A Local-to-Global Framework: Simplicial Complex

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1 Markov Chains and Local Properties

Given a distribution μ over state space Ω, let P be a reversible Markov chain with respect to Ω. We define the mixing time of P at initial state $x \in \Omega$ as

$$t_{\mathrm{mix}}(P, x, \varepsilon) = \inf\left\{ t \geq 0 \;\middle|\; \mathcal{D}_{\mathrm{TV}}\left(P^t(x, \cdot) \;\middle\|\; \mu\right) \leq \varepsilon \right\}.$$

The functional inequality is introduced to bound the mixing time of P. For two functions $f, g : \Omega \to \mathbb{R}$, define the Dirichlet form with assumption that all terms are well-defined to be

$$\mathcal{E}_P(f, g) := \langle f, (I - P)g \rangle_\mu = \int_{x \in \Omega} f(x)(I - P)g(x) \, \mathrm{d}\mu(x). \tag{1}$$

Definition 1.1 (Functional Inequalities)**.** Given a reversible Markov chain P with respect to its stationary distribution μ over Ω, we define the *spectral gap* of P as

$$\mathrm{Gap}(P) := \inf_{f:\Omega \to \mathbb{R}} \frac{\mathcal{E}_P(f, f)}{\mathbf{Var}_\mu(f)}$$

and we define the *modified log-Sobolev inequality constant* (MLSI) of P as

$$\rho_{\mathrm{LS}}(P) := \inf_{f:\Omega \to \mathbb{R}_{\geq 0}} \frac{\mathcal{E}_P(f, \log f)}{\mathbf{Ent}_\mu[f]}$$

where the variance and the entropy of f with respect to μ are defined as

$$\mathbf{Var}_\mu(f) = \mathbf{E}_\mu\left[f^2\right] - \mathbf{E}_\mu[f]^2, \quad \mathbf{Ent}_\mu[f] = \mathbf{E}_\mu[f \log f] - \mathbf{E}_\mu[f] \log \mathbf{E}_\mu[f].$$

Moreover, for every reversible P, it holds that $\mathrm{Gap}(P) = 1 - \lambda_2(P)$.

Previously several works have used the functional inequalities to bound the mixing time.

Lemma 1.2 (Theorem 12.4 in [LP17])**.** *There exists a universal constant $C > 0$ such that the followings hold for all $x \in \Omega$,*

$$t_{\mathrm{mix}}(P, x, \varepsilon) \leq \frac{C}{\mathrm{Gap}(P)}\left(\log \frac{1}{\mu(x)} + \log \frac{1}{\varepsilon}\right),$$
$$t_{\mathrm{mix}}(P, x, \varepsilon) \leq \frac{C}{\rho_{\mathrm{LS}}(P)}\left(\log\log \frac{1}{\mu(x)} + \log \frac{1}{\mu(x)}\right).$$

Dirichlet form and continuous-time random walk

The Dirichlet form is introduced in Bobkov and Tetali [BT06] to analyze the mixing time of Markov chains. To briefly see this, consider the following Markov process:

$$P_t = e^{-(I-P)t}, \quad \forall t \geq 0.$$

Let μ_0 be the initial distribution and $\mu_t = \mu_0 P_t$. Consider the function f_t supported on Ω as $f_t = \frac{\mathrm{d}\mu_t}{\mathrm{d}\mu}$. Thus $\langle f_t, \mathbf{1}\rangle_\mu = 1$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} f_t = -(I-P)f_t, \quad \frac{\mathrm{d}}{\mathrm{d}t}\log f_t = -(I-P)\mathbf{1}.$$

Then,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{Var}_\mu(f_t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\langle f_t, f_t\rangle_\mu - \langle f_t, \mathbf{1}\rangle_\mu^2\right) \\
&= \left\langle \frac{\mathrm{d}}{\mathrm{d}t} f_t, f_t \right\rangle_\mu + \left\langle f_t, \frac{\mathrm{d}}{\mathrm{d}t} f_t \right\rangle_\mu - 2\langle f_t, \mathbf{1}\rangle_\mu \left\langle \frac{\mathrm{d}}{\mathrm{d}t} f_1, \mathbf{1}\right\rangle_\mu \\
&= 2\langle f_t, -(I-P)f_t\rangle_\mu - 2\langle f_t, \mathbf{1}\rangle_\mu \langle -(I-P)f_t, \mathbf{1}\rangle_\mu \\
&= -2\mathcal{E}_P(f_t, f_t).
\end{aligned}$$

Similarly for the relative entropy we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{D}_{\mathrm{KL}}(\mu_t \parallel \mu) &= \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{Ent}_\mu[f_t] \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\langle f_t, \log f_t\rangle_\mu \\
&= \left\langle \frac{\mathrm{d}}{\mathrm{d}t} f_t, \log f_t \right\rangle_\mu + \left\langle f_t, \frac{\mathrm{d}}{\mathrm{d}t}\log f_t \right\rangle_\mu \\
&= \langle -(I-P)f_t, \log f_t\rangle_\mu + \langle f_t, -(I-P)\mathbf{1}\rangle_\mu \\
&= -2\mathcal{E}_P(f_t, \log f_t).
\end{aligned}$$

The two inequalities drive us to bound the spectral gap and MLSI constant.

1.1 Spectral independence

Now we consider the case $\Omega \subseteq [q]^n$ for a positive integer $q \geq 2$. It makes common sense since we focus on the mixing rate of the Glauber dynamics for the Gibbs distribution of q-spin systems.

The local property named *spectral independence* is firstly introduced in Anari, Liu and Oveis Gharan [ALOG20] to evaluate the local dependence in hard-core models.

