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Physics 151 Problem Set 7

Problem 33

Problem 3.43. Other probability distributions (Gould and Tobochnik)

Not all probability densities have a finite variance as you will find in the following.

(a) Sketch the *Lorentz* or *Cauchy distribution* given by

$$p(x) = \frac{1}{\pi} \frac{\gamma}{(x-a)^2 + \gamma^2} \quad (-\infty < x < \infty).$$

Choose $a = 0$ and $\gamma = 1$ and compare the form of $p(x)$ in (3.127) to the Gaussian distribution given by (3.124).

Solution:

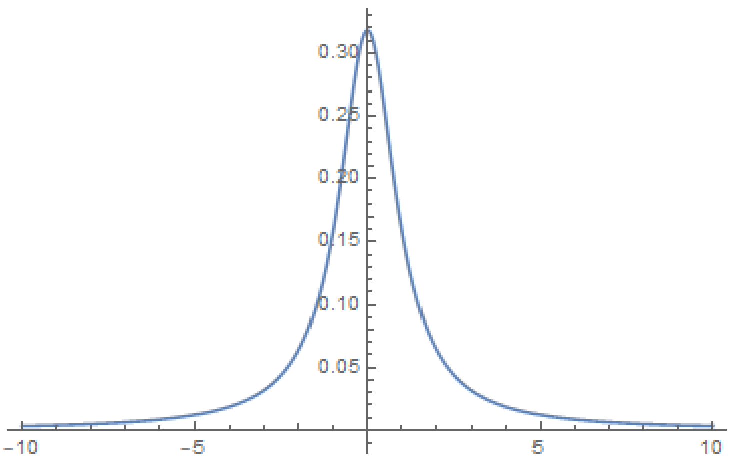

Figure 1: Lorentz Distribution

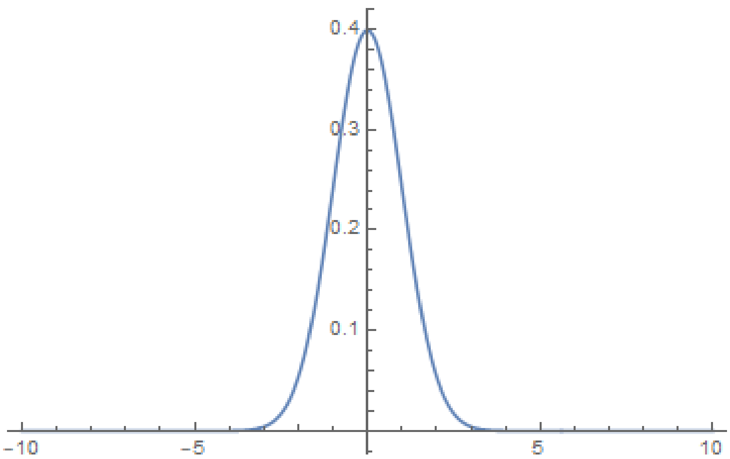

Figure 2: Gaussian Distribution

The plots of the Lorentz and Gaussian distributions are shown in Figures 1 and 2, respectively. We can see that the Gaussian distribution has a higher peak and a narrower curve than the Lorentz distribution. The difference in width is due to the forms of the two distributions. The $e^{-x^2/2}$ term in the Gaussian distribution corresponds to a more steeper climb and fall compared to the $1/(x^2+1)$ term in the Lorentz.

(b) Calculate the first moment of the Lorentz distribution assuming that $a = 0$ and $\gamma = 1$.

Solution:

With $a = 0$ and $\gamma = 1$, the Lorentz distribution becomes

$$p(x) = \frac{1}{\pi(x^2+1)} \tag{1}$$

The first moment of a probability distribution is equivalent to the mean and is given by

$$\overline{x} = \int_{-\infty}^{\infty} xp(x)dx \tag{2}$$

so the mean of the Lorentz distribution is

$$\overline{x} = \int_{-\infty}^{\infty} \frac{x}{\pi(x^2+1)}dx \tag{3}$$

Note that this integral does not converge so the mean $\overline{x}$ is undefined.

(c) Does the second moment exist?

Solution:

The second moment of a probability distribution is given by

$$\overline{x^2} = \int_{-\infty}^{\infty} x^2 p(x)dx \tag{4}$$

so for the Lorentz distribution this becomes

$$\overline{x^2} = \int_{-\infty}^{\infty} \frac{x^2}{\pi(x^2+1)}dx \tag{5}$$

This integral does not converge as well so the second moment of the Lorentz distribution does not exist.