Parallactic Angle Calculation

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April 2019

1 Introduction

The parallactic angle defines the rotation between RA/Dec and Az/ZA coordinates. It is calculated in FHD in the `parallactic_angle.pro` function and is used in `rotate_jones_matrix.pro` to construct the Jones matrix, which converts between instrumental and Stokes polarization modes.

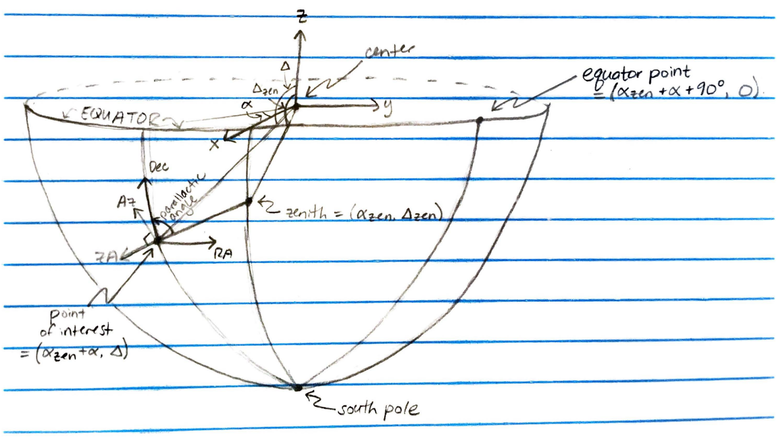
This calculation does not account for aberration.

2 Parallactic Angle Calculation

Here we consider a point with (RA, Dec) coordinates $(\alpha_{\mathrm{zen}} + \alpha, \Delta)$ and consider the angle that rotates a vector at that point between (Az, ZA) coordinates and (RA, Dec) coordinates.

The (RA, Dec) coordinates of zenith are $(\alpha_{\mathrm{zen}}, \Delta_{\mathrm{zen}})$. We can define a Cartesian coordinate system with the origin at the center of the sphere and the x-axis aligned with RA=α_{zen}. We define the radius of the sphere to be 1.

We define our coordinates looking *down*, such that at zenith RA increases in the $+y$ direction and Dec increases in the $+z$ direction. ZA increases away from zenith and Az increases clockwise from the $+y$ direction at zenith.



Converting (RA, Dec) to Cartesian coordinates, the vector to our point $(\alpha_{\mathrm{zen}} + \alpha, \Delta)$ is

$$\text{point} = \cos\alpha \cos\Delta \hat{x} + \sin\alpha \cos\Delta \hat{y} + \sin\Delta \hat{z}. \tag{1}$$

and to zenith is

$$\text{zenith} = \cos\Delta_{\mathrm{zen}} \hat{x} + \sin\Delta_{\mathrm{zen}} \hat{z}. \tag{2}$$

At our point, the Az/ZA coordinate system is aligned with the plane that includes the center of the sphere, the point, and zenith. A vector normal to this plane can be calculated by taking the cross product between the vectors point and zenith:

$$\begin{aligned}&\text{zenith} \times \text{point} = \\ &\sin\Delta_{\text{zen}}\cos\Delta\sin\alpha\hat{x} + (\sin\Delta_{\text{zen}}\cos\Delta\cos\alpha - \cos\Delta_{\text{zen}}\sin\Delta)\hat{y} + \cos\Delta_{\text{zen}}\cos\Delta\sin\alpha\hat{z}\end{aligned} \tag{3}$$

To normalize, we find the amplitude of this vector:

$$\begin{aligned}&|\text{zenith} \times \text{point}|^2 \\ &= \sin^2\alpha\cos^2\Delta + \sin^2\Delta_{\text{zen}}\cos^2\Delta\cos^2\alpha + \cos^2\Delta_{\text{zen}}\sin^2\Delta \\ &- 2\sin\Delta_{\text{zen}}\cos\Delta_{\text{zen}}\cos\alpha\sin\Delta\cos\Delta \\ &= \cos^2\Delta_{\text{zen}}(\cos\Delta\tan\Delta_{\text{zen}} - \sin\Delta\cos\alpha)^2 + \cos^2\Delta_{\text{zen}}\sin^2\alpha,\end{aligned} \tag{4}$$

so the unit vector normal to the plane is

$$\begin{aligned}&\frac{\text{zenith} \times \text{point}}{|\text{zenith} \times \text{point}|} = \\ &\frac{\sin\Delta_{\text{zen}}\cos\Delta\sin\alpha\hat{x} + (\sin\Delta_{\text{zen}}\cos\Delta\cos\alpha - \cos\Delta_{\text{zen}}\sin\Delta)\hat{y} + \cos\Delta_{\text{zen}}\cos\Delta\sin\alpha\hat{z}}{\sqrt{\cos^2\Delta_{\text{zen}}(\cos\Delta\tan\Delta_{\text{zen}} - \sin\Delta\cos\alpha)^2 + \cos^2\Delta_{\text{zen}}\sin^2\alpha}}.\end{aligned} \tag{5}$$

This vector is parallel to the -Az axis.