Definition 1.3 (Spectral Independence - Boolean Domain)**.** Let μ be a distribution over $\Omega \subseteq \{-1,+1\}^n$. We define the *influence matrix* $\Psi_\mu \in \mathbb{R}^{n\times n}$ to be

$$\Psi_\mu(i,j) = \frac{1}{2}\mathbf{E}_{X\sim\mu}\left[X_i \mid X_j = 1\right] - \frac{1}{2}\mathbf{E}_{X\sim\mu}\left[X_i \mid X_j = -1\right], \quad \forall i,j \in [n].$$

For $\eta > 0$, we say μ is η-spectrally independent if $\left\|\Psi_\mu\right\|_{\mathrm{OP}} \leq 1+\eta$.

For arbitrary $q \geq 2$, Feng, Guo, Yin and Zhang [FGYZ22] extend the definition of the influence matrix of μ and introduce the generalized version of the spectral independence.

Definition 1.4 (Spectral Independence)**.** Let μ be a distribution over $\Omega \subseteq [q]^n$. For any $\Lambda \subseteq [n]$ and every feasible pinning $\tau \in [q]^\Lambda$, the *absolute influence matrix* $\Psi_\mu^\tau \in \mathbb{R}_{\geq 0}^{n\times n}$ is defined as, for every distinct $u, v \in [n]$,

$$\Psi_\mu^\tau(u,v) := \inf_{i,j\in[q]} \mathcal{D}_{\mathrm{TV}}\left(\mu_v^{\tau\cup\{u\leftarrow i\}} \;\middle\|\; \mu_v^{\tau\cup\{u\leftarrow j\}}\right).$$

For $\eta > 0$, we say μ is η-spectrally independent if for all $\Lambda \subseteq [n]$ and $\tau \in [q]^\Lambda$, the spectral radius of the absolute influence matrix satisfies $\rho(\Psi_\mu^\tau) \leq 1+\eta$.

Remark 1.5. In some cases, we also define the influence matrix $\widetilde{\Psi_\mu}$ as

$$\widetilde{\Psi_\mu}((i,s),(j,t)) := \mathbf{Pr}_{\omega\sim\mu}\left[\omega(j)=t \mid \omega(i)=s\right] - \mathbf{Pr}_{\omega\sim\mu}\left[\omega(j)=t\right].$$

It is well-known that $\lambda_{\max}\left(\widetilde{\Psi_\mu}\right) \le \rho(\Psi_\mu)$.

The following argument relates the influence matrix to the correlation of the distribution. This might explain the motivation and the intuition that we take the spectral independence into account and serve it as a local property of the distribution.

Lemma 1.6. *Given a distribution μ over $\Omega \subseteq \{-1,+1\}^n$, define the correlation matrix of μ as*

$$\mathbf{Cor}(\mu) := \mathrm{diag}\left(\mathbf{Cov}(\mu)\right)^{-1/2}\mathbf{Cov}(\mu)\,\mathrm{diag}\left(\mathbf{Cov}(\mu)\right)^{-1/2}.$$

Then $\Psi_\mu = \mathbf{Cov}(\mu)\,\mathrm{diag}\left(\mathbf{Cov}(\mu)\right)^{-1}$ and

$$\left\|\Psi_\mu\right\|_{\mathrm{OP}} = \|\mathbf{Cor}(\mu)\|_{\mathrm{OP}}.$$

Proof. Let X be a random variable drawn from μ. For $i, j \in [n]$, by calculation,

$$\begin{aligned}
\mathbf{Cov}(\mu)_{i,j} &= \mathbf{E}\left[X_iX_j\right] - \mathbf{E}\left[X_i\right]\mathbf{E}\left[X_j\right] \\
&= \mathbf{E}\left[X_i \mid X_j = 1\right]\mathbf{Pr}\left[X_j=1\right]\left(1-\mathbf{E}\left[X_j\right]\right) - \mathbf{E}\left[X_i \mid X_j=-1\right]\mathbf{Pr}\left[X_j=1\right]\left(1+\mathbf{E}\left[X_j\right]\right) \\
&= \Psi_\mu(i,j)\left(1-\mathbf{E}\left[X_j\right]^2\right)
\end{aligned}$$

thus leading to the identity $\Psi_\mu = \mathbf{Cov}(\mu)\,\mathrm{diag}\left(\mathbf{Cov}(\mu)\right)^{-1}$. To prove the second identity, let $\mathbf{v}$ be an eigenvector of $\mathbf{Cor}(\mu)$ and its associated eigenvalue is λ. For simplicity let $D = \mathrm{diag}\left(\mathbf{Cov}(\mu)\right)$. Then,

$$\lambda\mathbf{v} = D^{-1/2}\mathbf{Cov}(\mu)\,D^{-1/2}\mathbf{v}.$$

Let $\mathbf{u} = D^{1/2}\mathbf{v}$. Thus we obtain

$$\lambda\mathbf{u} = \mathbf{Cov}(\mu)\,D^{-1/2}\mathbf{v} = \mathbf{Cov}(\mu)\,D^{-1}\mathbf{u} = \Psi_\mu\mathbf{u}.$$

Then we know Ψ_μ and $\mathbf{Cor}(\mu)$ share the same spectrum, meaning that their operator norms are equal. □

2 High-Dimensional Expander: Simplicial Complex

Now we introduce a framework relate the local property to the global rate of the mixing of Markov chains.

Definition 2.1 (Simplicial Complex)**.** A simplicial complex $\mathscr{C}$ is a non-empty downwards closed collection of sets (called faces) over a finite ground set of elements. It satisfies

- $\varnothing \in \mathscr{C}$;
- if $S \in \mathscr{C}$ and $T \subseteq S$, then $T \in \mathscr{C}$.

