l = k + 1$:

assume that $d_l \leq n/2^l$ it is satisfied for $l = k$

$$d_k \leq \frac{n}{2^k}$$
$$\frac{d_k}{2} \leq \frac{1}{2}\left(\frac{n}{2^k}\right)$$
$$\frac{d_k}{2} \leq \frac{1}{2^{k+1}} \tag{3}$$

on the other hand, from (2) we have:

$$d_k \leq \frac{d_{k-1}}{2}$$

then

$$d_{k+1} \leq \frac{d_k}{2} \tag{4}$$

from (3) and (4)

$$d_{k+1} \leq \frac{d_k}{2} \leq \frac{1}{2^{k+1}}$$
$$d_{k+1} \leq \frac{1}{2^{k+1}} \quad \blacksquare$$

Problem 4.24. Complete the solution of Example 4.7.7 by determining the value of c_1 as function of t_0.
Solution:
From the solution of the Example 4.7.7 we have: $c_3 = -1$ and $c_2 = -2$.
The equation 4.20

$$t_n = c_1 2^n + c_2 1^n + c_3 n 1^n \tag{4.20}$$

when $n = 0$

$$\begin{aligned} t_0 &= c_1 2^0 + c_2 1^0 + c3(0)1^0 \\ t_0 &= c_1 + c2 \\ t_0 &= c_1 - 2 \\ c_1 &= t_0 + 2 \end{aligned}$$

Problem 4.25. Complete the solution of Example 4.7.11 by determining the value of c_1 as function of t_0.
Solution:
We have:

$$\begin{aligned} t_i &= c_1 4^i + c_2 i 4^i \\ t_0 &= c_1 4^0 + c_2(0)4^0 \\ c_1 &= t_0 \end{aligned}$$

Problem 4.29. Solve the following recurrence exactly.

$$t(n) = \begin{cases} n & \text{if } n = 0 \text{ or } n = 1 \\ 5t_{n-1} - 6t_{n-2} & \text{otherwise} \end{cases}$$

Solution:
First, rewrite the recurrence

$$\begin{aligned} t_n &= 5t_{n-1} - 6t_{n-2} \\ t_n - 5t_{n-1} + 6t_{n-2} &= 0 \end{aligned}$$

the characteristic polynomial is

$$x^2 - 5x + 6 = 0$$

compute the roots

$$x^2 - 5x + 6 = (x-3)(x-2)$$

the roots are: $r_1 = 3$ and $r_2 = 2$
The general solution is:

$$t_n = c_1(3)^n + c_2(2)^n$$

use the initial conditions to calculate the constants c_1 and c_2. We have $t_0 = 0$ and $t_1 = 1$

$$c_1 + c_2 = 0 \qquad n = 0 \tag{1}$$
$$3c_1 + 2c_2 = 1 \qquad n = 1 \tag{2}$$

from (1)

$$c_1 = -c_2 \tag{3}$$

substituting in (2)

$$3(-c_2) + 2c_2 = 1 \Rightarrow -c_2 = 1$$
$$c_2 = -1$$

substituting in (3)

$$c1 = -(-1)$$
$$c1 = 1 \tag{4}$$

then, the solution is:

$$t_n = 3^n - 2^n$$

Problem 4.30. Solve the following recurrence exactly.

$$t(n) = \begin{cases} 9n^2 - 15n + 106 & \text{if } n = 0, 1 \text{ or } n = 2 \\ t_{n-1} + 2t_{n-2} - 2t_{n-3} & \text{otherwise} \end{cases}$$

Solution:
the recurrence equation is:

$$t_n = t_{n-1} + 2t_{n-2} - 2t_{n-3}$$
$$t_n - t_{n-1} - 2t_{n-2} + 2t_{n-3} = 0$$

