Physics 195 Problem Set 3

Problem 6

Derive the exact solution to the Lane-Emden equation for $n = 5$. Derive an expression for its total mass, and show that although its first root $\xi_1 \to \infty$, the mass is finite.

Solution:

We start with the general form of the Lane-Emden equation given by

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi} = -\theta^n \tag{1}$$

Doing a change of variables (Kelvin's transformation), we get

$$x = \frac{1}{\xi} \tag{2}$$

and

$$\frac{d}{d\xi} = -x^2\frac{d}{dx} \tag{3}$$

With these, the Lane-Emden equation becomes

$$x^2(-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\left[x^{-2}(-x^2)\frac{\mathrm{d}\theta}{\mathrm{d}x}\right] = -\theta^n \tag{4}$$

or

$$x^4\frac{d^2\theta}{dx^2} = -\theta^n \tag{5}$$

Using another change of variables (Emden's transformations),

$$\theta = Ax^\omega z \tag{6}$$

where

$$\omega = \frac{2}{n-1} \tag{7}$$

The second-order derivative of θ with respect to x becomes

$$\frac{d^2\theta}{dx^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}}{\mathrm{d}x}(Ax^\omega z)\right] \tag{8}$$

$$\frac{d^2\theta}{dx^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(A\omega x^{\omega-1}z + Ax^\omega\frac{\mathrm{d}z}{\mathrm{d}x}\right)\right] \tag{9}$$

$$\frac{d^2\theta}{dx^2} = A\left[x^\omega\frac{d^2z}{dx^2} + 2\omega x^{\omega-1}\frac{\mathrm{d}z}{\mathrm{d}x} + \omega(\omega-1)x^{\omega-2}z\right] \tag{10}$$

Substituting this into Equation (5),

$$x^4A\left[x^\omega\frac{d^2z}{dx^2} + 2\omega x^{\omega-1}\frac{\mathrm{d}z}{\mathrm{d}x} + \omega(\omega-1)x^{\omega-2}z\right] = -(Ax^\omega z)^n \tag{11}$$

Note that

$$\omega + 2 = \frac{2}{n-1} + 2 = \frac{2+2(n-1)}{n-1} = \frac{2n}{n-1} = n\omega \tag{12}$$

so

$$x^2\frac{d^2z}{dx^2}+2\omega x\frac{dz}{dx}+\omega(\omega-1)z+A^{n-1}z^n=0 \tag{13}$$

We use another substitution of variable given by

$$x=\frac{1}{\xi}=e^t \tag{14}$$

such that the first second derivatives are given by

$$\frac{dz}{dx}=e^{-t}\frac{\mathrm{d}z}{\mathrm{d}t} \tag{15}$$

and

$$\begin{aligned}\frac{d^2z}{dx^2}&=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)\\&=\frac{\mathrm{d}t}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}t}{\mathrm{d}x}\frac{\mathrm{d}z}{\mathrm{d}t}\right)\\&=e^{-t}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-t}\frac{\mathrm{d}z}{\mathrm{d}t}\right)\\&=e^{-2t}\left(\frac{d^2z}{dt^2}-\frac{dz}{dt}\right)\end{aligned} \tag{16}$$

Thus,

$$e^{2t}\left[e^{-2t}\left(\frac{d^2z}{dt^2}-\frac{dz}{dt}\right)\right]+2\omega e^t\left(e^{-t}\frac{\mathrm{d}z}{\mathrm{d}t}\right)+\omega(\omega-1)z+A^{n-1}z^n=0 \tag{17}$$

This simplifies to

$$\frac{\mathrm{d}^2z}{\mathrm{d}t^2}+(2\omega-1)\frac{\mathrm{d}z}{\mathrm{d}t}+\omega(\omega-1)z+A^{n-1}z^n=0 \tag{18}$$

For $n>3$, we choose

$$A^{n-1}=\omega(\omega-1) \tag{19}$$

so the equation becomes

$$\frac{\mathrm{d}^2z}{\mathrm{d}t^2}+(2\omega-1)\frac{\mathrm{d}z}{\mathrm{d}t}+\omega(\omega-1)z+\omega(\omega-1)z^n=0 \tag{20}$$

$$\frac{\mathrm{d}^2z}{\mathrm{d}t^2}+(2\omega-1)\frac{\mathrm{d}z}{\mathrm{d}t}+\omega(\omega-1)z(1-z^{n-1})=0 \tag{21}$$

