Notes on *Introduction to Random Graphs* [FK23]

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Contents

1 Mathematical Symbols and Technique Tools

Before we start our discussion on random graphs, it is of great necessity to state some mathematical symbols and technique tools for completeness.

1.1 Probabilistic methods

The most common tools we use are the moment methods, especially the *first moment method (the Markov inequality)* and *the second moment method (the Chebyshev inequality).*

Lemma 1.1 (The Markov Inequality)**.** *Let X be a non-negative random variable. Then for all $t > 0$,*

$$\mathbf{Pr}\left[X \geq t\right] \leq \frac{\mathbf{E}\left[X\right]}{t}.$$

Theorem 1.2 (The First Moment Method)**.** *Let X be a non-negative integer-valued random variable. Then*

$$\mathbf{Pr}\left[X > 0\right] \leq \mathbf{E}\left[X\right].$$

Lemma 1.3 (The Chebyshev Inequality)**.** *Let X be a random variable with finite mean and finite variance. Then for $t > 0$, it holds that*

$$\mathbf{Pr}\left[\left|X - \mathbf{E}\left[X\right]\right| \geq t\right] \leq \frac{\mathbf{Var}\left[X\right]}{t^2}.$$

Theorem 1.4 (The Second Moment Method)**.** *Let X be a non-negative integer valued random variable. Then*

$$\mathbf{Pr}\left[X = 0\right] \leq \frac{\mathbf{Var}\left[X\right]}{\mathbf{E}\left[X\right]^2}. \tag{1}$$

Furthermore, it holds that

$$\mathbf{Pr}\left[X = 0\right] \leq \frac{\mathbf{Var}\left[X\right]}{\mathbf{E}\left[X^2\right]}. \tag{2}$$

Proof. The first inequality is quite easy to show by Lemma 1.3. For the second one, note that

$$X = X \cdot \mathbb{1}\left[X \geq 1\right].$$

Then by the Cauchy-Schwarz inequality,

$$\mathbf{E}\left[X\right]^2 = \left(\mathbf{E}\left[X \cdot \mathbb{1}\left[X \geq 1\right]\right]\right)^2 \leq \mathbf{E}\left[X^2\right]\mathbf{Pr}\left[X \geq 1\right].$$

$\square$

2 Basic Models of Random Graphs

Before we begin all studies on properties, firstly we introduce the models that we usually take into account.

Let $\mathscr{G}_{n,m}$ be the collection of all graphs $G = (V, E)$ with $|V| = n$ and $|E| = m$. For convenience, we assume that $V = \{1, \ldots, n\}$. To ensure that $\mathscr{G}_{n,m}$ is well-defined, always suppose that $0 \leq m \leq \binom{n}{2}$. For every $G \in \mathscr{G}_{n,m}$, we equip it with probability

$$\mathbb{P}(G) = \binom{\binom{n}{2}}{m}^{-1}.$$

It's easy to note that following the probability, we draw a graph with n vertices and m edges uniformly at random. We denote this random graph by $\mathcal{G}_{n,m} = (V = [n], E_{n,m})$ and call it a *uniform random graph.*

Another random graph model we consider is similar. Given a real $p \in [0, 1]$. For $0 \leq m \leq \binom{n}{2}$ and every graph $G = (V, E)$ with $|V| = n$ and $|E| = m$, we assign to G the probability

$$\mathbb{P}(G) = p^m (1-p)^{\binom{n}{2}-m}.$$

We denote this random graph by $\mathcal{G}_{n,p} = (V = [n], E_{n,p})$ and call it an *Erdős-Rényi random graph.*

The two models are strongly related to each other.