At our point, the RA/Dec coordinate system is aligned with a plane that includes the center of the sphere, the point, and a point we'll call "equator point" at

$$\text{equator point} = \cos(\alpha + 90^\circ)\hat{x} + \sin(\alpha + 90^\circ)\hat{y} = -\sin\alpha\hat{x} + \cos\alpha\hat{y}. \tag{6}$$

A vector normal to this plane can be calculated by taking the cross product between the vectors point and equator point:

$$\text{point} \times \text{equator point} = -\sin\Delta\cos\alpha\hat{x} - \sin\Delta\sin\alpha\hat{y} + \cos\Delta\hat{z}. \tag{7}$$

To normalize, we find that the amplitude of this vector is

$$|\text{point} \times \text{equator point}|^2 = 1, \tag{8}$$

so the unit vector normal to the plane is

$$\frac{\text{point} \times \text{equator point}}{|\text{point} \times \text{equator point}|} = -\sin\Delta\cos\alpha\hat{x} - \sin\Delta\sin\alpha\hat{y} + \cos\Delta\hat{z}. \tag{9}$$

This vector is parallel to the Dec axis.

Taking the dot product of these two vectors gives

$$\begin{aligned}&\left(\frac{\text{zenith} \times \text{point}}{|\text{zenith} \times \text{point}|}\right) \cdot \left(\frac{\text{point} \times \text{equator point}}{|\text{point} \times \text{equator point}|}\right) \\ &= \frac{\cos\Delta_{\text{zen}}\sin\alpha}{\sqrt{\cos^2\Delta_{\text{zen}}(\cos\Delta\tan\Delta_{\text{zen}} - \sin\Delta\cos\alpha)^2 + \cos^2\Delta_{\text{zen}}\sin^2\alpha}}\end{aligned} \tag{10}$$

so the angle between the planes is

$$\cos^{-1}\left[\frac{\cos\Delta_{\text{zen}}\sin\alpha}{\sqrt{\cos^2\Delta_{\text{zen}}(\cos\Delta\tan\Delta_{\text{zen}}-\sin\Delta\cos\alpha)^2+\cos^2\Delta_{\text{zen}}\sin^2\alpha}}\right]. \tag{11}$$

This corresponds to the angle that rotates the -Az axis counterclockwise into the +Dec axis. The parallactic angle is defined as the angle that rotates the -ZA axis into the +Dec, so

$$\begin{aligned}&\text{parallactic angle}\\&=\sin^{-1}\left[\frac{-\cos\Delta_{\text{zen}}\sin\alpha}{\sqrt{\cos^2\Delta_{\text{zen}}(\cos\Delta\tan\Delta_{\text{zen}}-\sin\Delta\cos\alpha)^2+\cos^2\Delta_{\text{zen}}\sin^2\alpha}}\right].\end{aligned} \tag{12}$$

To simplify, we can use some handy trig identities to rewrite

$$\text{parallactic angle}=\tan^{-1}\left(\frac{-\sin\alpha}{\cos\Delta\tan\Delta_{\text{zen}}-\sin\Delta\cos\alpha}\right). \tag{13}$$

The parallactic angle is defined as the counterclockwise angle between the -ZA axis and the +Dec axis. It lies in quadrants III and IV when $\alpha > 0$ and quadrants I and II when $\alpha < 0$.

3 Implementation in FHD

The parallactic angle is calculated in FHD in `parallactic_angle.pro`. In that function, `latitude` $=\Delta_{\text{zen}}$, `dec` $=\Delta$, and `hour_angle` $=-\alpha$ as defined in this memo (note that the hour angle is defined as increasing to the West, the opposite of the Right Ascension).

The IDL inverse tangent function `atan` has a range of $(-\pi/2, \pi/2]$ when only one argument is provided. When two arguments are provided, the range is $(-\pi, \pi]$: the range is $(0, \pi]$ when the first argument is positive and $(-\pi, 0)$ when the first argument is negative.

`parallactic_angle.pro` is called by `projection_slant_orthographic.pro` and by `rotate_jones_matrix.pro`, which is used in constructing the Jones matrix for converting between Stokes and instrumental polarization modes. In FHD, the Jones matrix is defined such that

$$J\begin{bmatrix}\text{RA}\\\text{Dec}\end{bmatrix}=\begin{bmatrix}p\\q\end{bmatrix}, \tag{14}$$

where p is the East-West dipole orientation, q is the North-South dipole orientation, and

$$J=\begin{bmatrix}J[0,0] & J[1,0]\\J[0,1] & J[1,1]\end{bmatrix} \tag{15}$$

Before rotation, the function `fhd_struct_init_antenna.pro` returns a Jones matrix (that we will call J') that converts from Az/ZA to instrumental polarizations:

$$J'\begin{bmatrix}\text{ZA}\\\text{Az}\end{bmatrix}=\begin{bmatrix}p\\q\end{bmatrix}. \tag{16}$$

`rotate_jones_matrix.pro` applies a rotation to calculate $J = J'R$ where R is the rotation

$$R\begin{bmatrix}\mathrm{RA}\\ \mathrm{Dec}\end{bmatrix}=\begin{bmatrix}\mathrm{ZA}\\ \mathrm{Az}\end{bmatrix}. \tag{17}$$

The parallactic angle is the angle between the -ZA axis (pointing towards zenith) and the Dec axis. The rotation matrix is

$$R=\begin{bmatrix}R[0,0] & R[1,0]\\ R[0,1] & R[1,1]\end{bmatrix}=\begin{bmatrix}\sin\delta & -\cos\delta\\ -\cos\delta & -\sin\delta\end{bmatrix}. \tag{18}$$

Note that $R^{-1}=R$. Under IDL convention, the matrix product AB is given by `matrix_multiply(B, A)`.