Additionally, we assume that $\mathscr{C}$ is pure, *i.e.*, for all maximal elements $S \in \mathscr{C}$, they share the same size denoted by $d = \mathrm{rank}(\mathscr{C})$. For all $S \in \mathscr{C}$, let $\mathrm{rank}(S) := |S|$. According to the rank function, we partition $\mathscr{C}$ into $d+1$ parts as: for every $0 \le k \le d$, define the k-skeleton as

$$\mathscr{C}(k) := \{S \in \mathscr{C} \mid \mathrm{rank}(S) = k\}.$$

For every face $S \in \mathscr{C}$, define the link at S as

$$\mathscr{C}_S := \{T \in \mathscr{C} \mid S \cap T = \varnothing, S \cup T \in \mathscr{C}\}$$

and for all $0 \le k \le d - \mathrm{rank}(S)$, define the k-skeleton at S as

$$\mathscr{C}_S(k) := \{T \in \mathscr{C}_S \mid \mathrm{rank}(T) = k\}.$$

2.1 Weight functions and random walks on the simplicial complex

Given a distribution μ over $\Omega = \mathscr{C}(d)$, we define the weight function $w : \mathscr{C} \to \mathbb{R}_{\geq 0}$ as

$$w(S) = \begin{cases} \mu(S) & S \in \mathscr{C}(d); \\ \sum_{T \supseteq S, T \in \mathscr{C}(k+1)} w(T) & S \in \mathscr{C}(k), k < d. \end{cases}$$

For every link $\mathscr{C}_S$ at face $S \in \mathscr{C}$, we define $w_S(T) = w(S \cup T)$ for every $T \in \mathscr{C}_S$.

To see the random walks on $\mathscr{C}$, firstly we introduce the distribution on it. For every $0 \leq k \leq d$, we define the distribution π_d on $\mathscr{C}(k)$ as

$$\pi_k(S) = \frac{w(S)}{\sum_{T \in \mathscr{C}(k)} w(T)}, \quad \forall S \in \mathscr{C}(k).$$

Similarly for the link at S, we can also define the distribution $\pi_{S,k}$ over $\mathscr{C}_S(k)$.

For $0 \leq k \leq d$ and every $S \in \mathscr{C}(k)$, by calculation,

$$w(S) = \frac{d!}{k!}\mu(S).$$

This leads to the identity

$$\pi_k(S) = \frac{w(S)}{\sum_{T \in \mathscr{C}(k)} w(T)} = \frac{\mu(S)}{\sum_{T \in \mathscr{C}(k)} \mu(T)} = \frac{1}{\binom{d}{k}}\mu(S).$$

For simplicity of notations and analysis, we assume that the dimension of all the matrices is $\mathscr{C}(1)$, and we add zeros to appropriate positions. For every $0 \leq k \leq d$, we define $\Pi_k := \operatorname{diag}(\pi_k)$ to be the diagonal matrix induced by π_k, and similarly define $\Pi_{S,k} \in \mathbb{R}^{\mathscr{C}_\tau(k) \times \mathscr{C}_\tau(k)}$ for all links at $S \in \mathscr{C}$ and $0 \leq k \leq d - \operatorname{rank}(S)$, and the inverse of them means taking inverse only on their non-zero entries. Additionally, we use the operator $\bar{\cdot}$ to denote the actual vector or matrix in the simplicial complex. Precisely speaking, for a matrix A supported on $\mathscr{C}_\tau(k) \times \mathscr{C}_\tau(k)$,

$$\overline{A}(S \cup T, S \cup R) := A(T, R), \quad \forall T, R \in \mathscr{C}_\tau(k)$$

and 0 otherwise, meanwhile for a vector $\mathbf{v}$ supported on $\mathscr{C}_\tau(k)$,

$$\overline{\mathbf{v}}(S \cup T) := \mathbf{v}(T), \quad \forall T \in \mathscr{C}_\tau(k)$$

and 0 otherwise.

There are two natural random walks on the simplicial complex $\mathscr{C}$: up-walk and down-walk.

- 'Up-Walk' $\mathrm{P}_k^{\uparrow}$: starting from $S \in \mathscr{C}(k)$, we add an element $x \in \mathscr{C}_S(1)$ as $\pi_{S,1}$.
- 'Down-Walk' $\mathrm{P}_k^{\downarrow}$: starting from $S \in \mathscr{C}(k)$, we remove an element $x \in S$ uniformly at random.

We write them in a explicit form: for $0 \leq k \leq d-1$, $S \in \mathscr{C}(k)$, $T \in \mathscr{C}(k+1)$,

$$\mathrm{P}_k^{\uparrow}(S, T) = \frac{w(T)}{w(S)} \mathbb{1}\left[S \subseteq T\right]$$

and for $1 \leq k \leq d$, $S \in \mathscr{C}(k)$, $T \in \mathscr{C}(k-1)$,

$$\mathrm{P}_k^{\downarrow}(S, T) = \frac{1}{k} \mathbb{1}\left[T \subseteq S\right].$$

Based on the two walks, we define the following up-down walk and down-up walk (note that they are all lazy random walks):

$$\mathrm{P}_k^{\Delta} = \mathrm{P}_k^{\uparrow}\mathrm{P}_{k+1}^{\downarrow}, \quad \forall 0 \leq k \leq d-1,$$
$$\mathrm{P}_k^{\nabla} = \mathrm{P}_k^{\downarrow}\mathrm{P}_{k-1}^{\uparrow}, \quad \forall 1 \leq k \leq d.$$

For the up-down walks, usually we consider its non-lazy version $\mathrm{P}_k^{\wedge} := \frac{k+1}{k}\mathrm{P}_k^{\Delta} - \frac{1}{k}I$. For the link $\mathscr{C}_\tau$ at $\tau \in \mathscr{C}$, it is similar to define the random walks $\mathrm{P}_{\tau,k}^{\Delta}$, $\mathrm{P}_{\tau,k}^{\nabla}$ and $\mathrm{P}_{\tau,k}^{\wedge}$. Among all these walks, we pay quite a special attention to the local walk $\mathrm{P}_{\tau,1}^{\wedge}$ and $\mathrm{P}_{\tau,1}^{\nabla}$. Define the matrix $W_{\tau,2}$ supported on $\mathscr{C}_\tau(1) \times \mathscr{C}_\tau(1)$ as $W_{\tau,2}(x,y) = \pi_{\tau,2}(\{x,y\})$ for $x, y \in \mathscr{C}_\tau(1)$ and $\{x,y\} \in \mathscr{C}_\tau(2)$. By definition, it holds that

$$\mathrm{P}_{\tau,1}^{\wedge} = \frac{1}{2}\Pi_{\tau,1}^{-1}W_{\tau,2}, \tag{2}$$

$$\mathrm{P}_{\tau,1}^{\nabla} = \mathbf{1}\pi_{\tau,1}^{\top}. \tag{3}$$

Moreover, directly from the definition, for the distributions of the two adjacent layers, it holds that

$$\Pi_{k+1}\mathrm{P}_{k+1}^{\downarrow} = \left(\mathrm{P}_k^{\uparrow}\right)^{\top}\Pi_k, \quad \forall 0 \le k \le d-1. \tag{4}$$

