Topics on Continua

Sergio Macías

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Second Edition

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To Elsa

León Felipe escribió un
tributo, no al héroe de la
historia, sino a su fiel
caballo Rocinante,
quien lo llevó en su lomo
por las tierras de España.

El héroe es, por supuesto,
Don Quijote de la Mancha:

"El Caballero de la Triste Figura"

¡Yo quería ese nombre!
pero me lo ganaron,
llegué a este mundo casi
trescientos cincuenta
años tarde...

Ya sólo me queda ser:

"El Caballero de la Triste Locura..."

S. M.

Preface to the Second Edition

After 12 years of the publication of *Topics on Continua* many things have happened. As it is well known, it is impossible to include everything. This Second Edition contains two new chapters which appear for the first time in a book, namely: *n-fold Hyperspace Suspensions* and *Induced Maps on n-fold Hyperspaces*. We include recent developments.

The first two chapters have very few modifications. In the first one, we prepare the way to prove the monotone-light factorization theorem, which appears later in chapter eight. We also add the notions of freely decomposable continuum and more concepts of aposyndesis. We include the notions of arc-smoothness of continua and arcwise decomposable continua too. For the second chapter we have not included much because of the two books on inverse limits and generalized inverse limits that appeared in 2012, namely: the book by Professors W. T. Ingram and William S. Mahavier *Inverse Limits: From Chaos to Continua*, Developments in Mathematics, Vol. 25, Springer, 2012 and the book by Professor W. T. Ingram *An Introduction to Inverse Limits with Set-valued Functions*, Springer Briefs in Mathematics, 2012. If the reader is interested in such topics, please refer to the mentioned books. We add the notions of confluent and weakly confluent maps to show that the bonding maps of an inverse limit are confluent if and only if the projection maps are confluent and the fact that each surjective map onto a chainable continuum is weakly confluent. By using inverse limits, it can be shown that the Cantor set is a topological group.

Chapter 3 has four new sections, namely: *Idempotency of* $\mathcal{T}$, *Three Decomposition Theorems*, *Examples*, and $\mathcal{T}$*-closed sets*. Throughout the chapter, we present characterizations of locally connected continua using the distinct forms of aposyndesis added in the first chapter. A sufficient condition for the idempotency on closed sets is given we also present an example showing that the condition is not necessary. We present a study of the relation between arc-smoothness and strict point $\mathcal{T}$-asymmetry. In particular, we show that Question 9.2.9 has a negative answer. We include results about the idempotency of $\mathcal{T}$ on products, cones, and suspensions. In particular, we prove that the first part of Question 9.2.3 has always a negative answer. We present three decomposition theorems using $\mathcal{T}$. In the strongest of the theorems, we obtain a continuous decomposition of the continuum with a locally connected quotient space and many of the elements of the decomposition are indecomposable continua. We present several classes of continua for which $\mathcal{T}$ is continuous and we study the family of $\mathcal{T}$-closed sets.

Chapter 4 remains essentially the same; we add three more consequences of the Property of Effros. The same happens with Chap. 5 where we include a few characterizations of the continuity of the set function $\mathcal{T}$ for homogeneous continua.

Chapter 6 has two new sections, namely: *Z-sets* and *Strong Size Maps*. Throughout the chapter, we include several bounds for the dimension of the n-fold hyperspace of certain classes of continua. We show that the n-fold hyperspaces are zero-dimensional aposyndetic. We give the correct statement and proof of Theorem 6.5.14. We give basic properties of Z-sets and sufficient conditions in order to show that the n-fold symmetric product of a continuum is a Z-set of the n-hyperspace of such continuum. We add several results that indicate when the n-fold symmetric product is a strong deformation retract of the m-fold hyperspace or of the hyperspace of closed sets. Also, we include properties of the continuum and the n-fold symmetric product when this is a retract of the m-fold hyperspace. We add a characterization of the graphs for which their n-fold hyperspace is a Cantor manifold. We also characterize the class of continua for which its n-fold hyperspace is a k-cell. We include results about suspensions and products related to the ones already given for cones. We end the chapter with a study of strong size maps, which are a nice generalization of Whitney maps to n-fold hyperspaces.

Chapter 7 is new. It is about hyperspace suspensions. We present most of what is known about n-fold hyperspace suspensions. We prove several properties of these spaces. We give sufficient conditions in order to obtain that n-fold hyperspace suspensions are contractible. We show that they are zero-dimensional aposyndetic. We study these hyperspaces when the continuum is locally connected. In particular, we give a sufficient condition to obtain that the n-fold hyperspace suspension of a locally connected continuum is the Hilbert cube. We characterize indecomposable continua by showing that their n-fold hyperspace suspensions are arcwise disconnected by removing two points. We present a description of the arc components of arcwise disconnected n-fold hyperspace suspensions when those two points are removed. We study properties of the n-fold hyperspace suspensions when they are homeomorphic to cones, suspensions, or products of continua. We present several results about the fixed point property of these hyperspaces. We study absolute n-fold hyperspace suspensions. We end this chapter by proving that hereditarily indecomposable continua have unique n-fold hyperspace suspensions.

Chapter 8 is also new. It is about induced maps between n-fold hyperspaces; these include hyperspace suspensions. We start with the definition of all the classes of maps that we study. Then we continue with general properties about the induced maps and present results about homeomorphisms, atomic maps, ε-maps, refinable maps, and almost monotone maps. We continue with results about confluent, monotone, open, light, and freely decomposable maps.