Lemma 2.1. *A random graph $\mathcal{G}_{n,p}$ given that the number of its edge is m, is equally likely to be one of the graph $G \sim \mathcal{G}_{n,m}$.*

Proof. For every $G = (V, E)$ with $|E| = m$, simply we can observe that

$$\left\{\mathcal{G}_{n,p} = G\right\} \subseteq \left\{\left|E_{n,p}\right| = m\right\}.$$

Then by calculation,

$$\begin{aligned}
\mathbf{Pr}\left[\mathcal{G}_{n,p} = G \;\middle|\; \left|E_{n,p}\right| = m\right] &= \frac{\mathbf{Pr}\left[\mathcal{G}_{n,p} = G \wedge \left|E_{n,p}\right| = m\right]}{\mathbf{Pr}\left[\left|E_{n,p}\right| = m\right]} \\
&= \frac{p^m (1-p)^{\binom{n}{2}-m}}{p^m (1-p)^{\binom{n}{2}-m} \binom{\binom{n}{2}}{m}} \\
&= \binom{\binom{n}{2}}{m}^{-1} \\
&= \mathbf{Pr}\left[\mathcal{G}_{n,m} = G\right].
\end{aligned}$$

□

Intuitively, the two random graphs perform a similar fashion when m is closed to the expected number of the edges of $\mathcal{G}_{n,p}$, *i.e.*,

$$m = \binom{n}{2} p = (1 + o(1)) \frac{n^2 p}{2}$$

or

$$p = \frac{m}{\binom{n}{2}} = (1 + o(1)) \frac{2m}{n^2}.$$

To generate the random graphs, we usually apply a coupling technique. Suppose that $p_1 < p$ and p_2 is defined by

$$1 - p = (1 - p_1)(1 - p_2).$$

Now we independently draw $\mathcal{G}(n,p_1)$ and $\mathcal{G}(n,p_2)$, and let $\mathcal{G}_{n,p} = \mathcal{G}(n,p_1) \cup \mathcal{G}(n,p_2)$. So when we write

$$\mathcal{G}(n,p_1) \subseteq \mathcal{G}_{n,p},$$

it means that the two graphs are coupled so that $\mathcal{G}_{n,p}$ is obtained from $\mathcal{G}(n,p_1)$ by the method described above.

To introduce a similar coupling process for $\mathcal{G}_{n,m}$, firstly consider $m_1 < m$. Then let

$$\mathcal{G}_{n,m} = \mathcal{G}(n,m_1) \cup \mathcal{H}$$

where $\mathcal{H}$ is a random graph with exactly $m_2 = m - m_1$ edges uniformly generated from $\binom{[n]}{2} \setminus E_{n,m_1}$.

Pseudo-random graphs

Besides the 'real' random graph models, the following two models will be taken into account.

- **Model A:** Let $\mathbf{x} = (x_1, \ldots, x_{2m})$ be chosen uniformly at random from $[n]^{2m}$.
- **Model B:** Let $\mathbf{x} = (x_1, \ldots, x_{2m})$ be chosen uniformly at random from $\binom{[n]}{2}^m$.

For $X \in \{A, B\}$, we construct the random graph $\mathcal{G}_{n,m}^{(X)}$ with the vertex set $[n]$ and edge set $E_m = \{(x_{2i-1}, x_{2i}) \ : \ i = 1, \ldots, m\}$. Note that the graph might be a multi-graph. To generate the simple graph $\mathcal{G}_{n,m}^{(X,-)}$ with m^- edges, we remove all self-loops and multiple edges. It can be seen that conditional the value of m^-, the simple graphs generated by the above two models are distributed the same as $\mathcal{G}_{n,m}$.

Also, it holds that, by symmetry for every $G_1 \in \mathscr{G}_{n,m}$ and $G_2 \in \mathscr{G}_{n,m}$,

$$\mathbf{Pr}\left[\mathcal{G}_{n,m}^{(X)} = G_1 \,\middle|\, \mathcal{G}_{n,m}^{(X)} \text{ is simple}\right] = \mathbf{Pr}\left[\mathcal{G}_{n,m}^{(X)} = G_2 \,\middle|\, \mathcal{G}_{n,m}^{(X)} \text{ is simple}\right]$$

for $X \in \{A, B\}$.

