Physics 195 Problem Set 4

Problem 2 (Problem 2.5 of Ryden.)

Consider blackbody radiation at a temperature T. Show that for an energy threshold $E_0 >> kT$, the fraction of the blackbody photons that have energy $hf > E_0$ is

$$\frac{n\left(hf > E_0\right)}{n_\gamma} \approx 0.42 \left(\frac{E_0}{kT}\right)^2 \exp\left(-\frac{E_0}{kT}\right)$$

The cosmic background radiation is currently called the "cosmic *microwave* background." However, photons with $\lambda < 1$ mm actually lie in the *far infrared* range of the electromagnetic spectrum. It's time for truth in advertising: what fraction of the photons in today's "cosmic microwave background" are actually far infrared photons?

Solution:

The number density of photons is given by

$$n(f)df = \frac{\varepsilon(f)df}{hf} = \frac{8\pi}{c^3}\frac{f^2 df}{e^{hf/kT}-1} \tag{1}$$

for a given frequency range f and $f + df$.

To get the number of the blackbody photons that have energy $hf > E_0$, we integrate this from $E_o/hto\infty$:

$$n\left(hf > E_o\right) = \frac{8\pi}{c^3}\int_{E_o/h}^{\infty}\frac{f^2 df}{e^{hf/kT}-1} \tag{2}$$

Using the substitution

$$u = f - \frac{E_o}{h} \tag{3}$$

we can transform the integral into

$$n\left(hf > E_o\right) = \frac{8\pi}{c^3}\int_0^{\infty}\frac{\left(u + E_o/h\right)^2 du}{\exp\left(\frac{hu}{kT} + \frac{E_o}{kT}\right) - 1} \tag{4}$$

Multiplying the numerator and denominator by $(h/kT)^2$,

$$n\left(hf > E_0\right) = \frac{8\pi}{c^3}\left(\frac{kT}{h}\right)^3\int_0^{\infty}\left(\frac{hu}{kT} + \frac{E_o}{kT}\right)^2\left[\exp\left(\frac{hu}{kT} + \frac{E_o}{kT}\right) - 1\right]^{-1} du\left(\frac{h}{kT}\right) \tag{5}$$

Let the whole integral be I such that

$$n\left(hf > E_0\right) = \frac{8\pi}{c^3}\left(\frac{kT}{h}\right)^3 I \tag{6}$$

The number density of photons in blackbody radiation is

$$n_\gamma = \beta T^3 \tag{7}$$

where

$$\beta = \frac{2.4041}{\pi^2} \frac{k^3}{\hbar^3 c^3} \tag{8}$$

Thus, we can expand n_γ into

$$n_\gamma = \frac{8\pi k^3}{h^3 c^3}(2.4041)T^3 \tag{9}$$

The fraction of the blackbody photons that have energy $hf > E_0$ will be

$$\frac{n\left(hf > E_0\right)}{n_\gamma} = \frac{\frac{8\pi}{c^3}\left(\frac{kT}{h}\right)^3}{\frac{8\pi k^3 T^3}{h^3 c^3}} \cdot \frac{1}{2.4041} I = 0.42I \tag{10}$$

Going back to the intgral, we have defined I to be

$$I = \int_0^\infty \left(\frac{hu}{kT} + \frac{E_o}{kT}\right)^2 \left[\exp\left(\frac{hu}{kT} + \frac{E_o}{kT}\right) - 1\right]^{-1} du \left(\frac{h}{kT}\right) \tag{11}$$

Using another change of variables

$$x = \frac{hu}{kT}, \tag{12}$$

we transform the integral into

$$I = e^{-E_o/kT} \int_0^\infty \left(x + \frac{E_o}{kT}\right)^2 e^{-x} dx \tag{13}$$

This expands to

$$I = e^{-E_o/kT} \left\{ \int_0^\infty x^2 e^{-x} dx + \frac{2E_o}{kT} \int_0^\infty x e^{-x} dx + \left(\frac{E_o}{kT}\right)^2 \int_0^\infty e^{-x} dx \right\} \tag{14}$$

We can evaluate the integrals using the gamma function and get

$$I = e^{-E_o/kT} \left[2 + \frac{2E_o}{kT} + \left(\frac{E_o}{kT}\right)^2\right] \tag{15}$$

Note that for $E_0 >> kT$, we can ignore the first two terms:

$$I \approx \left(\frac{E_0}{kT}\right)^2 \exp\left(-\frac{E_0}{kT}\right) \tag{16}$$

Therefore, the fraction of the blackbody photons that have energy $hf > E_0$ is

$$\frac{n\left(hf > E_0\right)}{n_\gamma} \approx 0.42 \left(\frac{E_0}{kT}\right)^2 \exp\left(-\frac{E_0}{kT}\right) \tag{17}$$