Lizbeth Joy G. Tan March 1, 2019

Physics 151 Problem Set 5

Problem 24

Problem 2.54. Black hole thermodynamics (Gould and Tobochnik)

A black hole is created from the collapse of a massive object into one so dense that nothing can escape beyond a certain radius, including light. The measurable properties of a black hole depend only on its mass, charge, and angular momentum. In this problem we estimate the entropy and temperature of a charge neutral nonrotating black hole.

(a) Because the properties of the black hole depend only on its mass, use dimensional analysis to estimate the radius R of a black hole in terms of its mass M, the gravitational constant G, and the speed of light c. (Black holes are a consequence of the general theory of relativity, and thus we expect that the radius depends only on M, G, and c.)

Solution:

The gravitational constant G has units

$$[G] = \frac{L^3}{MT^2} \tag{1}$$

We estimate the radius R of a black hole using M, G, and c as

$$[R] = L = \frac{L^3}{MT^2} \cdot \frac{T^2}{L^2} \cdot M = [G]\left[\frac{1}{c^2}\right] M \tag{2}$$

Thus R is approximated to be

$$R \sim \frac{GM}{c^2} \tag{3}$$

(b) Assume that the entropy is of order Nk, where N is the number of particles in the black hole. The maximum entropy occurs when the particles are photons of wavelength λ of the order of the diameter of the black hole. Take $\lambda = 2R$ and determine the entropy S as a function of M (the total energy is Mc^2 and the energy of a photon is hc/λ). More detailed theoretical arguments give the correct relation

$$S = k\frac{8\pi^2 GM^2}{hc}$$

Your approximate result should have the correct dependence on G, M, h, c, and k. Calculate a numerical value for the entropy for a one solar mass black hole using (2.249). (The solar mass $M_\odot \approx 2 \times 10^{30}$kg.)

Solution:

Given the total energy E_T to be

$$E_T = Mc^2 \tag{4}$$

and the energy of a photon E_γ

$$E_\gamma = \frac{hc}{\lambda} \tag{5}$$

We can approximate the "number of particles" in a black hole using the ratio of these two energies

$$N = \frac{E_T}{E_\gamma} = \frac{Mc^2}{hc/\lambda} \tag{6}$$

Since the maximum entropy occurs when the particles are photons with $\lambda = 2R$, this expression becomes

$$N = \frac{Mc^2(2R)}{hc} \sim \frac{2Mc}{h}\left(\frac{GM}{c^2}\right) \sim \frac{2GM^2}{hc} \tag{7}$$

Note that we used the relation for the radius R from Equation (3).
With the entropy S assumed to be of order Nk, we get

$$S \sim Nk \sim \frac{2kGM^2}{hc} \tag{8}$$

Using the correct relation for the entropy of a black hole given by

$$S = k\frac{8\pi^2 GM^2}{hc}, \tag{9}$$

we can calculate a numerical value for the entropy for one solar mass black hole:

$$S = \frac{8\pi^2(1.38 \text{ x } 10^{-23} \text{ m}^2 \text{ kg s}^{-2} \text{ K}^{-1})(6.67 \text{ x } 10^{-11} \text{ m}^3 \text{ kg }^{-1}\text{s}^{-2})(2 \text{ x } 10^{30} \text{ kg})^2}{(6.626 \text{ x } 10^{-34} \text{ m}^2 \text{ kg s}^{-1})(299792458 \text{ m/s})} \tag{10}$$

This evaluates to

$$S = 1.46 \text{ x } 10^{54} \text{ m}^2 \text{ kg s}^{-2} \text{ K}^{-1} \tag{11}$$

(c) Express the entropy in terms of the surface area A of the black hole instead of M. Note that the area is a direct measure of the entropy. Does the entropy increase or decrease when two black holes coalesce into one?

Solution:

The surface area A of a spherical black hole is given by

$$A = 4\pi R^2 \tag{12}$$

Note that the correct relation for the radius R of a black hole is

$$R = \frac{2GM}{c^2} \tag{13}$$

so the area becomes

$$A = 4\pi\left(\frac{2GM}{c^2}\right)^2 = \frac{16\pi G^2 M^2}{c^4} \tag{14}$$

Thus the mass M in terms of the area A is

$$M^2 = \frac{Ac^4}{16\pi G^2} \tag{15}$$

Substituting this into the expression for entropy (9),

$$S = k\frac{8\pi^2 G}{hc}\frac{Ac^4}{16\pi G^2} = \frac{\pi k A c^3}{2Gh} \tag{16}$$

Note that the entropy decreases when two black holes merge since the surface area of the new black hole is less than the sum of the surface areas of the two original black holes.

(d) Express S in terms of the total energy E instead of M and determine the temperature for a one solar mass black hole. Use the approximation for R obtained in part (a) to find the temperature in terms of the gravitational field g at the radius R.

Solution:

Given the total energy E_T from Equation (4), we can write the mass M in terms of this energy such that

$$M = \frac{E_T}{c^2} \tag{17}$$

In terms of E_T, the expression of entropy becomes

$$S = k\frac{8\pi^2 G}{hc}\left(\frac{E_T}{c^2}\right)^2 = \frac{8\pi^2 k G E^2}{hc^5} \tag{18}$$

Modelling the black hole as a blackbody, we get its temperature T to be

$$T = \frac{\hbar c^3}{8\pi G M k} \tag{19}$$

The temperature of a one solar mass black hole is

$$T = \frac{(6.626 \text{ x } 10^{-34} \text{ m}^2 \text{ kg s}^{-1})(299792458 \text{ m/s})^3}{16\pi^2(6.67 \text{ x } 10^{-11} \text{ m}^3 \text{ kg }^{-1}\text{s}^{-2})(2 \text{ x } 10^{30} \text{ kg})(1.38 \text{ x } 10^{-23} \text{ m}^2 \text{ kg s}^{-2} \text{ K}^{-1})} \tag{20}$$

$$T = 6.15 \text{ x } 10^{-8} \text{ K} \tag{21}$$

The gravitational constant g is given by

$$g = \frac{GM}{R^2} \tag{22}$$

With this, we can rewrite the temperature T to

$$T = \frac{hc^3}{16\pi^2 GMk}\cdot\frac{R^2}{R^2} = \frac{\hbar c^3}{8\pi k g R^2} \tag{23}$$