When $m = cn$ with constant parameter $c > 0$, it holds that

$$\mathbf{Pr}\left[\mathcal{G}_{n,m}^{(X)} \text{ is simple}\right] \geq \binom{\binom{n}{2}}{m} \frac{m! 2^m}{n^{2m}} \geq (1 - o(1)) \exp\left(-c^2 - c\right).$$

Then we know that

$$\mathbf{Pr}\left[\mathcal{G}_{n,m} \in \mathscr{P}\right] = \mathbf{Pr}\left[\mathcal{G}_{n,m}^{(X)} \in \mathscr{P} \,\middle|\, \mathcal{G}_{n,m}^{(X)} \text{ is simple}\right] \leq (1 + o(1)) e^{c^2+c} \mathbf{Pr}\left[\mathcal{G}_{n,m}^{(X)} \in \mathscr{P}\right].$$

Then to show the random graph does not satisfy some graph property, when $m = O(n)$, it is feasible to turn to the pseudo-random graph models.

2.1 Results on random graph properties

Now we consider the property of graphs.

Definition 2.2 (Graph Property)**.** Fix a vertex set $V = [n]$. A *graph property* $\mathscr{P}$ is a collection of graphs $G = (V, E)$ where $E \subseteq \binom{[n]}{2}$.

Lemma 2.3. *Let $\mathscr{P}$ be any graph property and $p = m/\binom{n}{2}$ where $m = m(n) \to \infty$ and $\binom{n}{2} - m \to \infty$ as $n \to \infty$. Then for sufficiently large n,*

$$\mathbf{Pr}\left[\mathcal{G}_{n,m} \in \mathscr{P}\right] \leq 10 m^{1/2} \mathbf{Pr}\left[\mathcal{G}_{n,p} \in \mathscr{P}\right].$$

Proof. By the law of total probability,

$$\begin{aligned}
\mathbf{Pr}\left[\mathcal{G}_{n,p} \in \mathcal{P}\right] &= \sum_{k=0}^{\binom{n}{2}} \mathbf{Pr}\left[\mathcal{G}_{n,p} \in \mathcal{P} \mid \left|E_{n,p} = k\right|\right] \mathbf{Pr}\left[\left|E_{n,p}\right| = k\right] \\
&= \sum_{k=0}^{\binom{n}{k}} \mathbf{Pr}\left[\mathcal{G}(n,k) \in \mathcal{P}\right] \mathbf{Pr}\left[\left|E_{n,p}\right| = k\right] \\
&\geq \mathbf{Pr}\left[\mathcal{G}_{n,m} \in \mathcal{P}\right] \mathbf{Pr}\left[\left|E_{n,p}\right| = m\right]
\end{aligned}$$

where the second equality holds by Lemma 2.1. Now it is suffices to estimate the term $\mathbf{Pr}\left[\left|E_{n,p}\right| = m\right]$. By definition,

$$\mathbf{Pr}\left[\left|E_{n,p}\right| = m\right] = \binom{\binom{n}{2}}{m} p^m (1-p)^{\binom{n}{2}-m}.$$

By Stirling's formula,

$$k! = (1+o(1))\sqrt{2\pi k}\frac{k^k}{e^k}.$$

Then when $m = m(n) \to \infty$ and $\binom{n}{2} - m \to \infty$ as $n \to \infty$,

$$\begin{aligned}
\mathbf{Pr}\left[\left|E_{n,p}\right| = m\right] &= (1+o(1))\sqrt{\frac{\binom{n}{2}}{2\pi m(\binom{n}{2} - m)}} \\
&\geq \frac{1}{10\sqrt{m}}.
\end{aligned}$$

Putting it into the above inequality we conclude the lemma. □

When the property $\mathcal{P}$ is so called *monotone increasing*, the result of Lemma 2.3 can be tightened.

Definition 2.4 (Monotone Increasing Graph Property)**.** A graph property $\mathcal{P}$ is said to be *monotone increasing* if $G \in \mathcal{P}$ implies $G + e \in \mathcal{P}$. Furthermore, it is said to be *non-trivial* if the empty graph $\emptyset \notin \mathcal{P}$ and the complete graph $K_n \in \mathcal{P}$.

