Physics 195 Problem Set 4

Problem 6

Consider a homogeneous, isotropic, cosmological model described by the line element

$$\mathrm{d}s^2 = \mathrm{d}t^2 + \frac{t}{t_*}\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right),$$

where t_* is a constant.

(a) Is this model closed, open, or flat?

Solution:

The Robertson-Walker metric is given by

$$ds^2 = -c^2dt^2 + a(t)^2\left[dr^2 + S_\kappa(r)^2 d\Omega^2\right] \tag{1}$$

We can change the given line element into polar coordinates such that

$$\mathrm{d}s^2 = \mathrm{d}t^2 + \frac{t}{t_*}\mathrm{d}r^2 \tag{2}$$

Comparing the two equations, we get the curvature of this model to be flat ($\kappa = 0$).

(b) Is this a matter-dominated universe? Explain.

Solution:

From (1) and (2), we see that the scale factor $a(t)$ is

$$a(t) = \left(\frac{t}{t_*}\right)^{1/2} \tag{3}$$

At time t_o, this becomes

$$a(t_o) = 1 = \left(\frac{t_o}{t_*}\right)^{1/2} \tag{4}$$

so we get the relation $t_o = t_*$.

Thus, the scale factor becomes

$$a(t) = \left(\frac{t}{t_o}\right)^{1/2} \tag{5}$$

Note that this is also the scale factor that we get for a spacially flat, single-component universe that is dominated by radiation. Therefore, the universe in this model is radiation-dominated, not matter-dominated.

(c) Assuming the Friedman equation holds for this universe, find the energy density $\epsilon(t)$ as a function of time.

Solution:

The Friedmann equation is given by

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon(t) - \frac{\kappa c^2}{R_0^2}\frac{1}{a(t)^2} \tag{6}$$

For a spatially flat universe, this reduces to

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon(t) \tag{7}$$

From this equation, we can get the energy density $\epsilon(t)$ given the scale factor:

$$\epsilon(t) = \frac{3c^2}{8\pi G}\left(\frac{\dot{a}}{a}\right)^2 = \frac{3c^2}{8\pi G}\cdot\frac{1}{4}\left(\frac{1}{t_*}\right)\left(\frac{t_*}{t}\right) = \frac{3c^2}{8\pi G}t^{-2} \tag{8}$$