Physics 195 Problem Set 5

Problem 1 (Problem 5.1 of Ryden.)

A light source in a flat, single-component universe has a redshift z when observed at a time t_0. Show that the observed redshift z changes at a rate

$$\frac{dz}{dt_0} = H_0(1+z) - H_0(1+z)^{3(1+w)/2}$$

For what values of w does the observed redshift increase with time?

Solution:

In a flat, single-component universe, the redshift z is given by the relation

$$1+z = \frac{a\left(t_0\right)}{a\left(t_e\right)} = \left(\frac{t_0}{t_e}\right)^{2/(3+3w)} \tag{1}$$

In terms of the Hubble constant, the observed time is

$$t_0 = \frac{2}{3(1+w)H_0} \tag{2}$$

To get the change in redshift, we differentiate the expression wrt t_0:

$$\frac{dz}{dt_0} = \frac{d}{dt_0}\left(\frac{t_0}{t_e}\right)^{2/(3+3w)} + \frac{dt_e}{dt_0}\frac{d}{dt_e}\left(\frac{t_0}{t_e}\right)^{2/(3+3w)} \tag{3}$$

$$\frac{dz}{dt_0} = \left(\frac{2}{3+3w}\right) t_0^{-1}\left(\frac{t_0}{t_e}\right)^{2/(3+3w)} - \frac{dt_e}{dt_0}\left(\frac{2}{3+3w}\right) t_e^{-1}\left(\frac{t_0}{t_e}\right)^{2/(3+3w)} \tag{4}$$

From the first equation, we get t_e to be

$$t_e = \frac{t_0}{(1+z)^{3(1+w)/2}} \tag{5}$$

Substituting t_0 and t_e into the expression for the change in redshift, we get

$$\frac{dz}{dt_0} = \left(\frac{2}{3+3w}\right)\left[\frac{3+3w}{2}H_o\right](1+z) - \frac{dt_e}{dt_0}\left(\frac{2}{3+3w}\right)\left(\frac{2}{3+3w}\right)(1+z)^{3(1+w)/2} t_0^{-1}(1+z) \tag{6}$$

$$\frac{dz}{dt_0} = H_o(1+z) - \frac{dt_e}{dt_0}H_0(1+z)^{3(1+w)/2}(1+z) \tag{7}$$

Consider two photons emitted in the same direction, dt_e will be the time interval between their emissions and the observed time would be dt_o. The distance due to these two times will be proportional to each other by a factor of $(1+z)$ as shown in the first equation. Thus,

$$cdt_o = c(1+z)dt_e \tag{8}$$

$$\frac{dt_e}{dt_0} = (1+z)^{-1} \tag{9}$$

Using this, we get the change in redshift to be

$$\frac{dz}{dt_0} = H_0(1+z) - H_0(1+z)^{3(1+w)/2} \tag{10}$$

An increase in the observed redshift corresponds to this change being greater than zero:

$$\frac{dz}{dt_0} = H_0(1+z) - H_0(1+z)^{3(1+w)/2} > 0 \tag{11}$$

From this, we get the inequality

$$H_0(1+z) > H_0(1+z)^{3(1+w)/2} \tag{12}$$

Thus, the values of w are restricted to

$$3(1+w)/2 < 1 \tag{13}$$

or

$$w < -\frac{1}{3} \tag{14}$$