In `fhd_struct_init_jones.pro`, `Jmat` is the tensor product of J with its conjugate:

$$\texttt{Jmat}=J\otimes J^*=\begin{bmatrix}|J[0,0]|^2 & |J[1,0]|^2 & J[0,0]J^*[1,0] & J[1,0]J^*[0,0]\\ |J[0,1]|^2 & |J[1,1]|^2 & J[0,1]J^*[1,1] & J[1,1]J^*[0,1]\\ J[0,0]J^*[0,1] & J[1,0]J^*[1,1] & J[0,0]J^*[1,1] & J[1,0]J^*[0,1]\\ J[0,1]J^*[0,0] & J[1,1]J^*[1,0] & J[0,1]J^*[1,0] & J[1,1]J^*[0,0]\end{bmatrix} \tag{19}$$

where

$$\texttt{Jmat}=\begin{bmatrix}\texttt{Jmat}[0,0] & \texttt{Jmat}[1,0] & \texttt{Jmat}[2,0] & \texttt{Jmat}[3,0]\\ \texttt{Jmat}[0,1] & \ddots & & \\ \texttt{Jmat}[0,2] & & & \\ \texttt{Jmat}[0,3] & & & \end{bmatrix}. \tag{20}$$

`Jmat` converts from RA/Dec coordinates to instrumental polarization:

$$\texttt{Jmat}\begin{bmatrix}\alpha\alpha^*\\ \Delta\Delta^*\\ \alpha\Delta^*\\ \Delta\alpha^*\end{bmatrix}=\begin{bmatrix}pp^*\\ qq^*\\ pq^*\\ qp^*\end{bmatrix}, \tag{21}$$

and its inverse, `Jinv`, does the opposite transformation:

$$\texttt{Jinv}\begin{bmatrix}pp^*\\ qq^*\\ pq^*\\ qp^*\end{bmatrix}=\begin{bmatrix}\alpha\alpha^*\\ \Delta\Delta^*\\ \alpha\Delta^*\\ \Delta\alpha^*\end{bmatrix}. \tag{22}$$

4 Conversion to Stokes

The conversion between sky coordinates and Stokes parameters is done by `stokes_cnv.pro` in FHD. Stokes parameters are inherently tied to a sky coordinate system, preferably

RA/Dec. The parameters are defined as follows:

$$\begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & i & -i \end{bmatrix} \begin{bmatrix} \alpha\alpha^* \\ \Delta\Delta^* \\ \alpha\Delta^* \\ \Delta\alpha^* \end{bmatrix} \tag{23}$$

and the inverse

$$\begin{bmatrix} \alpha\alpha^* \\ \Delta\Delta^* \\ \alpha\Delta^* \\ \Delta\alpha^* \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & -i/2 \\ 0 & 0 & 1/2 & i/2 \end{bmatrix} \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix}. \tag{24}$$

$$\begin{bmatrix} \alpha\alpha^* \\ \Delta\Delta^* \\ \alpha\Delta^* \\ \Delta\alpha^* \end{bmatrix} = \begin{bmatrix} \sin^2\delta & \cos^2\delta & -\sin\delta\cos\delta & -\sin\delta\cos\delta \\ \cos^2\delta & \sin^2\delta & \sin\delta\cos\delta & \sin\delta\cos\delta \\ -\sin\delta\cos\delta & \sin\delta\cos\delta & -\sin^2\delta & \cos^2\delta \\ -\sin\delta\cos\delta & \sin\delta\cos\delta & \cos^2\delta & -\sin^2\delta \end{bmatrix} \begin{bmatrix} |Z|^2 \\ |A|^2 \\ ZA^* \\ AZ^* \end{bmatrix} \tag{25}$$

where Z is the ZA coordinate and A is the Az coordinate. For an unpolarized source, such that $Q = U = V = 0$, we find that

$$\begin{bmatrix} |Z|^2 \\ |A|^2 \\ ZA^* \\ AZ^* \end{bmatrix} = \begin{bmatrix} \alpha\alpha^* \\ \Delta\Delta^* \\ \alpha\Delta^* \\ \Delta\alpha^* \end{bmatrix} = \begin{bmatrix} I/2 \\ I/2 \\ 0 \\ 0 \end{bmatrix}, \tag{26}$$

where I is the source flux, so the Az/ZA vector components have no dependence on the the parallactic angle.