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Physics 152 Problem Set 2

Problem 4

Problem 6.37. Numerical evaluation of μ for an ideal Bose gas (Gould and Tobochnik)

In this problem we study the behavior of μ as a function of the temperature. Program `IdealBoseGasIntegral` numerically evaluates the integral on the right-hand side of (6.206) for particular values of β and μ. The goal is to find the value of μ for a given value of β that yields the desired value of ρ.

To put (6.206) in a convenient form we introduce dimensionless variables and let $\epsilon = kT_c y$, $T = T_c T^*$, and $\mu = kT_c\mu^*$ and rewrite (6.206) as

$$1 = \frac{2}{2.612\sqrt{\pi}} \int_0^\infty \frac{y^{1/2}dy}{e^{(y-\mu^*)/T^*} - 1},$$

or

$$1 = 0.432 \int_0^\infty \frac{y^{1/2}dy}{e^{(y-\mu^*)/T^*} - 1},$$

where we have used (6.210).

(a) Fill in the missing steps and derive (6.211).

Solution:

Equation (6.206) is given by

$$\rho = \frac{\bar{N}}{V} = \frac{(2m)^{3/2}}{4\pi^2\hbar^3} \int_0^\infty \frac{\epsilon^{1/2}d\epsilon}{e^{\beta(\epsilon-\mu)} - 1} \tag{1}$$

Using the following change of variables:

$$\epsilon = kT_c y \tag{2}$$
$$d\epsilon = kT_c dy \tag{3}$$
$$T = T_c T^* \tag{4}$$
$$\mu = kT_c\mu^* \tag{5}$$

we can rewrite the integral in the expression into

$$\int_0^\infty \frac{\epsilon^{1/2}d\epsilon}{e^{\beta(\epsilon-\mu)} - 1} = \int_0^\infty \frac{(kT_c y)^{1/2}(kT_c dy)}{\exp\left[kT_c(y-\mu^*)/kT_cT^*\right] - 1} = (kT_c)^{3/2} \int_0^\infty \frac{y^{1/2}dy}{e^{(y-\mu^*)/T^*} - 1} \tag{6}$$

noting that $\beta = 1/kT$.

Substituting this back into ρ,

$$\rho = \frac{(2m)^{3/2}}{4\pi^2\hbar^3}(kT_c)^{3/2} \int_0^\infty \frac{y^{1/2}dy}{e^{(y-\mu^*)/T^*} - 1} \tag{7}$$

To simplify the constants, we use the relation

$$kT_c = \frac{1}{2.612^{2/3}} \frac{2\pi\hbar^2}{m} \rho^{2/3} \tag{8}$$

from the book.

Thus, the expression for ρ becomes

$$\rho = \frac{(2m)^{3/2}}{4\pi^2\hbar^3} \left(\frac{1}{2.612^{2/3}} \frac{2\pi\hbar^2}{m} \rho^{2/3} \right)^{3/2} \int_0^\infty \frac{y^{1/2} dy}{e^{(y-\mu^*)/T^*} - 1} \tag{9}$$

which reduces to

$$\rho = \frac{(2m)^{3/2}}{4\pi^2\hbar^3} \frac{1}{2.612} \frac{(2\pi)^{3/2}\hbar^3}{m^{3/2}} \rho \int_0^\infty \frac{y^{1/2} dy}{e^{(y-\mu^*)/T^*} - 1} \tag{10}$$

or

$$1 = \frac{2}{2.612\sqrt{\pi}} \int_0^\infty \frac{y^{1/2} dy}{e^{(y-\mu^*)/T^*} - 1} \tag{11}$$

(b) The program evaluates the left-hand side of (6.211b). The idea is to find μ^* for a given value of T^* such the left-hand side of (6.211b) equals 1. Begin with $T^* = 10$. First choose $\mu^* = -10$ and find the value of the integral. Do you have to increase or decrease the value of μ^* to make the numerical value of the left-hand side of (6.211b) closer to 1? Change μ^* by trial and error until you find the desired result. You should find that $\mu^* \approx -25.2$.

Solution:

The evaluated value started greater than 1 so μ^* must be decreased in order for the exponential term to increase and the value of the whole term in the integral to decrease and the whole expression to reach 1. By trial and error, we got a numerical value of $\mu^* = -25.23$.

(c) Next choose $T^* = 5$ and find the value of μ^* so that the left-hand side of (6.211b) equals 1. Does μ^* increase or decrease in magnitude? You can generate a plot of μ^* versus T^* by clicking on the `Plot` button each time you find an approximately correct value of μ.

Solution:

The magnitude of μ^* must decrease to get a value of 1 in the left-hand side since we should get the same value for the ratio between $(y - \mu^*)$ and T^*. For this case, we got $\mu^* = -7.683$.

(d) Discuss the qualitative behavior of μ as a function of T for fixed density.

Solution:

Doing the same procedure for different values of T^* and μ^* and then plotting them, we get a linear and decreasing behavior for μ as a function of T for fixed density.