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Sergio Macías

Set Function $\mathcal{T}$

An Account on F. B. Jones' Contributions to Topology

Springer

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Elsa:

¿Cómo poder expresarte

Todo lo que hay dentro de mí?

Tu forma de ser me indica

Que todavía hay gente hermosa.

Tu presencia me motiva a seguir

Luchando por lo que creo.

En este momento me gustaría,

además de ser lo que soy,

Ser el más grande poeta

Y así poder escribirte

Los versos más bellos

Que nunca nadie

En la tierra escribiera

Y decirte, de esa manera

Muy hermosa,

Todo lo que por ti siento…

Te doy gracias porque eres

Mi compañera.

S. M.

In Memoriam

Alfredo Macías

María Alvarez

James T. Rogers, Jr.

Sam B. Nadler, Jr.

Preface

This book is timely. The literature on this subject has become so extensive that it is an overwhelming task for a new researcher to become familiar with what is known. It is nice to see a book organized so elegantly, which will make the subject more readily accessible. There are also a lot of opportunity for further work on the subject, so this book is much more than a complete history of the topic. Such a volume as that could not be written now, as there are too many gaps in our knowledge.

I am not including any definitions in this Preface, as they can be found in the text itself. The notion of aposyndesis, introduced by F. B. Jones, dates back more than three quarters of a century, to the early 1940's. This concept is a useful tool for the study of continua, especially those of intermediate complexity, that is to say, neither locally connected nor indecomposable. The set-valued set function $\mathcal{T}$ arises from the analysis of fine detail of when and how a continuum may be, or fail to be, aposyndetic. Professor Macías has provided this book just when it is needed. There is a substantial amount of information available in the literature on the set function $\mathcal{T}$ now, and this book provides a coherent and organized presentation of it, along with the most comprehensive bibliography I have seen on the subject. Both current researchers and students will find it to be a helpful reference book. It is also written in such a way that it can be used as a basis for a seminar, or possibly even a single individual, working alone, wishing to learn about the subject from scratch.

Through a quirk of history, the set function $\mathcal{T}$, in its earliest form, was called $\mathcal{L}$ and was also introduced by Professor Jones. The notation was mostly changed from $\mathcal{L}$ to $\mathcal{T}$ in the 1950's and early 1960's. Jones also introduced a companion set function $\mathcal{K}$; the name of $\mathcal{K}$ has not changed. In conversation, at least, $\mathcal{T}$ and $\mathcal{K}$ have been referred to as dual functions or adjoint functions; I have encountered both terms, but never in print, I think. The function $\mathcal{T}$ has been the primary focus of research over the years; $\mathcal{K}$ typically has been studied only when its use sheds light on properties of $\mathcal{T}$. The present volume continues that convention; I hope that it may lead to more study of $\mathcal{K}$ in its own right.

The function $\mathcal{T}$ has been both a subject of research in its own right and, increasingly, a powerful tool for investigating problems that, at first glance, seem to have nothing to do with $\mathcal{T}$.

The first chapter provides general background in topology and specifically the theory of continua. The discussion of the function $\mathcal{T}$ begins in earnest in the second chapter. I am very glad to have had an opportunity to peruse this volume. I expect that others will also find it interesting and useful.

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Introduction

The main purpose of this book is to present Professor F. Burton Jones' set function $\mathcal{T}$. This topic was treated in [92, Chapters 3 and 5] for metric continua. Here, we present most of those results for Hausdorff continua throughout the book. Whenever we could not find the way to remove the metric hypothesis, the result is presented in its original form. We include new topics not contained in [92, Chapters 3 and 5], as we mention below. The set function $\mathcal{T}$ has been used by many authors, for example as a tool to prove results about the semigroup structure of continua [71, 72] and [62], about the existence of a metric continuum that cannot be mapped onto its cone [10] or to characterize spheres [34]. In fact, it seems that [72] is the first paper to use the notation $\mathcal{T}$ in print. Even though [71] appeared before [72], the latter was written before than the former.

The book has eight chapters. In Chapter 1, we present the basic knowledge needed for the rest of the book on continuous decompositions, topological products and inverse limits (including generalized inverse limits with a single bonding function), uniformities, (Hausdorff) continua, uniformly completely regular maps and hyperspaces. The part on decomposition, continua and hyperspaces is brought from [92, Sections 1.2, 1.7 and 1.8] with the appropriate changes to Hausdorff spaces.