Chapter 9 (former Chap. 7), the last chapter, which is about questions, has two new sections (one for each of the new chapters), with questions on n-fold hyperspace suspensions and induced maps between n-fold hyperspaces.

I thank Javier Camargo for letting me include part of his dissertation in the second edition of the book.

I thank Ms. Elsa Arroyo for preparing all the pictures of the second edition of the book.

I thank the people at Springer, especially Professor Dr. Jan Holland, Ms. Anne Comment, Mr. Tilton Edward Stanley, Ms. Uma Periasamy and Ms. Kathleen Moriarty, for all their help.

Ciudad de México, México Sergio Macías

Preface to the First Edition

My aim is to present four of my favorite topics in continuum theory: inverse limits, Professor Jones's set function $\mathcal{T}$, homogeneous continua, and n-fold hyperspaces.

Most topics treated in this book are not covered in Professor Sam B. Nadler Jr.'s book: *Continuum Theory: An Introduction*, Monographs and Textbooks in Pure and Applied Math., Vol. 158, Marcel Dekker, New York, Basel, Hong Kong, 1992.

The reader is assumed to have taken a one-year course on general topology.

The book has seven chapters. In Chap. 1, we include the basic background to be used in the rest of the book. The experienced readers may prefer to skip this chapter and jump right to the study of their favorite subject. This can be done without any problem. The topics of Chap. 1 are essentially independent of one another and can be read at any time.

Chapter 2 is for the most part about inverse limits of continua. We present the basic results on inverse limits. Some theorems are stated without proof in Professor W. Tom Ingram's book: *Inverse Limits*, Aportaciones Matemáticas, Textos # 18, Sociedad Matemática Mexicana, 2000. We show that the operation of taking inverse limits commutes with the operations of taking finite products, cones, and hyperspaces. We also include some applications of inverse limits.

In Chap. 3 we discuss Professor F. Burton Jones's set function $\mathcal{T}$. After giving the basic properties of this function, we present properties of continua in terms of $\mathcal{T}$, such as connectedness im kleinen, local connectedness, and semi-local connectedness. We also study continua for which the set function $\mathcal{T}$ is continuous. In the last section we present some applications of $\mathcal{T}$.

In Chap. 4 we start our study of homogeneous continua. We present a topological proof of a Theorem of Professor E. G. Effros given by F. D. Ancel. We include a brief introduction to topological groups and group actions.

Chapter 5 contains our main study of homogeneous continua. We present two Decomposition Theorems of such continua, whose proofs are applications of Professor Jones's set function $\mathcal{T}$ and Professor Effros's Theorem. These theorems have narrowed the study of homogeneous continua in such a way that they may hopefully be eventually classified. We also give examples of nontrivial homogeneous continua and their covering spaces.

In Chap. 6 we present most of what is known about n-fold hyperspaces. This chapter is slightly different from the other chapters because the proofs of many of the theorems are based on results in the literature that we do not prove; however, we give references to the appropriate places where proofs can be found. This chapter is a complement of the two existing books—Sam B. Nadler, Jr., *Hyperspaces of Sets: A Text with Research Questions*, Monographs and Textbooks in Pure and Applied Math., Vol. 49, Marcel Dekker, New York, Basel, 1978[1] and Alejandro Illanes and Sam B. Nadler, Jr., *Hyperspaces: Fundamentals and Recent Advances*, Monographs and Textbooks in Pure and Applied Math., Vol. 216, Marcel Dekker, New York, Basel, 1999, in which a thorough study of hyperspaces is done.

In Chap. 6, we also prove general properties of n-fold hyperspaces. In particular, we show that n-fold hyperspaces are unicoherent and finitely aposyndetic. We study the arcwise accessibility of points of the n-fold symmetric products from their complement in n-fold hyperspaces. We give a treatment of the points that arcwise disconnect n-fold hyperspaces of indecomposable continua. Then we study continua for which the operation of taking n-fold hyperspaces is continuous ($\mathcal{C}_n^*$-smoothness). We also investigate continua for which there exist retractions between their various hyperspaces. Next, we present some results about the n-fold hyperspaces of graphs. We end Chap. 6 by studying the relation between n-fold hyperspaces and cones over continua.

We end the book with a chapter (Chap. 7) containing open questions on each of the subjects presented in the book.

We include figures to illustrate definitions and aspects of proofs.

The book originates from two sources—class notes I took from the course on continuum theory given by Professor James T. Rogers, Jr. at Tulane University in the Fall Semester of 1988 and the one-year courses on continuum theory I have taught in the graduate program of mathematics at the Facultad de Ciencias of the Universidad Nacional Autónoma de México, since the spring of 1993. I thank all the students who have taken such courses.

I thank María Antonieta Molina and Juan Carlos Macías for letting me include part of their thesis in the book. Ms. Molina's thesis was based on two talks on the set function $\mathcal{T}$ given by Professor David P. Bellamy in the *IV Research Workshop on Topology*, celebrated in Oaxaca City, Oaxaca, México, November 14 through 16, 1996.

I thank Professors Sam B. Nadler, Jr. and James T. Rogers, Jr. for reading parts of the manuscript and making valuable suggestions. I also thank Ms. Gabriela Sanginés and Mr. Leonardo Espinosa for answering my questions about LaTeX, while I was typing this book.

I thank Professor Charles Hagopian and Marvi Hagopian for letting me use their living room to work on the book during my visit to California State University, Sacramento.

[1] This book has been reprinted in: Aportaciones Matemáticas de la Sociedad Matemática Mexicana, Serie Textos # 33, 2006.

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