Multiplying all-ones vector on both sides, we obtain,

$$\pi_{k+1}\mathrm{P}_{k+1}^{\downarrow} = \pi_k, \tag{5}$$

$$\pi_k\mathrm{P}_k^{\uparrow} = \pi_{k+1}. \tag{6}$$

For every $0 \le \ell \le k \le d$ and $\sigma \in \mathscr{C}(k)$, $\tau \in \mathscr{C}(\ell)$ with $\tau \subseteq \sigma$, by definition,

$$\begin{aligned}
\pi_k(\sigma) &= \frac{1}{\binom{d}{k}}\mu(\sigma) \\
&= \frac{1}{\binom{d}{k}}\mu(\tau)\mu^{\tau}(\sigma \setminus \tau) \\
&= \binom{k}{\ell}\pi_\ell(\tau)\pi_{\tau,k-\ell}(\sigma \setminus \tau).
\end{aligned}$$

2.2 Garland's method

The kernel of the local-to-global theorem is to establish the relationship between local walks and global walks. The Garland's method is implicit in the work of Oppenheim [Opp18] and we put it in a more direct and explicit form.

Lemma 2.2 (Garland's Method)**.** *The following identities hold:*

1. $\Pi_1 = \mathbf{E}_{\tau\sim\pi_1}\left[\Pi_{\tau,1}\right]$.
2. $\Pi_1\mathrm{P}_1^{\wedge} = \mathbf{E}_{\tau\sim\pi_1}\left[\Pi_{\tau,1}\mathrm{P}_{\tau,1}^{\wedge}\right]$.
3. $\Pi_1(\mathrm{P}_1^{\wedge})^2 = \mathbf{E}_{\tau\sim\pi_1}\left[\pi_{\tau,1}\pi_{\tau,1}^{\top}\right]$.

Proof. We prove these identities entry by entry.

1. For every $x \in \mathscr{C}(1)$, by definition,

$$\begin{aligned}
\Pi_1(x,x) &= \pi_1(x) \\
&= \sum_{\{x,y\}\in\mathscr{C}(1)} \frac{1}{2}\pi_2(\{x,y\}) \\
&= \sum_{y\in\mathscr{C}(1)} \pi_1(y)\frac{\pi_2(\{x,y\})}{2\pi_1(y)} \\
&= \mathbf{E}_{y\sim\pi_1}\left[\Pi_{y,1}(x,x)\right].
\end{aligned}$$

2. By (2), it holds that

$$\mathbf{E}_{\tau\sim\pi_1}\left[\Pi_{\tau,1}\mathrm{P}_{\tau,1}^{\wedge}\right] = \mathbf{E}_{\tau\sim\pi_1}\left[\frac{1}{2}W_{\tau,2}\right] = 2W_2.$$

On the other hand, by definition,

$$\Pi_1\mathrm{P}_1^{\wedge} = 2\Pi_1\mathrm{diag}\left(\pi_1\right)^{-1}W_2 = 2W_2.$$

Then we conclude the identity.

3. For every $x, y \in \mathscr{C}(1)$,

$$\Pi_1(\mathrm{P}_1^{\wedge})^2(x,y) = \sum_{\tau\in\mathscr{C}(1)} \pi_1(x)\pi_{x,1}(\tau)\pi_{\tau,1}(y).$$

On the other hand,

$$\begin{aligned}\mathbf{E}_{\tau\sim\pi_1}\left[\pi_{\tau,1}\pi_{\tau,1}^{\top}\right](x,y) &= \sum_{\tau\in\mathscr{C}(1)} \pi_1(\tau)\pi_{\tau,1}(x)\pi_{\tau,1}(y)\\ &= \sum_{\tau\in\mathscr{C}(1)} \pi_1(x)\pi_{x,1}(\tau)\pi_{\tau,1}(y).\end{aligned}$$

Thus we conclude the identity.

$\square$

Additionally the following two identities between two skeletons are important.

Lemma 2.3. *The following identities hold*

1. $\Pi_k\mathrm{P}_k^{\wedge} = \mathbf{E}_{\tau\sim\pi_{k-1}}\left[\overline{\Pi_{\tau,1}\mathrm{P}_{\tau,1}^{\wedge}}\right]$.
2. $\Pi_k\mathrm{P}_k^{\nabla} = \mathbf{E}_{\tau\sim\pi_{k-1}}\left[\overline{\Pi_{\tau,1}\mathrm{P}_{\tau,1}^{\nabla}}\right] = \mathbf{E}_{\tau\sim\pi_{k-1}}\left[\overline{\pi_{\tau,1}\pi_{\tau,1}^{\top}}\right]$.

Proof. The proofs of two identities are similar.