Chapter 2 contains the main properties of the set function $\mathcal{T}$. We consider its symmetry, additivity, idempotency and finite powers of it. In Chapter 3, we consider decomposition theorems using $\mathcal{T}$. We present three general decomposition theorems. Also, we give a weak version of Professor F. Burton Jones' Aposyndetic Decomposition Theorem for homogeneous Hausdorff continua using only the property of Kelley and then present the full version for such continua using the uniform property of Effros. We use the set function $\mathcal{T}$ to prove Professor Janusz R. Prajs' Mutual Aposyndetic Decomposition Theorem for Hausdorff homogeneous continua using the uniform property of Effros and give some relationships between this and Jones' theorem. We include the statements of most of the results of [92, Section 5.3] to prove Professor James T. Rogers, Jr.'s Terminal Decomposition Theorem for metric homogeneous continua.

Chapter 4 is about $\mathcal{T}$-closed sets, a topic studied by several authors, for instance [46] and [122]. We present the main properties of this class of sets, a couple of characterizations of $\mathcal{T}$-closed sets: one for Hausdorff continua and the other one for metric continua. A necessary condition to be a $\mathcal{T}$-closed set is given, and we prove that this condition is also sufficient for the class of metric continua with the property of Kelley. We study the class of minimal $\mathcal{T}$-closed sets and the set function $\mathcal{T}^{\infty}$. We end with the concept of $\mathcal{T}$-growth bound and give an upper bound of the $\mathcal{T}$-growth bound for the classes of homogeneous Hausdorff continua with the property of Kelley and type λ continua. We present a relationship of the $\mathcal{T}$-growth bound of continua and monotone and open monotone maps.

Chapter 5 deals with the continuity of the set function $\mathcal{T}$. The main difference between the results presented in Section 5.1 and the ones in [92, Section 3.3] is that new results appeared after the publication of [92], and these allow us to remove the hypothesis of *point $\mathcal{T}$-symmetry* of the continuum in many of the theorems. We present classes of metric continua for which $\mathcal{T}$ is continuous. In addition, we include results about the continuity of $\mathcal{T}$ on continua. In particular, we show the equivalence of the continuity of $\mathcal{T}$ and the continuity of its restriction to continua for the product of two continua. We end this chapter proving that the continuity of the set function $\mathcal{T}$ implies the continuity of Professor Jones' set function $\mathcal{K}$.

In Chapter 6, we study the images under the set function $\mathcal{T}$ of the hyperspace of closed subsets, 2^X, and the hyperspace of singletons, known as the first symmetric product, $\mathcal{F}_1(X)$, of a metric continuum X. We show that $\mathcal{T}(2^X)$ is an analytic set. We consider finite and countable images of $\mathcal{T}$. We also study connected and compact images of $\mathcal{T}$.

Chapter 7 contains applications of the set function $\mathcal{T}$. We study continuously irreducible continua and their hyperspace of subcontinua. Also, we characterize continuously type A' metric θ-continua. We present sufficient conditions for continua to be not contractible. We prove that arc-smooth metric continua are strict point $\mathcal{T}$-asymmetric and the reverse implication is true for fans. We consider R-subcontinua in dendroids. We present two characterizations of local connectedness. We give sufficient conditions for the image under $\mathcal{T}$ of a nonempty closed subset of a metric continuum to be a shore set. We consider generalized inverse limits of nonaposyndetic homogeneous metric continua X using the set function $\mathcal{T}|_{\mathcal{F}_1(X)}$ as a bonding function. We end the chapter giving more relationships between the set functions $\mathcal{T}$ and $\mathcal{K}$.

Chapter 8 contains the questions from [92, Section 9.2]. We give an update on the advances in answering those questions and include new open ones.

I thank Javier Camargo for letting me use some of the pictures he produced.

I thank Professor David P. Bellamy for the valuable suggestions he made to improve the book. I really appreciate the time and effort he spent reading the whole book in order to write a very nice Preface.

I thank my wife, Elsa, for all her love, help and support during the preparation of this manuscript.

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