Change of variable 1D case

Theorem 0.1. *[1] Let X be a continuous random variable with PDF f_X, and let $Y = g(X)$, where g is differentiable and strictly increasing (or strictly decreasing). Then the PDF of Y is given by*

$$f_Y(y) = f_X(x)\Big|\frac{\mathrm{d}x}{\mathrm{d}y}\Big|$$

Proof. If g is strictly increasing. Then CDF of Y is:

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(g(X) \leq y) = \mathbb{P}(X \leq g^{-1}(y)) = F_X(g^{-1}(y)) = F_X(x)$$

Then, by the Fundamental Theorem of Calculus and chain rule:

$$\begin{aligned}
f_Y(y) &= \frac{\mathrm{d}}{\mathrm{d}y}F_Y(y) \quad \text{by the Fundamental Theorem of Calculus} \\
&= \frac{\mathrm{d}}{\mathrm{d}y}F_X(x) \quad \text{since } F_Y(y) = F_X(x) \\
&= \frac{\mathrm{d}}{\mathrm{d}y}F_X(g^{-1}(y)) \quad \text{since } x = g^{-1}(y) \\
&= F_X'(g^{-1}(y)) \cdot (g^{-1}(y))' \quad \text{by definition of chain rule} \\
&= F_X'(x) \cdot (g^{-1}(y))' \\
&= \frac{\mathrm{d}}{\mathrm{d}x}F_X(x) \cdot \frac{\mathrm{d}}{\mathrm{d}y}g^{-1}(y) \quad \text{by chain rule, 1st derivative taken wrt x, while the second taken wrt to y} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\Big(\int_a^x f_X(t)dt\Big) \cdot \frac{\mathrm{d}x}{\mathrm{d}y} \quad \text{by definitions of integration and derivation} \\
&= f_X(x)\,\frac{\mathrm{d}x}{\mathrm{d}y} \quad \text{, where } \frac{\mathrm{d}x}{\mathrm{d}y} > 0, \quad \text{by the Fundamental Theorem of Calculus}
\end{aligned}$$

If g is strictly decreasing, then CDF of Y is:

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(g(X) \leq y) = \mathbb{P}(X \geq g^{-1}(y)) = 1 - F_X(g^{-1}(y)) = 1 - F_X(x)$$

Then, by chain rule:

$$\begin{aligned}
f_Y y &= \frac{\mathrm{d}}{\mathrm{d}y}F_Y(y) \quad \text{by the Fundamental Theorem of Calculus} \\
&= \frac{\mathrm{d}}{\mathrm{d}y}(1 - F_X(x)) \\
&= -\frac{\mathrm{d}}{\mathrm{d}y}F_X(x) \quad \text{where } x = g^{-1}(y) \text{ ,so need to use chain rule} \\
&= -\frac{\mathrm{d}}{\mathrm{d}x}F_X(x) \cdot \frac{\mathrm{d}x}{\mathrm{d}y} \\
&= -f_X(x)\,\frac{\mathrm{d}x}{\mathrm{d}y} \quad \text{, where } \frac{\mathrm{d}x}{\mathrm{d}y} < 0, \quad \text{by the Fundamental Theorem of Calculus} \\
&= f_X(x)\,\Big|\frac{\mathrm{d}x}{\mathrm{d}y}\Big|
\end{aligned}$$

General equation to both cases, $f_Y(y) = f_X(x)\Big|\frac{\mathrm{d}x}{\mathrm{d}y}\Big|$ □

References

[1] Joseph K. Blitzstein and Jessica Hwang. *Introduction of Probability*. CRC press Boca Raton, FL, 2015.