1. By direct calculation,

$$\begin{aligned}\Pi_k\mathrm{P}_k^{\wedge} &= \Pi_k\cdot\frac{1}{k}\sum_{\tau\in\mathscr{C}(k-1)}\overline{\mathrm{P}_{\tau,1}^{\wedge}}\\ &= \sum_{\tau\in\mathscr{C}(k-1)}\frac{1}{k}\Pi_k\overline{\mathrm{P}_{\tau,1}^{\wedge}}\\ &= \sum_{\tau\in\mathscr{C}(k-1)}\pi_{k-1}(\tau)\frac{\Pi_k}{k\pi_{k-1}(\tau)}\overline{\mathrm{P}_{\tau,1}^{\wedge}}\\ &= \sum_{\tau\in\mathscr{C}(k-1)}\pi_{k-1}(\tau)\overline{\Pi_{\tau,1}\mathrm{P}_{\tau,1}^{\wedge}}.\end{aligned}$$

2. Similarly to above, we have

$$\begin{aligned}\Pi_k\mathrm{P}_k^{\nabla} &= \Pi_k\cdot\frac{1}{k}\sum_{\tau\in\mathscr{C}(k-1)}\overline{\mathrm{P}_{\tau,1}^{\nabla}}\\ &= \sum_{\tau\in\mathscr{C}(k-1)}\pi_{k-1}(\tau)\frac{\Pi_k}{k\pi_{k-1}(\tau)}\overline{\mathrm{P}_{\tau,1}^{\nabla}}\\ &= \sum_{\tau\in\mathscr{C}(k-1)}\pi_{k-1}(\tau)\overline{\Pi_{\tau,1}\mathrm{P}_{\tau,1}^{\nabla}}.\end{aligned}$$

$\square$

2.3 Trickling-down theorem

Based on the identities in Section 2.2, we establish more properties of the local walks.

Definition 2.4 (Local Spectral Expander)**.** For a simplicial complex $\mathscr{C}$ equipped with distribution μ, for $0 \le k \le d-2$ and $\gamma_k \in [0,1]$, we say $\mathscr{C}(k)$ is a γ_k-*local spectral expander* if it holds that

$$\lambda_2(\mathrm{P}^{\wedge}_{\tau,1}) \le \gamma_k, \quad \forall \tau \in \mathscr{C}(k),$$

or equivalently,

$$\Pi_{\tau,1}\mathrm{P}^{\wedge}_{\tau,1} - \pi_{\tau,1}\pi_{\tau,1}^{\top} \preceq \gamma_k \left(\Pi_{\tau,1} - \pi_{\tau,1}\pi_{\tau,1}^{\top}\right).$$

Moreover, we say $\mathscr{C}$ is a $(\gamma_0, \dots, \gamma_{d-2})$-*local spectral expander* if $\mathscr{C}(k)$ is a γ_k-local spectral expander for all $0 \le k \le d-2$.

The following lemma shows, if $\mathscr{C}(k)$ is a local spectral expander, then we can see $\mathscr{C}(k-1)$ is also a local spectral expander.

Theorem 2.5 (Oppenheim's Trickling-Down Theorem, [Opp18])**.** *Suppose that $\mathscr{C}(k)$ is a γ-local spectral expander for some $1 \le k \le d-2$. Then $\mathscr{C}(k-1)$ is a $\frac{\gamma}{1-\gamma}$-local spectral expander (assuming the total connectivity of the random walk).*

Proof. When $k > 1$, we can only focus the link at each face in $\mathscr{C}(k)$ and this is the case $k = 1$. Then we assume that $k = 1$. By Lemma 2.2,

$$\begin{aligned}
\Pi_1\mathrm{P}^{\wedge}_1 &= \mathbf{E}_{\tau\sim\pi_1}\left[\overline{\Pi_{\tau,1}\mathrm{P}^{\wedge}_{\tau,1}}\right] \\
&\preceq \mathbf{E}_{\tau\sim\pi_1}\left[\gamma\overline{\Pi_{\tau,1}} + (1-\gamma)\overline{\pi_{\tau,1}\pi_{\tau,1}^{\top}}\right] \\
&= \gamma\Pi_1 + (1-\gamma)\Pi_1(\mathrm{P}^{\nabla}_1)^2
\end{aligned}$$

where the inequality comes from the fact that the local spectral expander means

$$\Pi_{\tau,1} - \Pi_{\tau,1}\mathrm{P}^{\wedge}_{\tau,1} \succeq (1-\gamma)(\Pi_{\tau,1} - \pi_{\tau,1}\pi_{\tau,1}^{\top}).$$

Now we consider the eigenvector $\mathbf{v}_2$ of P^{∇}_1 with respect to the second largest eigenvalue λ_2. Then,

$$\lambda_2 \left\langle \mathbf{v}_2, \mathbf{v}_2 \right\rangle_{\pi_1} \le \gamma \left\langle \mathbf{v}_2, \mathbf{v}_2 \right\rangle_{\pi_1} + (1-\gamma)\lambda_2^2 \left\langle \mathbf{v}_2, \mathbf{v}_2 \right\rangle_{\pi_1}.$$

This means $(1-\lambda_2)((1-\gamma)\lambda_2 - \gamma) \le 0$. Since $\lambda_2 < 1$ (otherwise the bound is meaningless), we have $\lambda_2 \le \frac{\gamma}{1-\gamma}$. □

Observe that in Theorem 2.5, we only consider the second largest eigenvalue of the local walk. When we take more eigenvalues into account, the improved trickling-down theorem is introduced in Abdolazimi, Liu and Oveis Gharan [ALOG21].

Theorem 2.6 (Matrix Trickling-Down Theorem, [ALOG21])**.** *Given a simplicial complex $\mathscr{C}$ equipped with distribution μ, suppose that the following holds:*

1. $\lambda_2(\mathrm{P}^{\wedge}_1) < 1$, i.e., $\mathrm{P}^{\wedge}_1$ *is irreducible.*
2. *There exists a family of symmetric matrices $\left\{M_\tau \in \mathbb{R}^{\mathscr{C}(1)\times\mathscr{C}(1)}\right\}_{\tau\in\mathscr{C}(1)}$ such that*

   $$\Pi_{\tau,1}\mathrm{P}^{\wedge}_{\tau,1} - \alpha\pi_{\tau,1}\pi_{\tau,1}^{\top} \preceq M_\tau \preceq \frac{1}{2\alpha+1}\Pi_{\tau,1}$$

   for all $\tau \in \mathscr{C}(1)$.