Remark 2.5. From the view of coupling, if $\mathcal{P}$ is monotone increasing, then whenever $p \leq p'$ or $m < m'$, if $\mathcal{G}_{n,p} \in \mathcal{P}$ or $\mathcal{G}_{n,m} \in \mathcal{P}$, then

$$\mathcal{G}(n,p') \in \mathcal{P}, \quad \mathcal{G}(n,m_1) \in \mathcal{P}.$$

Lemma 2.6. *Let $\mathcal{P}$ be a monotone increasing graph property. Given integers $n, m > 0$, fix $p = \frac{m}{N}$ where $N = \binom{n}{2}$. Then for large n and $p = o(1)$ such that Np, $\frac{N(1-p)}{\sqrt{Np}} \to \infty$ as $n \to \infty$,*

$$\mathbf{Pr}\left[\mathcal{G}_{n,m} \in \mathcal{P}\right] \leq 3\mathbf{Pr}\left[\mathcal{G}_{n,p} \in \mathcal{P}\right].$$

Proof. Since $\mathcal{P}$ is monotone increasing, we know

$$\mathbf{Pr}\left[\mathcal{G}_{n,p} \in \mathcal{P}\right] \geq \sum_{k=m}^{N} \mathbf{Pr}\left[\mathcal{G}(n,k) \in \mathcal{P}\right] \mathbf{Pr}\left[\left|E_{n,p}\right| = k\right].$$

By Remark 2.5, for $m \leq k \leq N$,

$$\mathbf{Pr}\left[\mathcal{G}(n,k)\right] \geq \mathbf{Pr}\left[\mathcal{G}_{n,m} \in \mathcal{P}\right].$$

Then we know

$$\mathbf{Pr}\left[\mathcal{G}_{n,p} \in \mathcal{P}\right] \geq \mathbf{Pr}\left[\mathcal{G}_{n,m} \in \mathcal{P}\right] \sum_{k=m}^{N} u_k$$

where

$$u_k = \binom{N}{k} p^k (1-p)^{N-k}.$$

Using Stirling's formula, we know

$$u_m = \frac{1+o(1)}{(2\pi m)^{1/2}}.$$

For $0 \leq k - m \leq m^{1/2}$, we know

$$\frac{u_{k+1}}{u_k} = \frac{(N-k)p}{(k+1)(1-p)} \geq \exp\left(-\frac{k-m}{N-k} - \frac{m-k+1}{m}\right).$$

Then it follows that for $0 \leq t \leq m^{1/2}$,

$$u_{m+t} \geq \frac{\exp\left(-\frac{t^2}{2m} - o(1)\right)}{(2\pi m)^{1/2}}.$$

Then we know

$$\sum_{k=m}^{N} u_k \geq \sum_{t=0}^{m^{1/2}} u_{m+t} \geq \frac{1-o(1)}{(2\pi)^{1/2}} \int_0^1 e^{-x^2/2}\,\mathrm{d}x \geq \frac{1}{3}.$$

This conclude our lemma. □

Lemmas 2.3 and 2.6 show us that if we want to prove $\mathbf{Pr}\left[\mathcal{G}_{n,m} \in \mathcal{P}\right] \to 0$, it suffices to show $\mathbf{Pr}\left[\mathcal{G}_{n,p} \in \mathcal{P}\right] \to 0$. In most cases, $\mathbf{Pr}\left[\mathcal{G}_{n,p} \in \mathcal{P}\right]$ is much easier to compute.

To get rid of the limit between m and p, we have the following asymptotic version.