Changing back ω to in terms of n,

$$\frac{\mathrm{d}^2z}{\mathrm{d}t^2}+\left(\frac{5-n}{n-1}\right)\frac{\mathrm{d}z}{\mathrm{d}t}-\frac{2(n-3)}{(n-2)^2}z(1-z^{n-1})=0 \tag{22}$$

For $n=5$, we get

$$\frac{\mathrm{d}^2z}{\mathrm{d}t^2}=\frac{1}{4}z\left(1-z^4\right) \tag{23}$$

Multiplying both sides by $\mathrm{d}z/\mathrm{d}t$,

$$\frac{\mathrm{d}z}{\mathrm{d}t}\left(\frac{d^2z}{dt^2}\right)=\frac{1}{4}z\left(1-z^4\right)\frac{\mathrm{d}z}{\mathrm{d}t} \tag{24}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\right] = 2\frac{\mathrm{d}z}{\mathrm{d}t}\left(\frac{d^2z}{dt^2}\right) \tag{25}$$

so

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\right] = \frac{1}{4}z\left(1-z^4\right)\frac{\mathrm{d}z}{\mathrm{d}t} \tag{26}$$

Integrating both sides wrt t,

$$\frac{1}{2}\left(\frac{dz}{dt}\right)^2 = \frac{1}{8}z^2 - \frac{1}{24}z^6 + C_3 \tag{27}$$

We can rewrite this into

$$\frac{\mathrm{d}z}{\pm\left(2C_3 + \frac{1}{4}z^2 - \frac{1}{12}z^6\right)^{1/2}} = \mathrm{d}t \tag{28}$$

When $C_3 = 0$,

$$\frac{\mathrm{d}z}{\pm\left(\frac{1}{4}z^2 - \frac{1}{12}z^6\right)^{1/2}} = \mathrm{d}t \tag{29}$$

We choose the negative one (-) so that $t \to \infty$:

$$\frac{\mathrm{d}z}{z\left(1-\frac{1}{3}z^4\right)^{1/2}} = -\frac{1}{2}\mathrm{d}t \tag{30}$$

Using the trig substitution,

$$\frac{1}{3}z^4 = \sin^2\alpha \tag{31}$$

we get

$$\frac{4}{3}z^3\mathrm{d}z = 2\sin\alpha\cos\alpha\mathrm{d}\alpha \tag{32}$$

Dividing both sides by Equation (31),

$$4\frac{\mathrm{d}z}{z} = 2\frac{\sin\alpha}{\cos\alpha}\mathrm{d}\alpha \tag{33}$$

Thus,

$$\frac{1}{2}\frac{\sin\alpha}{\cos\alpha}(1-\sin^2\alpha)^{-1/2}\mathrm{d}\alpha = -\frac{1}{2}\mathrm{d}t \tag{34}$$

This simplifies to

$$\csc\alpha\mathrm{d}\alpha = -\mathrm{d}t \tag{35}$$

Integrating both sides,

$$\ln|\csc\alpha - \cot\alpha| = -t + C_4 \tag{36}$$

Getting the exponential of both sides, we get

$$\csc\alpha - \cot\alpha = C_5e^{-t} \tag{37}$$

or

$$\tan\frac{\alpha}{2} = C_5e^{-t} \tag{38}$$

Using the trigonometric relation,

$$\sin\alpha = \frac{2\tan(\alpha/2)}{1+\tan^2(\alpha/2)} \tag{39}$$

We can rewrite Equation (31) into

$$\frac{1}{3}z^4 = \frac{4\tan^2(\alpha/2)}{(1+\tan^2(\alpha/2))^2} \tag{40}$$

or

$$\frac{1}{3}z^4 = \frac{4C_5^2 e^{-2t}}{(1+C_5^2 e^{-2t})^2} \tag{41}$$

Isolating z, we get

$$z = \pm\left[\frac{12C_5^2 e^{-2t}}{(1+C_5^2 e^{-2t})^2}\right]^{1/4} \tag{42}$$

Note that

$$\theta = \left(\frac{1}{2}e^t\right)^{1/2} z \tag{43}$$

so

$$\left(\frac{1}{2}e^t\right)^{-1/2}\theta = \left[\frac{12C_5^2 e^{-2t}}{(1+C_5^2 e^{-2t})^2}\right]^{1/4} \tag{44}$$

