Change of variable 1D case

Theorem 0.1. *[1] Let X be a continuous random variable with PDF f_x, and let $Y = g(X)$, where g is differentiable and strictly increasing (or strictly decreasing). Then the PDF of Y is given by*

$$f_Y(y) = f_X(x)\Big|\frac{\mathrm{d}x}{\mathrm{d}y}\Big|$$

Proof. Let g be strictly increasing. Then CDF of Y is:

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(g(X) \leq y) = \mathbb{P}(X \leq g^{-1}(y)) = F_X(g^{-1}(y)) = F_X(x)$$

Then, by chain rule:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}F_Y(y) &= \frac{\mathrm{d}}{\mathrm{d}x}F_X(x) \\ \frac{\mathrm{d}y}{\mathrm{d}x}f_Y(y) &= f_X(x) \\ f_Y(y) &= f_X(x)\,\frac{\mathrm{d}x}{\mathrm{d}y} \quad \text{, where } \frac{\mathrm{d}x}{\mathrm{d}y} > 0 \end{aligned}$$

Next, let g be strictly decreasing. Then CDF of Y is:

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(g(X) \leq y) = \mathbb{P}(X \geq g^{-1}(y)) = 1 - F_X(g^{-1}(y)) = 1 - F_X(x)$$

Then, by chain rule:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}F_Y(y) &= \frac{\mathrm{d}}{\mathrm{d}x}(1 - F_X(x)) \\ \frac{\mathrm{d}y}{\mathrm{d}x}f_Y(y) &= -f_X(x) \\ f_Y(y) &= -f_X(x)\,\frac{\mathrm{d}x}{\mathrm{d}y} \quad \text{, where } \frac{\mathrm{d}x}{\mathrm{d}y} < 0 \end{aligned}$$

Combining two cases,

$$f_Y(y) = f_x(x)\Big|\frac{\mathrm{d}x}{\mathrm{d}y}\Big|$$

□

References

[1] Joseph K. Blitzstein and Jessica Hwang. *Introduction of Probability*. CRC press Boca Raton, FL, 2015.