Theorem 2.7 ([Łuc90]). *Let $0 \leq p_0 \leq 1$ be a real, $s(n) = n\sqrt{p(1-p)} \to \infty$, and $\omega(n) \to \infty$ arbitrary slowly as $n \to \infty$.*

1. *Suppose that $\mathcal{P}$ is a graph property such that $\mathbf{Pr}\left[\mathcal{G}_{n,m} \in \mathcal{P}\right] \to p_0$ for all*

$$m \in \left[\binom{n}{2}p - \omega(n)s(n), \binom{n}{2}p + \omega(n)s(n)\right].$$

   Then $\mathbf{Pr}\left[\mathcal{G}_{n,p} \in \mathcal{P}\right] \to p_0$ as $n \to \infty$.

2. *Let $p_- = p - \omega(n)s(n)/n^2$ and $p_+ = p + \omega(n)s(n)/n^2$. Suppose that $\mathcal{P}$ is a monotone increasing graph property such that $\mathbf{Pr}\left[\mathcal{G}(n,p_-)\right] \to p_0$ and $\mathbf{Pr}\left[\mathcal{G}(n,p_+)\right] \to p_0$. Then $\mathbf{Pr}\left[\mathcal{G}_{n,m} \in \mathcal{P}\right] \to p_0$ for $m = \left\lfloor \binom{n}{2}p \right\rfloor$.*

2.2 Thresholds and sharp thresholds

One of the most important observation is that, for a monotone increasing graph property, there might exist a 'threshold'.

Definition 2.8 (Thresholds for $\mathcal{G}_{n,m}$). A function $m^* = m^*(n)$ is called a *threshold* for a monotone increasing property $\mathcal{P}$ in the random graph $\mathcal{G}_{n,m}$ if

$$\lim_{n\to\infty} \mathbf{Pr}\left[\mathcal{G}_{n,m} \in \mathcal{P}\right] = \begin{cases} 0 & m/m^* \to 0, \\ 1 & m/m^* \to \infty. \end{cases}$$

Definition 2.9 (Thresholds for $\mathcal{G}_{n,p}$). A function $p^* = p^*(n)$ is called a *threshold* for a monotone increasing property $\mathcal{P}$ in the random graph $\mathcal{G}_{n,p}$ if

$$\lim_{n\to\infty} \mathbf{Pr}\left[\mathcal{G}_{n,p} \in \mathcal{P}\right] = \begin{cases} 0 & p/p^* \to 0, \\ 1 & p/p^* \to \infty. \end{cases}$$

Remark 2.10. The threshold is not unique since any function which differs from $m^*(n)$ (or $p^*(n)$) by only a constant factor is also a threshold.

Theorem 2.11. *Every non-trivial monotone graph property has a threshold.*

Proof. Without loss of generality we assume that $\mathcal{P}$ is monotone increasing. Given $0 < \varepsilon < 1$, we define $p(\varepsilon)$ by

$$\mathbf{Pr}\left[\mathcal{G}_{n,p(\varepsilon)} \in \mathcal{P}\right] = \varepsilon.$$

Before the proof, firstly we argue that $p(\varepsilon)$ exists. Note that, for every $0 \le p \le 1$,

$$\mathbf{Pr}\left[\mathcal{G}_{n,p} \in \mathcal{P}\right] = \sum_{G\in\mathcal{P}} p^{|E(G)|}(1-p)^{N-|E(G)|}$$

is a polynomial increasing from 0 to 1. Then we know $p(\varepsilon)$ exists.

Now we will show $p(1/2)$ is a threshold for $\mathcal{P}$. Let $G_1, \ldots, G_k$ be k independent copies of $\mathcal{G}_{n,p}$. Then the graph $G = G_1 \cup \ldots \cup G_k$ is distributed as $\mathcal{G}_{n,1-(1-p)^k}$. Note that $1-(1-p)^k \le kp$. By the coupling argument,

$$\mathcal{G}_{n,1-(1-p)^k} \subseteq \mathcal{G}_{n,kp}.$$

And so, $\mathcal{G}_{n,kp} \notin \mathcal{P}$ implies $G_1, \ldots, G_k \notin \mathcal{P}$ (by monotonicity). Hence,

$$\mathbf{Pr}\left[\mathcal{G}_{n,kp} \notin \mathcal{P}\right] \le \mathbf{Pr}\left[\mathcal{G}_{n,p} \notin \mathcal{P}\right]^k.$$