Then for every $M \in \mathbb{R}^{\mathscr{C}(1)\times\mathscr{C}(1)}$ satisfying $M \preceq \frac{1}{2\alpha}\Pi_1$ and $\mathbf{E}_{\tau\sim\pi_{\tau,1}}[M_\tau] \preceq M - \alpha M\Pi_1^{-1}M$, it holds that

$$\Pi_1\mathrm{P}^{\wedge}_1 - \left(2 - \frac{1}{\alpha}\right)\pi_1\pi_1^{\top} \preceq M.$$

In particular, $\lambda_2(\mathrm{P}^{\wedge}_1) \le \rho(\Pi_1^{-1}M)$.

Proof. We take expectation on all sides of the assumption and by Lemma 2.2,

$$\Pi_1 \mathrm{P}_1^{\wedge} - \alpha \Pi_1 (\mathrm{P}_1^{\wedge})^2 \preceq \mathbf{E}_{\tau \sim \pi_1}[M_\tau] \preceq \frac{1}{2\alpha+1}\Pi_1.$$

Therefore, $\Pi_1 \mathrm{P}_1^{\wedge} - \alpha \Pi_1 (\mathrm{P}_1^{\wedge})^2 \preceq M - \alpha M \Pi_1^{-1} M$. Set $Q = \mathrm{P}_1^{\wedge} - \beta \cdot \mathbf{1}\pi_1^\top$ with $\beta = 2 - \frac{1}{\alpha}$. Then we know

$$\Pi_1 \mathrm{P}_1^{\wedge} - \alpha \Pi_1 (\mathrm{P}_1^{\wedge})^2 = \Pi_1 Q - \alpha \Pi_1 Q^2.$$

Thus we know

$$\Pi_1 Q - \alpha \Pi_1 Q^2 \preceq M - \alpha M \Pi_1^{-1} M.$$

Since $\lambda_2(\mathrm{P}_{\tau,1}^{\wedge}) \leq \frac{1}{2\alpha+1}$ for every $\tau \in \mathscr{C}(1)$, by Theorem 2.5, $\lambda_2(\mathrm{P}_1^{\wedge}) \leq \frac{1}{2\alpha}$. Combined with $\beta = 2 - \frac{1}{\alpha} \geq 1 - \frac{1}{2\alpha}$ we have $Q \preceq \frac{1}{2\alpha} I$. By Lemma 2.3 in [ALOG21], we have $\Pi_1 Q \preceq M$. □

Commonly in use we apply the following proposition induced by Theorem 2.6.

Proposition 2.7. *Given a simplicial complex $\mathscr{C}$ equipped with distribution μ, if there exists a family of symmetric matrices $\left\{M_\tau \in \mathbb{R}^{\mathscr{C}(1)\times\mathscr{C}(1)}\right\}_{\tau\in\mathscr{C}}$ satisfying*

1. ***Base Cases:*** *For every $\tau \in \mathscr{C}(d-2)$,*

$$\Pi_{\tau,1}\mathrm{P}_{\tau,1}^{\wedge} - 2\pi_{\tau,1}\pi_{\tau,1}^\top \preceq M_\tau \preceq \frac{1}{5}\Pi_{\tau,1}.$$

2. ***Recursive Conditions:*** *For every $\tau \in \mathscr{C}(d-k)$ with $k \geq 3$, one of the followings holds:*

   - *The matrices satisfy*

$$M_\tau \preceq \frac{k-1}{3k-1}\Pi_{\tau,1}, \quad \mathbf{E}_{x\sim\pi_{\tau,1}}\left[M_{\tau\cup\{x\}}\right] \preceq M_\tau - \frac{k-1}{k-2} M_\tau \Pi_{\tau,1}^{-1} M_\tau.$$

   - *$(\mathscr{C}_\tau, \pi_{\tau,k})$ is the product of m pure weighted simplicial complexes $(\mathscr{C}^{(1)}, \pi^{(1)}), \ldots, (\mathscr{C}^{(m)}, \pi^{(m)})$ of dimension $d_1, \ldots, d_m$ respectively and,*

$$M_\tau = \sum_{j=1}^{m} \frac{d_j(d_j-1)}{k(k-1)} M_{\tau\cup\eta_j}$$

   where $\eta_j = \eta \setminus \mathscr{C}^{(j)}(1)$ for an arbitrary $\eta \in \mathscr{C}_\tau(k)$.

Then for every $\tau \in \mathscr{C}(d-k)$ with $k \geq 2$, it holds that

$$\Pi_{\tau,1}\mathrm{P}_{\tau,1}^{\wedge} - \frac{k}{k-1}\pi_{\tau,1}\pi_{\tau,1}^\top \preceq M_\tau \preceq \frac{k-1}{3k-1}\Pi_{\tau,1}.$$

In particular, $\lambda_2(\mathrm{P}_{\tau,1}^{\wedge}) \leq \rho(\Pi_{\tau,1}^{-1} M_\tau)$ for all $\tau \in \mathscr{C}(d-k)$ with $k \geq 2$.

2.4 The local-to-global theorem

Now we are ready to introduce the method named *Alev-Lau's Local-to-Global Theorem* introduced in Alev and Lau [AL20].

Theorem 2.8 (Alev-Lau's Local-to-Global Theorem, [AL20]). *Assume that $\mathscr{C}$ is an $(\alpha_0, \ldots, \alpha_{d-2})$-local spectral expander. Then for any $1 \leq k \leq d$, it holds that*

$$\mathrm{Gap}(\mathrm{P}_k^{\nabla}) = \mathrm{Gap}(\mathrm{P}_{k-1}^{\Delta}) \geq \frac{1}{k}\prod_{i=0}^{k-2}(1-\alpha_i).$$

Proof. It suffices to show that for all $1 \le k \le d-1$,

$$\mathrm{Gap}(\mathsf{P}_{k+1}^{\nabla}) = \mathrm{Gap}(\mathsf{P}_k^{\Delta}) \ge \frac{k}{k+1}(1-\alpha_{k-1})\mathrm{Gap}(\mathsf{P}_k^{\nabla}). \tag{7}$$

Together with the hypothesis induction and $\mathrm{Gap}(\mathsf{P}_1^{\nabla}) = 1$ we can conclude the theorem.

