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Physics 152 Problem Set 2

Problem 3

Problem 6.28. Landau potential at zero temperature (Gould and Tobochnik)
From (6.107) the Landau potential for an ideal Fermi gas at arbitrary T can be expressed as

$$\Omega = -kT \int_0^\infty g(\epsilon) \ln\left[1 + e^{-\beta(\epsilon-\mu)}\right] d\epsilon.$$

To obtain the $T = 0$ limit of Ω, we have that $\epsilon < \mu$ in (6.156), $\beta \to \infty$, and hence $\ln\left[1 + e^{-\beta(\epsilon-\mu)}\right] \to \ln e^{-\beta(\epsilon-\mu)} = -\beta(\epsilon - \mu)$. Hence show that

$$\Omega = \frac{(2m)^{3/2} V}{2\pi^2\hbar^3} \int_0^{\epsilon_F} \epsilon^{1/2} \left(\epsilon - \epsilon_F\right) d\epsilon.$$

Calculate Ω and determine the pressure at $T = 0$.

Solution:
In the limit $T \to 0$, we can reduce the Landau potential into

$$\Omega = -kT \int_0^\infty g(\epsilon) \ln\left[1 + e^{-\beta(\epsilon-\mu)}\right] d\epsilon \implies \Omega = -kT \int_0^\infty g(\epsilon) \left[-\beta(\epsilon - \mu)\right] d\epsilon \tag{1}$$

Note that $\beta = 1/kT$ so

$$\Omega = \int_0^\infty g(\epsilon)(\epsilon - \mu) d\epsilon \tag{2}$$

We can separate the integral into two intervals such that

$$\Omega = \int_0^{\mu(T=0)} g(\epsilon)(\epsilon - \mu) d\epsilon + \int_{\mu(T=0)}^\infty g(\epsilon)(\epsilon - \mu) d\epsilon \tag{3}$$

The second integral will vanish and the chemical potential μ at $T = 0$ is denoted as the Fermi energy:

$$\epsilon_F \equiv \mu(T = 0) \tag{4}$$

Thus, the expression reduces to

$$\Omega = \int_0^{\epsilon_F} g(\epsilon)(\epsilon - \epsilon_F) d\epsilon \tag{5}$$

We recall the density of states for nonrelativistic particles:

$$g(\epsilon) d\epsilon = \frac{V}{2\pi^2\hbar^3} (2m)^{3/2} \epsilon^{1/2} d\epsilon \tag{6}$$

and we substitute it into Ω to get

$$\Omega = \frac{(2m)^{3/2} V}{2\pi^2\hbar^3} \int_0^{\epsilon_F} \epsilon^{1/2} \left(\epsilon - \epsilon_F\right) d\epsilon \tag{7}$$

Consider the integral part of the expression. We can evaluate it to

$$\int_0^{\epsilon_F} \epsilon^{1/2} \left(\epsilon - \epsilon_F\right) d\epsilon = \int_0^{\epsilon_F} \left(\epsilon^{3/2} - \epsilon_F \epsilon^{1/2}\right) = \frac{2}{5}\epsilon_F^{5/2} - \frac{2}{3}\epsilon_F^{5/2} = -\frac{4}{15}\epsilon_F^{5/2} \tag{8}$$

Using this, we can simplify Ω:

$$\Omega = \frac{(2m)^{3/2}V}{2\pi^2\hbar^3} \cdot \left(-\frac{4}{15}\epsilon_F^{5/2}\right) = -\frac{(2m)^{3/2}2V}{15\pi^2\hbar^3}\epsilon_F^{5/2} \tag{9}$$

To get the pressure, we use the relation $\Omega = -PV$ and substitute Ω:

$$-PV = -\frac{(2m)^{3/2}2V}{15\pi^2\hbar^3}\epsilon_F^{5/2} \tag{10}$$

$$P = \frac{2(2m)^{3/2}}{15\pi^2\hbar^3}\epsilon_F^{5/2} \tag{11}$$