Then, for any $\omega(n) \to \infty$ arbitrarily slowly as $n \to \infty$ and $\omega(n) \ll \log\log n$, we know

$$\mathbf{Pr}\left[\mathcal{G}_{n,\omega(n)p(1/2)} \notin \mathcal{P}\right] \le 2^{-\omega} = o(1).$$

On the other hand, for $p = p(1/2)/\omega(n)$, we know

$$\mathbf{Pr}\left[\mathcal{G}_{n,p(1/2)/\omega(n)} \notin \mathcal{P}\right] \ge 2^{-1/\omega} = 1 - o(1).$$

□

By observation, there exists a more subtle threshold for some monotone graph properties.

Definition 2.12 (Sharp Thresholds for $\mathcal{G}_{n,m}$). A function $m^* = m^*(n)$ is called a *sharp threshold* for a monotone increasing property $\mathcal{P}$ in the random graph $\mathcal{G}_{n,m}$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbf{Pr}\left[\mathcal{G}_{n,m} \in \mathcal{P}\right] = \begin{cases} 0 & m/m^* \le 1-\varepsilon, \\ 1 & m/m^* \ge 1+\varepsilon. \end{cases}$$

Definition 2.13 (Sharp Thresholds for $\mathcal{G}_{n,p}$). A function $p^* = p^*(n)$ is called a *sharp threshold* for a monotone increasing property $\mathcal{P}$ in the random graph $\mathcal{G}_{n,p}$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbf{Pr}\left[\mathcal{G}_{n,p} \in \mathcal{P}\right] = \begin{cases} 0 & p/p^* \le 1-\varepsilon, \\ 1 & p/p^* \ge 1+\varepsilon. \end{cases}$$

To illustrate Definitions 2.8 and 2.9 more precisely, we state the following simple example. We deal with the graph $\mathcal{G}_{n,p}$ and the property

$$\mathscr{P} = \{G = (V(G), E(G)) \mid V(G) = n, E(G) \neq \emptyset\}. \tag{3}$$

Now we will show $p^* = 1/n^2$ is a threshold.

Theorem 2.14. *Let $\mathscr{P}$ be the graph property defined as (3). Then*

$$\lim_{n\to\infty} \mathbf{Pr}\left[\mathcal{G}_{n,p} \in \mathscr{P}\right] = \begin{cases} 0 & p \ll n^{-2}, \\ 1 & p \gg n^{-2}. \end{cases}$$

Proof. Let X be the number of edges in $\mathcal{G}_{n,p}$. By the definition of the random model, it holds that

$$\mathbf{E}\left[X\right] = \binom{n}{2} p, \quad \mathbf{Var}\left[X\right] = \binom{n}{2} p(1-p) = (1-p)\mathbf{E}\left[X\right].$$

By the Markov inequality, it holds that

$$\mathbf{Pr}\left[X > 0\right] \leq \mathbf{E}\left[X\right] \leq \frac{n^2}{2} p.$$

When $p \ll n^{-2}$, it holds that $\lim_{n\to\infty} \mathbf{Pr}\left[X > 0\right] = 0$. Thus we conclude the first part of the theorem.

To show the second result, we consider the concentration of the random variable X. By the Chebyshev inequality,

$$\mathbf{Pr}\left[X > 0\right] \geq 1 - \frac{\mathbf{Var}\left[X\right]}{\mathbf{E}\left[X\right]^2} = 1 - \frac{1-p}{\mathbf{E}\left[X\right]}.$$

When $p \gg n^{-2}$, it holds that $\frac{1-p}{\mathbf{E}[X]} \to 0$ and we know $\lim_{n\to\infty} \mathbf{Pr}\left[X > 0\right] = 1$. $\square$