The equation is homogeneus, and $a_0 = 1$, $a_1 = -1$, $a_2 = -2$, $a_3 = 2$ and $k = 3$.
The characteristic polynomial is:

$$x^3 - x^2 - 2x + 2 = 0 \tag{1}$$

we can check that $x = 1$ is one of the roots of the third order equation

$$1^3 - (1)^2 - 2(1) + 2 = 1 - 1 - 2 + 2 = 0$$

then, $x - 1$ is a factor of the equation. We divide equation (1) between $x + 1$ by syntetic division

$$\begin{array}{c|cccc} 1 & 1 & -1 & -2 & 2 \\ \hline & & 1 & 0 & -2 \\ \hline & 1 & 0 & -2 & 0 \end{array}$$

$$x^2 - 2 = 0$$
$$r_{1,2} = \pm\sqrt{2}$$

then, the roots are: $r_1 = \sqrt{2}, r_2 = -\sqrt{2}$ and $r_3 = 1$
The general solution is:

$$t_n = c_1(\sqrt{2})^n + c_2(-\sqrt{2})^n + c_3(1)^n$$
$$t_n = c_1(\sqrt{2})^n + c_2(-\sqrt{2})^n + c_3$$

we use the initial conditions in t_0, t_1 and t_2 to compute c_1, c_2 and c_3

$$\begin{aligned} c_1 + c_2 + c_3 &= 106 && n = 0 \\ c_1\sqrt{2} + c_2(-\sqrt{2}) + c_3 &= 9 - 15 + 106 && n = 1 \\ \sqrt{2}c_1 - \sqrt{2}c_2 + c_3 &= 100 \\ c_1(\sqrt{2})^2 + c_2(-\sqrt{2})^2 + c_3 &= 9(2)^2 - 15(2) + 106 && n = 2 \\ 2c_1 + 2c_2 + c_3 &= 112 \end{aligned}$$

then, we have to solve this system of equations

$$c_1 + c_2 + c_3 = 106 \tag{2}$$
$$\sqrt{2}c_1 - \sqrt{2}c_2 + c_3 = 100 \tag{3}$$
$$2c_1 + 2c_2 + c_3 = 112 \tag{4}$$

solving we get $c_1 = 3$, $c_2 = 3$ and $c_3 = 100$
then, the solution is:

$$t_n = 3(\sqrt{2})^n + 3(-\sqrt{2})^n + 100$$

Problem 4.31. Consider the following recurrence.

$$t(n) = \begin{cases} n & \text{if } n = 0 \text{ or } n = 1 \\ 2t_{n-1} - 2t_{n-2} & \text{otherwise} \end{cases}$$

Prove that $t_n = 2^{n/2} \sin n\pi/4$, not by mathematical induction but by using the technique of the characteristic equation.
Solution:
The recurrence equation is:

$$t_n = 2t_{n-1} - 2t_{n-2}$$
$$t_n - 2t_{n-1} + 2t_{n-2} = 0$$

the equation is homogeneus, and $a_0 = 1$, $a_1 = -2$, $a_2 = 2$ and $k = 2$. Then, the characteristic polynomial is:

$$x^2 - 2x + 2 = 0$$

whose roots are:

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)}}{2} = \frac{2 \pm \sqrt{-4}}{2}$$
$$r_{1,2} = \frac{2 \pm j\sqrt{4}}{2} = \frac{2 \pm 2j}{2} = 1 \pm j$$
$$r_1 = 1 + j \text{ and } r_2 = 1 - j$$

then, the general solution is

$$t_n = c_1(1+j)^n + c_2(1-j)^n$$

to calculate c_1 and c_2 we use the initial conditions. We have $t_0 = 0$ and $t_1 = 1$

$$c_1 + c_2 = 0 \quad n = 0 \tag{1}$$
$$(1+j)c_1 + (1-j)c_2 = 1 \quad n = 1 \tag{2}$$

from (1)

$$c_1 = -c_2 \tag{3}$$

substituting (3) in (2)

$$1 = (1+j)(-c_2) + (1-j)c_2 = -c_2 - jc_2 + c_2 - jc_2 = -2jc_2$$
$$c_2 = -\frac{1}{2j} = \frac{j}{2} \tag{4}$$

substituting (4) in (3)

$$c1 = -\frac{j}{2}$$

then, we get

$$t_n = -\frac{j}{2}(1+j)^n + \frac{j}{2}(1-j)^n$$

De Moivres's Theorem:
For any complex number x and any integer n,

$$(r(\cos\theta + j\sin\theta))^n = r^n(\cos n\theta + j\sin n\theta).$$

The Euler's form af a complex number for any real number x

$$e^{jx} = \cos x + j\sin x$$

Let's represent the complex numbers $1 + j$ and $1 - j$ in its polar form:

$1 + j = r(\cos\theta + j\sin\theta)$
$r = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\theta = \tan^{-1}\frac{1}{1} = \tan^{-1} 1 = \frac{\pi}{4}$
then
$1 + j = \cos\frac{\pi}{4} + j\sin\frac{\pi}{4} = e^{j\pi/4}$

$1 - j = r(\cos\theta + j\sin\theta)$
$r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$ and $\theta = \tan^{-1}\frac{1}{-1} = \tan^{-1} -1 = -\frac{\pi}{4}$
then
$1 - j = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = e^{-j\pi/4}$

now, substituting in t_n

$$\begin{aligned} t_n &= -\frac{j}{2}\left(\sqrt{2}e^{j\pi/4}\right)^n + \frac{j}{2}\left(\sqrt{2}e^{-j\pi/4}\right)^n \\ &= -\frac{j}{2}\left(2^{n/2}e^{jn\pi/4}\right) + \frac{j}{2}\left(2^{n/2}e^{-jn\pi/4}\right) \\ &= -\frac{2^{n/2}}{2}j\left(e^{jn\pi/4} - e^{-jn\pi/4}\right) \end{aligned}$$

from the sum of a complex number and its conjugate we have

$$\begin{aligned} z - \bar{z} &= 2j\Im(z) \\ &= -\frac{2^{n/2}}{2}j\left(2j\sin n\frac{\pi}{4}\right) = -2^{n/2}j^2\sin n\frac{\pi}{4} \\ t_n &= 2^{n/2}\sin n\frac{\pi}{4} \quad \blacksquare \end{aligned}$$

Problem 4.32. Solve the following recurrence exactly.

$$t(n) = \begin{cases} n & \text{if } n = 0, 1, 2 \text{ or } n = 3 \\ t_{n-1} + t_{n-3} - t_{n-4} & \text{otherwise} \end{cases}$$

Express your answer as simply as possible using the Θ notation.
Solution:
The recurrence equation is:

$$\begin{aligned} t_n &= t_{n-1} + t_{n-3} - t_{n-4} \\ t_n - t_{n-1} - t_{n-3} + t_{n-4} &= 0 \end{aligned}$$

the equation is homogeneus, and $a_0 = 1$, $a_1 = -1$, $a_3 = -1$, $a_4 = 1$ and $k = 4$.
The characteristic polynomial is:

$$x^4 - x^3 - x + 1 = 0 \qquad (1)$$

we can check that 1 is a root of the equation. Then, we divide equation (2) between $x - 1$ by syntetic division

$$\begin{array}{r|rrrrr} 1 & 1 & -1 & 0 & -1 & 1 \\ \hline & & 1 & 0 & 0 & -1 \\ \hline & 1 & 0 & 0 & -1 & 0 \end{array}$$

$$x^3 - 1 = 0 \tag{2}$$
$$x^3 = 1 \tag{3}$$

Problem 4.36. Solve the following recurrence exactly for n power of 2.

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 4T(n/2) + n & \text{otherwise} \end{cases}$$

Express your answer as symply as possible using the Θ notation.
Solution:
We have the next recurrence equation:

$$T(n) = 4T(n/2) + n \tag{1}$$

sustituting $n = 2^i$, $\frac{n}{2} = 2^{i-1}$

$$\begin{aligned} t_i = T(2^i) = 4T(2^{i-1}) + 2^i &= 4T(2^{i-1}) + 2^i \\ t_i &= 4t_{i-1} + 2^i \\ t_i - 4t_{i-1} &= 2^i \end{aligned} \tag{2}$$

the former recurrence is of the form of Equation 4.10. Then the characteristic equation is:

$$(x-2)(x-4)$$

roots $r_1 = 2$ and $r_2 = 4$ has multiplicity 1. Then, all solutions for t_i are of the form

$$t_i = c_1 2^i + c_2 4^i$$

as $n = 2^i \Rightarrow \lg_2 n = i$, $T(2^i) = t_i \Rightarrow T(n) = t_{\lg_2 n}$

$$T(n) = c_1 2^{\log_2 n} + c_2 4^{\lg_2 n}$$

of properties of logarithms, we have: $x^{\lg_b y} = y^{\lg_b x}$

$$\begin{aligned} T(n) &= c_1 n^{\log_2 2} + c_2 n^{\lg_2 4} \\ &= c_1 n + c_2 n^2 \end{aligned} \tag{3}$$

