Physics 195 Problem Set 5

Problem 4 (Problem 5.6 of Ryden.)

Suppose you wanted to "pull an Einstein," and create a static universe ($\dot{a} = 0, \ddot{a} = 0$) in which the gravitational attraction of matter is exactly balanced by the gravitational repulsion of dark energy with equation-of-state parameter $-1/3 < w_q < -1$ and energy density ϵ_q. What is the necessary matter density (ϵ_m) required to produce a static universe, expressed in terms of ϵ_q and w_q? Will the curvature of this static universe be negative or positive? What will be its radius of curvature, expressed in terms of ϵ_q and w_q?

Solution:

The acceleration equation is given by

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} \sum_{i=1}^{N} \epsilon_i \left(1 + 3w_i\right) \tag{1}$$

For the conditions in the problem, this equation becomes

$$0 = -\frac{4\pi G}{3c^2} \left[\epsilon_m(1 + w_m) + \epsilon_q(1 + w_q)\right] \tag{2}$$

Note that $w_m = 0$ so

$$0 = -\frac{4\pi G}{3c^2} \left[\epsilon_m + \epsilon_q(1 + w_q)\right] \tag{3}$$

Thus, the energy density of matter is

$$\epsilon_m = -\epsilon_q(1 + w_q) \tag{4}$$

To get the curvature and radius of curvature for this universe, we use the Friedmann equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}(\epsilon_m + \epsilon_q) - \frac{\kappa c^2}{R_0^2 a^2} \tag{5}$$

For a static universe with matter and dark matter, this becomes

$$0 = \frac{8\pi G}{3c^2} \left[-\epsilon_q(1 + w_q) + \epsilon_q\right] - \frac{\kappa c^2}{R_0^2 a^2} \tag{6}$$

which reduces to

$$\frac{8\pi G}{3c^2} \epsilon_q w_q = -\frac{\kappa c^2}{R_0^2 a^2} \tag{7}$$

Since ϵ_q and w_q are both negative, κ should be negative as well so $\kappa = -1$.

We can get the radius of curvature from this expression by isolating it to one side of the equation:

$$R_o = \left(\frac{3c^4}{8\pi G a^2} \frac{1}{\epsilon_q w_q}\right)^{1/2} \tag{8}$$