Now we consider the degree of a fixed vertex $v \in V$ in random graphs. By definition, it is easy to show:

$$\mathbf{Pr}_{\mathcal{G}_{n,p}}\left[\deg(v) = d\right] = \binom{n-1}{d} p^d (1-p)^{n-1-d}.$$

and for the model $\mathcal{G}_{n,m}$,

$$\mathbf{Pr}_{\mathcal{G}_{n,m}}\left[\deg(v) = d\right] = \frac{\binom{n-1}{d}\binom{\binom{n-1}{2}}{m-d}}{\binom{\binom{n}{2}}{m}}.$$

Let $\mathscr{P}$ be the graph property such that the graph contains an isolated vertex, *i.e.*,

$$\mathscr{P} := \{G = (V(G), E(G)) \mid \exists v \in V(G), \deg(v) = 0\}.$$

Now we show $m = \frac{1}{2} n \log n$ is a sharp threshold for $\mathscr{P}$ in $\mathcal{G}_{n,m}$.

Lemma 2.15. *Let $\mathscr{P}$ be the property defined as above, and $m = \frac{1}{2} n(\log n + \omega(n))$. Then*

$$\lim_{n\to\infty} \mathbf{Pr}\left[\mathcal{G}_{n,m} \in \mathscr{P}\right] = \begin{cases} 1 & \omega(n) \to -\infty, \\ 0 & \omega(n) \to \infty. \end{cases}$$

Proof. We define a random variable X as the number of isolated vertices in $\mathcal{G}_{n,m}$, and for every $v \in V$, we define a random variable I_v to denote whether v is isolated. Then

$$X = \sum_{v \in V} I_v$$

and for each $v \in V$,

$$\begin{aligned}
\mathbf{E}[I_v] &= \mathbf{Pr}[I_v = 1] \\
&= \binom{\binom{n-1}{2}}{m} \Big/ \binom{\binom{n}{2}}{m} \\
&= \prod_{i=0}^{m-1} \left(\frac{\frac{(n-1)(n-2)}{2} - i}{\frac{n(n-1)}{2} - i} \right) \\
&= \left(\frac{n-2}{n} \right)^m \prod_{i=0}^{m-1} \left(1 - \frac{4i}{n(n-1)(n-2) - 2i(n-2)} \right).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\mathbf{E}[X] &= \sum_{v \in V} \mathbf{E}[I_v] \\
&= n \left(\frac{n-2}{n} \right)^m \prod_{i=0}^{m-1} \left(1 - \frac{4i}{n(n-1)(n-2) - 2i(n-2)} \right).
\end{aligned}$$

To bound the product, notice that, if $0 \le x_0, \dots, x_{m-1} \le 1$, it holds that

$$n \left(1 - \sum_{i=0}^{m-1} x_i \right) \le n \prod_{i=0}^{m-1} (1 - x_i) \le n.$$

Thus we obtain that, if we assume that $\omega(n) = o(\log n)$,

$$n \left(\frac{n-2}{n} \right)^m \prod_{i=0}^{m-1} \left(1 - \frac{4i}{n(n-1)(n-2) - 2i(n-2)} \right) \le n \left(1 - \frac{2}{n} \right)^m \le e^{-\omega(n)}.$$

When $\omega(n) \to \infty$, we know $\mathbf{E}[X] \to 0$ and by the first moment method, we know $X = 0$ with high probability.

For the counterpart, note that

$$\prod_{i=0}^{m-1} \left(1 - \frac{4i}{n(n-1)(n-2) - 2i(n-2)} \right) \ge 1 - \frac{4}{n-2} \sum_{i=0}^{m-1} \frac{i}{n(n-1) - 2i} = 1 - O\left(\frac{(\log n)^2}{n} \right).$$

Then it holds that

$$\mathbf{E}[X] = (1 - o(1)) n \left(\frac{n-2}{n} \right)^m \ge (1 - o(1)) n e^{-\frac{2m}{n-2}} \ge (1 - o(1)) e^{-\omega(n)} \to \infty.$$