To prove (7), firstly observe that $\mathsf{P}_{k+1}^{\nabla}$ and P_k^{Δ} share the same non-zero eigenvalues and thus their spectral gaps are the same. By Lemma 2.3, for every $1 \le k \le d-1$,

$$\Pi_k \mathsf{P}_k^{\wedge} - \Pi_k \mathsf{P}_k^{\nabla} = \mathbf{E}_{\tau\sim\pi_{k-1}}\left[\overline{\Pi_{\tau,1}\mathsf{P}_{\tau,1}^{\wedge}} - \overline{\pi_{\tau,1}\pi_{\tau,1}^{\top}}\right].$$

Since $\mathscr{C}$ is an $(\alpha_0, \dots, \alpha_{d-2})$-local spectral expander, for every $\tau \in \mathscr{C}(k-1)$, the local walks satisfy:

$$\Pi_{\tau,1}\mathsf{P}_{\tau,1}^{\wedge} - \pi_{\tau,1}\pi_{\tau,1}^{\top} \preceq \alpha_{k-1}(\Pi_{\tau,1} - \pi_{\tau,1}\pi_{\tau,1}^{\top}).$$

Plugging it into above, by Lemmas 2.2 and 2.3, we obtain

$$\begin{aligned}\Pi_k \mathsf{P}_k^{\wedge} - \Pi_k \mathsf{P}_k^{\nabla} &\preceq \alpha_{k-1}\mathbf{E}_{\tau\sim\pi_{k-1}}\left[\overline{\Pi_{\tau,1} - \pi_{\tau,1}\pi_{\tau,1}^{\top}}\right] \\ &= \alpha_{k-1}(\Pi_k - \Pi_k \mathsf{P}_k^{\nabla}).\end{aligned}$$

This means

$$\Pi_k(I_{\mathscr{C}(k)} - \mathsf{P}_k^{\wedge}) \succeq (1-\alpha_{k-1})\Pi_k(I_{\mathscr{C}(k)} - \mathsf{P}_k^{\nabla})$$

thus leading to $\mathrm{Gap}(\mathsf{P}_k^{\wedge}) \ge (1-\alpha_{k-1})\mathrm{Gap}(\mathsf{P}_k^{\nabla})$. To finish the proof, note that

$$\mathsf{P}_k^{\Delta} = \frac{k}{k+1}\mathsf{P}_k^{\wedge} + \frac{1}{k+1}I_{\mathscr{C}(k)},$$

meaning that

$$\mathrm{Gap}(\mathsf{P}_k^{\Delta}) = \frac{k}{k+1}\mathrm{Gap}(\mathsf{P}_k^{\wedge}) \ge \frac{k}{k+1}(1-\alpha_{k-1})\mathrm{Gap}(\mathsf{P}_k^{\nabla}).$$

□

Remark 2.9. Note that, the random walk P_d^{∇} is the typical single-site Glauber dynamics P^{GD}. When μ is η-spectrally independent, it was shown in [ALOG20] that the corresponding simplicial complex $\mathscr{C}$ equipped with μ is a $(\frac{\eta}{d-1}, \frac{\eta}{d-2}, \dots, \eta)$-local spectral expander. Then we know the spectral gap of P^{GD} is bounded by

$$\mathrm{Gap}(P^{\mathrm{GD}}) \ge \frac{1}{d}\prod_{i=0}^{k-2}\left(1 - \frac{\eta}{d-i-1}\right).$$

By Lemma 1.2 we can show the mixing rate of P^{GD}.

3 Variance and Entropy Contraction

Now we give an alternative view of Theorem 2.8. Given a simplicial complex $\mathscr{C}$ of dimension d, for $0 \le \ell < k \le d$, we define the following random walks

$$\begin{aligned}\mathsf{P}_{k\to\ell}^{\downarrow} &= \mathsf{P}_k^{\downarrow}\mathsf{P}_{k-1}^{\downarrow}\dots\mathsf{P}_{\ell+1}^{\downarrow}, \\ \mathsf{P}_{\ell\to k}^{\uparrow} &= \mathsf{P}_{\ell}^{\uparrow}\mathsf{P}_{\ell+1}^{\uparrow}\dots\mathsf{P}_{k-1}^{\uparrow}, \\ \mathsf{P}_{k\leftrightarrow\ell}^{\nabla} &= \mathsf{P}_{k\to\ell}^{\downarrow}\mathsf{P}_{\ell\to k}^{\uparrow}, \\ \mathsf{P}_{\ell\leftrightarrow k}^{\Delta} &= \mathsf{P}_{\ell\to k}^{\uparrow}\mathsf{P}_{k\to\ell}^{\downarrow}.\end{aligned}$$

Then we consider the Dirichlet form of $\mathrm{P}^{\nabla}_{k\leftrightarrow\ell}$. By definition,

$$\begin{aligned}
\mathcal{E}_{\mathrm{P}^{\nabla}_{k\leftrightarrow\ell}}(f,f) &= \left\langle f, \left(I-\mathrm{P}^{\nabla}_{k\leftrightarrow\ell}\right)f\right\rangle_{\pi_k} \\
&= f^{\top}\Pi_k f - f^{\top}\Pi_k \mathrm{P}^{\nabla}_{k\leftrightarrow\ell} f \\
&= f^{\top}\Pi_k f - f^{\top}\Pi_k \mathrm{P}^{\downarrow}_{k\to\ell}\mathrm{P}^{\uparrow}_{\ell\to k} f \\
&\overset{(a)}{=} f^{\top}\Pi_k f - f^{\top}\left(\mathrm{P}^{\uparrow}_{\ell\to k}\right)^{\top}\Pi_\ell \mathrm{P}^{\uparrow}_{\ell\to k} f \\
&\overset{(b)}{=} \mathbf{Var}_{\pi_k}(f) - \mathbf{Var}_{\pi_\ell}\left(\mathrm{P}^{\uparrow}_{\ell\to k} f\right)
\end{aligned}$$

