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Physics 151 Problem Set 6

Problem 30

Problem 3.37. Calculation of the normalization constant (Gould and Tobochnik)
Derive the form of A in (3.117) using Stirling's approximation (3.101). Note that the weaker form of Stirling's approximation in (3.102) yields the incorrect result that $\ln A = 0$.

Solution:
A is defined as

$$\ln A = \ln P_N(n = \tilde{n}) \tag{1}$$

Since $\tilde{n} = pN$, this becomes

$$\ln A = \ln P_N(n = pN) \tag{2}$$

The binomial distribution is given by

$$P_N(n) = \frac{N!}{n!(N-n)!} p^n q^{N-n} \tag{3}$$

Substituting this into $\ln A$, we get

$$\ln A = \ln \left[\frac{N!}{(pN)!(N-pN)!} \, p^{pN}(1-p)^{N-pN} \right] \tag{4}$$

or

$$\ln A = \ln N! - \ln(pN)! - \ln(N-pN)! + \ln \left[p^{pN}(1-p)^{N-pN} \right] \tag{5}$$

Using Stirling's approximation

$$\ln N! \approx N \ln N - N + \frac{1}{2} \ln(2\pi N), \tag{6}$$

we can expand $\ln A$ into

$$\begin{aligned} \ln A = &\left[N \ln N - N + \frac{1}{2} \ln(2\pi N) \right] - \left[pN \ln(pN) - pN + \frac{1}{2} \ln(2\pi pN) \right] \\ &- \left[(1-p)N \ln((1-p)N) - (1-p)N + \frac{1}{2} \ln(2\pi(1-p)N) \right] + \ln \left[p^{pN}(1-p)^{N-pN} \right] \end{aligned} \tag{7}$$

Getting the exponential on both sides,

$$A = N^N e^{-N} (2\pi N)^{1/2} (pN)^{pN} e^{pN} (2\pi pN)^{-1/2} (qN)^{-qN} e^{qN} (2\pi qN)^{-1/2} p^{pN} (1-p)^{N-pN} \tag{8}$$

Rearranging the terms and simplifying,

$$A = N^N (pN)^{pN} \left[(1-p)N \right]^{-(1-p)N} p^{pN} (1-p)^{N-pN} e^{-N+pN+N-pN} (2\pi pqN)^{-1/2} \tag{9}$$

$$A = (2\pi pqN)^{-1/2} \tag{10}$$

Therefore, we get A to be

$$A = \frac{1}{(2\pi Npq)^{1/2}} \tag{11}$$