to compute c_1 sustitute (3) in (1)

$$\begin{aligned} c_1 n + c_2 n^2 &= 4\left(c_1 \frac{n}{2} + c_2 \left(\frac{n}{2}\right)^2\right) + n \\ c_1 n + c_2 n^2 &= 2c_1 n + c_2 n^2 + n \\ 0 &= c_1 n + n \\ c_1 n &= -n \\ c_1 &= -1 \end{aligned} \tag{4}$$

to obtains c_2, evaluate $T(1)$ in (3)

$$\begin{aligned} T(1) &= c_1 + c_2 \\ 1 &= -1 + c_2 \\ c_2 &= 2 \end{aligned}$$

then, the solution is:

$$T(n) = 2n^2 - n$$

Problem 4.37. Solve the following recurrence exactly for n power of 2.

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2T(n/2) + \lg n & \text{otherwise} \end{cases}$$

Express your answer as symply as possible using the Θ notation.
Solution:
We have the next recurrence equation:

$$T(n) = 2T(n/2) + \lg n \tag{1}$$

sustituting $n = 2^i$, $\frac{n}{2} = 2^{i-1}$

$$\begin{aligned} t_i = T(2^i) = 2T(2^{i-1}) + \lg 2^i &= 2T(2^{i-1}) + i \\ t_i &= 2T(2^{i-1}) + i \\ t_i &= 2t_{i-1} + i \\ t_i - 2t_{i-1} &= i \end{aligned}$$

the former recurrence is of the form of Equation 4.10. Then the characteristic equation is:

$$(x-2)(x-1)^2$$

root $r_1 = 2$ has multiplicity 1 and $r_2 = 1$ has multiplicity 2. Then, all solutions for t_i are of the form

$$\begin{aligned} t_i &= c_1 2^i + c_2 1^i + c_3 i 1^i \\ &= c_1 2^i + c_2 + c_3 i \end{aligned} \tag{2}$$

as $n = 2^i \Rightarrow \lg_2 n = i$, $T(2^i) = t_i \Rightarrow T(n) = t_{\lg_2 n}$

$$T(n) = c_1 2^{\log_2 n} + c_2 + c_3 \lg_2 n$$

of properties of logarithms, we have: $x^{\lg_b y} = y^{\lg_b x}$

$$\begin{aligned} T(n) &= c_1 n^{\log_2 2} + c_2 + c_3 \lg_2 n \\ &= c_1 n + c_2 + c_3 \lg_2 n \\ T(n) &= n c_1 + c_2 + c_3 \lg_2 n \end{aligned} \tag{3}$$

to compute c_2 and c_3 substitute (3) in (1)

$$
\begin{aligned}
nc_1 + c_2 + c_3 \lg_2 n &= 2\left(\frac{n}{2}c_1 + c_2 + c_3 \lg \frac{n}{2}\right) + \lg \frac{n}{2} \\
&= nc_1 + 2c_2 + 2c_3 \lg \frac{n}{2} + \lg \frac{n}{2} \\
&= nc_1 + 2c_2 + 2c_3 \lg n - 2c_3 \lg 2 + \lg n - \lg 2 \\
nc_1 + c_2 + c_3 \lg n &= nc_1 + 2c_2 + 2c_3 \lg n - 2c_3 + \lg n - 1 \\
c_3 \lg n - 2c_3 \lg n - \lg n &= c_2 - 2c_3 - 1 \\
\lg n(c_3 + 1) &= 2c_3 - c_2 + 1
\end{aligned}
$$

then, we have

$$
\begin{aligned}
c_3 + 1 &= 0 \\
c_3 &= -1
\end{aligned}
$$

and

$$
\begin{aligned}
2c_3 - c_2 &= 0 \\
c_2 &= 2(-1) \\
c_2 &= -2
\end{aligned}
$$

to compute c_1, from (3) we have

$$
\begin{aligned}
T(1) &= c_1 + c_2 \\
1 &= c_1 + c_2 \\
c_1 &= 1 - c_2 \\
c_1 &= 1 - (-2) \\
c_1 &= 3
\end{aligned}
$$

then, the solution is:

$$
T(n) = 3n - 2 - \lg n
$$