Physics 195 Problem Set 4

Problem 5

For certain values of the cosmological density parameters, the expansion of the universe can change direction. For each of the following scenarios, find the value of a for the turnaround point, and indicate whether the universe changes from expansion to contraction or vice versa. Explain your analysis.

(a) $\Omega_\Lambda = 0$ and $\Omega_\mathrm{M} > 1$.

Solution:

In terms of the cosmological density parameters, the Friedmann equation takes the form

$$\frac{H^2}{H_0^2} = \frac{\Omega_R}{a^4} + \frac{\Omega_M}{a^3} + \Omega_\Lambda + \frac{1-\Omega_0}{a^2} \tag{1}$$

where

$$\Omega_0 = \Omega_R + \Omega_M + \Omega_\Lambda \tag{2}$$

Since $\Omega_\Lambda = 0$, this equation becomes

$$\frac{H^2}{H_0^2} = \frac{\Omega_R}{a^4} + \frac{\Omega_M}{a^3} + \frac{1-\Omega_0}{a^2} \tag{3}$$

Note that $\Omega_\mathrm{M} > 1$ so $\dfrac{1-\Omega_M-\Omega_R}{a^2} < 0$.

At $a \approx 0 \implies \dfrac{\Omega_R}{a^4}$ dominates and $\dfrac{\dot{a}}{a} > 0$ (universe expands).

At $a \to \infty \implies \dfrac{1-\Omega_M-\Omega_R}{a^2}$ dominates and $\dfrac{\dot{a}}{a} < 0$ (universe contracts).

We get the turnaround point by setting $\dot{a} = 0$ or $H = 0$:

$$0 = \frac{\Omega_R}{a_{max}^4} + \frac{\Omega_M}{a_{max}^3} + \frac{1-\Omega_M-\Omega_R}{a_{max}^2} \tag{4}$$

$$0 = \Omega_R + \Omega_M a_\mathrm{max} + (1-\Omega_M-\Omega_R)a_\mathrm{max}^2 \tag{5}$$

The solution to this equation is

$$a_\mathrm{max} = \frac{\Omega_M \pm \sqrt{(2\Omega_R+\Omega_M)^2 - 4\Omega_R}}{2(1-\Omega_M-\Omega_R)} \tag{6}$$

We only get the $(+)$ term because we want the maximum value of a:

$$a_\mathrm{max} = \frac{\Omega_M + \sqrt{(2\Omega_R+\Omega_M)^2 - 4\Omega_R}}{2(1-\Omega_M-\Omega_R)} \tag{7}$$

From this turnaround point, the universe changes from expansion to contraction.

(b) $\Omega_{\rm M} = 0$ and $\Omega_\Lambda > 1$.

Solution:

For this scenario, the Friedmann equation becomes

$$\frac{H^2}{H_0^2} = \frac{\Omega_R}{a^4} + \Omega_\Lambda + \frac{1 - \Omega_R - \Omega_\Lambda}{a^2} \tag{8}$$

Again, since $\Omega_\Lambda > 1$, we know that $\dfrac{1 - \Omega_R - \Omega_\Lambda}{a^2} < 0$.

At $a \approx 0 \implies \dfrac{\Omega_R}{a^4}$ dominates and $\dfrac{\dot{a}}{a} > 0$ (universe expands).

At $a \to \infty \implies \Omega_\Lambda$ dominates and $\dfrac{\dot{a}}{a} < 0$ (universe still expands).

Note that the term $\dfrac{1 - \Omega_R - \Omega_\Lambda}{a^2}$ can dominate at an intermediate value of a.

We again set $\dot{a} = 0$ or $H = 0$ to find the turnaround point:

$$0 = \frac{\Omega_R}{a_{\rm int}^4} + \Omega_\Lambda + \frac{1 - \Omega_R - \Omega_\Lambda}{a_{\rm int}^2} \tag{9}$$

$$\Omega_R + \Omega_\Lambda a_{\rm int}^4 + (1 - \Omega_R - \Omega_\Lambda) a_{\rm int}^2 = 0 \tag{10}$$

This will have the solution

$$a_{\rm int} = \frac{(\Omega_R + \Omega_\Lambda - 1) \pm \sqrt{(1 - \Omega_R - \Omega_\Lambda)^2 - 4\Omega_\Lambda \Omega_R}}{2\Omega_\Lambda} \tag{11}$$

and

$$a_{\rm int} = \left[\frac{(\Omega_R + \Omega_\Lambda - 1) \pm \sqrt{(1 - \Omega_R - \Omega_\Lambda)^2 - 4\Omega_\Lambda \Omega_R}}{2\Omega_\Lambda}\right]^{1/2} \tag{12}$$

At these points, the universe will first change from expansion to contraction and at the second point, the universe changes from contraction back to expansion.