If we isolate θ on one side, we get

$$\theta = \left[\frac{3C_5^2}{(1+C_5^2 e^{-2t})^2}\right]^{1/4} \tag{45}$$

We can change the variable t to ξ using the relation:

$$\xi = e^{-t} \tag{46}$$

Thus,

$$\theta = \left[\frac{3C_5^2}{(1+C_5^2 \xi^2)^2}\right]^{1/4} \tag{47}$$

Using the boundary condition,

$$\theta(0) = 1 \tag{48}$$

we can determine the integration constant to be

$$\theta(0) = (3C_5^2)^{1/4} = 1 \tag{49}$$

$$C_5^2 = \frac{1}{3} \tag{50}$$

Therefore, the solution to the Lane-Emden equation for the case n = 5 is

$$\theta = \frac{1}{\left(1+\frac{1}{3}\xi^2\right)^{1/2}} \tag{51}$$

To get the expression for the mass of this star, we will use the distance relations given by

$$x = \frac{r}{R_\star} \tag{52}$$

and

$$\xi = \frac{r}{a} \tag{53}$$

We can get an equation relating x and ξ using the relation between $R_\star$ and the first root $\xi_\star$ of θ in terms of ξ:

$$\xi_\star = \frac{R_\star}{a} \tag{54}$$

Note that this is the case since r goes from 0 to $R_\star$ and x and ξ go from 0 to 1. For $n = 5$, the first root of the solution is

$$\xi_\star = \infty \tag{55}$$

which implies that the radius of the star is infinite. Thus, the solution of the case $n = 5$ is not a physical star.

We can get the total mass $M_\star$ of the star by integrating the density over the whole volume of the spherical star:

$$M_\star = 4\pi \int_0^{R_\star} \rho r^2 \mathrm{d}r \tag{56}$$

With a change of variable from r to ξ by

$$r = a\xi \tag{57}$$

$$\mathrm{d}r = a\mathrm{d}\xi \tag{58}$$

The integral becomes

$$M_\star = 4\pi \int_0^{R_\star/a} \rho(\xi)(a\xi)^2(a\mathrm{d}\xi) \tag{59}$$

Note that

$$\rho = \rho_c \theta^n \tag{60}$$

so for $n = 5$, the expression for the mass becomes

$$M_\star = 4\pi\rho_c \int_0^{R_\star/a} \theta^5(\xi)(a\xi)^2(a\mathrm{d}\xi) \tag{61}$$

Substituting the θ term by the expression in the Lane-Emden equation, we get

$$M_\star = -4\pi\rho_c a^3 \int_0^{\xi_\star} \xi^2 \left[\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)\right] \mathrm{d}\xi \tag{62}$$

or

$$M_\star = -4\pi\rho_c a^3 \int_0^{\infty} \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) \mathrm{d}\xi \tag{63}$$

This integral evaluates to

$$M_\star = -4\pi\rho_c a^3 \left[\lim_{\xi\to\infty}\left(\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)\right] \tag{64}$$

The derivative of θ with respect to ξ is

$$\frac{\mathrm{d}\theta}{\mathrm{d}\xi} = -\frac{\xi}{3\left(1 + \frac{1}{3}\xi^2\right)^{3/2}} \tag{65}$$

Thus the limit expression is

$$\lim_{\xi\to\infty}\left(-\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = \lim_{\xi\to\infty} \frac{\xi^3}{3\left(1 + \frac{1}{3}\xi^2\right)^{3/2}} \tag{66}$$

Using L'Hopital's rule, we get

$$\lim_{\xi\to\infty}\left(-\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = \lim_{\xi\to\infty} \frac{\xi}{\left(1 + \frac{1}{3}\xi^2\right)^{1/2}} \tag{67}$$

Factoring out ξ in the denominator, the expression simplifies to

$$\lim_{\xi\to\infty}\left(-\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)=\lim_{\xi\to\infty}\frac{\xi}{\xi\left(\xi^{-2}+\frac{1}{3}\right)^{1/2}}=\lim_{\xi\to\infty}\frac{1}{\left(\xi^{-2}+\frac{1}{3}\right)^{1/2}} \tag{68}$$

and the limit evaluates to

$$\lim_{\xi\to\infty}\left(-\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)=\sqrt{3} \tag{69}$$

Therefore, the expression for the mass for the case $n=5$ is

$$M_\star=4\pi\sqrt{3}\rho_c a^3 \tag{70}$$

Note that we got a finite mass for the star even though its radius is infinite.