Also, to show the concentration of X, we compute the second moment of X. By calculation,

$$\begin{aligned}
\mathbf{E}\left[X^2\right] &= \mathbf{E}\left[\left(\sum_{v \in V} I_v \right)^2 \right] \\
&= \sum_{u,v \in V} \mathbf{Pr}[I_u = I_v = 1] \\
&= n(n-1) \binom{\binom{n-2}{2}}{m} \Big/ \binom{\binom{n}{2}}{m} + \mathbf{E}[X] \\
&\le (1 + o(1)) \mathbf{E}[X]^2 + \mathbf{E}[X].
\end{aligned}$$

Then we know

$$\mathbf{Pr}\left[X > 0\right] \geq \frac{\mathbf{E}\left[X\right]^2}{\mathbf{E}\left[X^2\right]} \geq \frac{1}{1 + o(1) + \mathbf{E}\left[X\right]^{-1}} = 1 - o(1)$$

whenever $\omega(n) \to -\infty$. □

At the end of the part, we show a more complicated example.

Theorem 2.16. *If $m/n \to \infty$, then with high probability the random graph $\mathcal{G}_{n,m}$ contains a triangle.*

Proof. It is easy to observe that the property is monotone increasing. Then it suffices to show that, when p satisfies the some regular requirements, the random graph $\mathcal{G}_{n,p}$ contains at least one triangle with high probability.

By coupling method, it suffices to show the case $\omega := np \leq \log n$. Let Z be the random variable denoting the number of triangles in $\mathcal{G}_{n,p}$. Then

$$\mathbf{E}\left[Z\right] = \binom{n}{3} p^3 \geq \frac{(1 - o(1))\omega^3}{6} \to \infty.$$

For the second moment, let $T_1, \ldots, T_M$ be the triangles of the complete graph K_n where $M = \binom{n}{3}$. Then,

$$\begin{aligned}
\mathbf{E}\left[Z^2\right] &= \sum_{i,j=1}^{M} \mathbf{Pr}\left[T_i, T_j \in \mathcal{G}_{n,p}\right] \\
&= \sum_{i=1}^{M} \mathbf{Pr}\left[T_i \in \mathcal{G}_{n,p}\right] \sum_{j=1}^{M} \mathbf{Pr}\left[T_j \in \mathcal{G}_{n,p} \mid T_i \in \mathcal{G}_{n,p}\right] \\
&= M\mathbf{Pr}\left[T_1 \in \mathcal{G}_{n,p}\right] \sum_{j=1}^{M} \mathbf{Pr}\left[T_j \in \mathcal{G}_{n,p} \mid T_1 \in \mathcal{G}_{n,p}\right] \\
&= \mathbf{E}\left[Z\right] \sum_{j=1}^{M} \mathbf{Pr}\left[T_j \in \mathcal{G}_{n,p} \mid T_1 \in \mathcal{G}_{n,p}\right].
\end{aligned}$$

Separating the summation according to the number of edges T_1, T_j share, we obtain

$$\begin{aligned}
\sum_{j=1}^{M} \mathbf{Pr}\left[T_j \in \mathcal{G}_{n,p} \mid T_1 \in \mathcal{G}_{n,p}\right] &= 1 + 3(n-3)p^2 + \left(\binom{n}{3} - 3n + 8\right) p^3 \\
&\leq 1 + \frac{3\omega^2}{n} + \mathbf{E}\left[Z\right].
\end{aligned}$$

Then we know

$$\mathbf{Var}\left[Z\right] \leq \mathbf{E}\left[Z\right]\left(1 + \frac{3\omega^2}{n} + \mathbf{E}\left[Z\right]\right) - \mathbf{E}\left[Z\right]^2 \leq 2\mathbf{E}\left[Z\right].$$

By the Chebyshev inequality, we conclude

$$\mathbf{Pr}\left[Z = 0\right] \leq \frac{\mathbf{Var}\left[Z\right]}{\mathbf{E}\left[Z\right]^2} \leq \frac{2}{\mathbf{E}\left[Z\right]} = o(1).$$

The result then comes immediately. □

References

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