# Physics 195 Problem Set 4

## Problem 1 (Problem 2.4 of Ryden.)

A hypothesis once used to explain the Hubble relation is the "tired light hypothesis." The tired light hypothesis states that the universe is not expanding, but that photons simply lose energy as they move through space (by some unexplained means), with the energy loss per unit distance being given by the law

$$\frac{dE}{dr} = -kE,$$

where $k$ is a constant. Show that this hypothesis gives a distance-redshift relation that is linear in the limit $z << 1$. What must the value of $k$ be in order to yield a Hubble constant of $H_0 = 68$ km s$^{-1}$ Mpc$^{-1}$?

**Solution:**

The redshift $z$ is defined to be

$$z \equiv \frac{\lambda_{\rm obs} - \lambda_{\rm em}}{\lambda_{\rm em}} \tag{1}$$

Using the relation $c = \lambda f$, we can write this as

$$z = \frac{f_{\rm em} - f_{\rm obs}}{f_{\rm obs}} \tag{2}$$

Note that the energy of a photon is $E = hf$. Thus, multiply the numerator and the denominator by $h$ to write the redshift $z$ in terms of the energy $E$:

$$z = \frac{hf_{\rm em} - hf_{\rm obs}}{hf_{\rm obs}} = \frac{E_{\rm em} - E_{\rm obs}}{E_{\rm obs}} \tag{3}$$

From the given, the energy loss per unit distance is given by the law

$$\frac{dE}{dr} = -kE \tag{4}$$

Solving this ODE, gives us a solution of

$$E(r) = Ce^{-kr} = E_{\rm em}e^{-kr} \tag{5}$$

since $E(r = 0) = E_{\rm em}$.
Substituting $E_{\rm obs} = E(r)$ into (3),

$$z = \frac{E_{\rm em} - E_{\rm em}e^{-kr}}{E_{\rm em}e^{-kr}} = \frac{1 - e^{-kr}}{e^{-kr}} = e^{kr} - 1 \tag{6}$$

We can use the Taylor expansion of the exponential,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \tag{7}$$

to rewrite (6) into

$$z = \left[1 + (kr) + \mathcal{O}(kr)^2\right] - 1 = kr + \mathcal{O}(kr)^2 \tag{8}$$

which can be approximated to

$$z \approx kr \tag{9}$$

Hubble's law gives us a relation between the redshift $z$ and the distance $r$ as well:

$$z = \frac{H_o}{c} r \tag{10}$$

Comparing these two expressions, we get

$$k = \frac{H_o}{c} \tag{11}$$

Using the value of the Hubble constant $H_0 = 68$ km s$^{-1}$ Mpc$^{-1}$, we get the value of k to be

$$k = \frac{68\, km\, \mathrm{s}^{-1}\, \mathrm{Mpc}^{-1}}{3 \times 10^8\, \mathrm{m/s}} \cdot \frac{1000\, \mathrm{m}}{1\, \mathrm{km}} = 226.7\, \mathrm{pc}^{-1} \tag{12}$$