where (a) holds from (4) and (b) holds from (6). Similarly, for the Dirichlet form of $\mathrm{P}^{\Delta}_{\ell\leftrightarrow k}$, we have

$$\begin{aligned}
\mathcal{E}_{\mathrm{P}^{\Delta}_{\ell\leftrightarrow k}}(f,f) &= \left\langle f, \left(I-\mathrm{P}^{\Delta}_{\ell\leftrightarrow k}\right)f\right\rangle_{\pi_\ell} \\
&= f^{\top}\Pi_\ell f - f^{\top}\Pi_\ell \mathrm{P}^{\Delta}_{\ell\leftrightarrow k} f \\
&= f^{\top}\Pi_\ell f - f^{\top}\Pi_\ell \mathrm{P}^{\uparrow}_{\ell\to k}\mathrm{P}^{\downarrow}_{k\to\ell} f \\
&= \mathbf{Var}_{\pi_\ell}(f) - \mathbf{Var}_{\pi_k}\left(\mathrm{P}^{\downarrow}_{k\to\ell} f\right).
\end{aligned}$$

Follow the similar routine together with Jenssen's inequality, and we obtain the following identities and inequalities:

$$\mathcal{E}_{\mathrm{P}^{\nabla}_{k\leftrightarrow\ell}}(f,f) = \mathbf{Var}_{\pi_k}(f) - \mathbf{Var}_{\pi_\ell}\left(\mathrm{P}^{\uparrow}_{\ell\to k} f\right), \tag{8}$$

$$\mathcal{E}_{\mathrm{P}^{\Delta}_{\ell\leftrightarrow k}}(f,f) = \mathbf{Var}_{\pi_\ell}(f) - \mathbf{Var}_{\pi_k}\left(\mathrm{P}^{\downarrow}_{k\to\ell} f\right), \tag{9}$$

$$\mathcal{E}_{\mathrm{P}^{\nabla}_{k\leftrightarrow\ell}}(f,\log f) \geq \mathbf{Ent}_{\pi_k}[f] - \mathbf{Ent}_{\pi_\ell}\left[\mathrm{P}^{\uparrow}_{\ell\to k} f\right], \tag{10}$$

$$\mathcal{E}_{\mathrm{P}^{\Delta}_{\ell\leftrightarrow k}}(f,\log f) \geq \mathbf{Ent}_{\pi_\ell}[f] - \mathbf{Ent}_{\pi_k}\left[\mathrm{P}^{\downarrow}_{k\to\ell} f\right]. \tag{11}$$

The identities or inequalities as above show us that, when we want to show a Poincaré's inequality, it suffices to show the variance/entropy contraction.

Lemma 3.1 ([CGM21]). *Let $0\leq\ell\leq k\leq d$ and $f^{(k)}:\mathscr{C}(k)\to\mathbb{R}_{\geq 0}$ be a function on $\mathscr{C}(k)$. Then*

$$\mathbf{Ent}_{\pi_k}\left[f^{(k)}\right] = \mathbf{E}_{\tau\sim\pi_\ell}\left[\mathbf{Ent}_{\pi_{\tau,k-\ell}}\left[f^{(k-\ell)}_\tau\right]\right] + \mathbf{Ent}_{\pi_\ell}\left[f^{(\ell)}\right]$$

where $f^{(k-\ell)}_\tau(\sigma) := f^{(k)}(\tau\cup\sigma)$ for every $\sigma\in\mathscr{C}_\tau(k-\ell)$, and $f^{(\ell)} := \mathrm{P}^{\uparrow}_{\ell}\mathrm{P}^{\uparrow}_{\ell+1}\ldots\mathrm{P}^{\uparrow}_{k-1}f^{(k)}$.

Moreover, for all $f^{(k)}:\mathscr{C}(k)\to\mathbb{R}$, it holds that

$$\mathbf{Var}_{\pi_k}\left(f^{(k)}\right) = \mathbf{E}_{\tau\sim\pi_\ell}\left[\mathbf{Var}_{\pi_{\tau,k-\ell}}\left(f^{(k-\ell)}_\tau\right)\right] + \mathbf{Var}_{\pi_\ell}\left(f^{(\ell)}\right).$$

Proof. We prove the identity for variance and the identity for entropy is similar. Note that, as a simple extended version of Lemma 2.3, it holds that

$$\Pi_k = \mathbf{E}_{\tau\sim\pi_\ell}\left[\overline{\Pi_{\tau,k-\ell}}\right].$$

Without loss of generality, assume that $\mathbf{E}_{\pi_k}\left[f^{(k)}\right]=0$. Then,

$$\begin{aligned}
\mathbf{Var}_{\pi_k}\left(f^{(k)}\right) &= \left(f^{(k)}\right)^{\top}\Pi_k f^{(k)} \\
&= \left(f^{(k)}\right)^{\top}\mathbf{E}_{\tau\sim\pi_\ell}\left[\overline{\Pi_{\tau,k-\ell}}\right]f^{(k)} \\
&= \mathbf{E}_{\tau\sim\pi_\ell}\left[\left(f^{(k-\ell)}_\tau\right)^{\top}\Pi_{\tau,k-\ell}f^{(k-\ell)}_\tau\right] \\
&= \mathbf{E}_{\tau\sim\pi_\ell}\left[\mathbf{Var}_{\pi_{\tau,k-\ell}}\left(f^{(k-\ell)}_\tau\right)\right] + \mathbf{E}_{\tau\sim\pi_\ell}\left[\mathbf{E}_{\pi_{\tau,k-\ell}}\left[\left(f^{(k-\ell)}_\tau\right)^2\right]\right] \\
&= \mathbf{E}_{\tau\sim\pi_\ell}\left[\mathbf{Var}_{\pi_{\tau,k-\ell}}\left(f^{(k-\ell)}_\tau\right)\right] + \mathbf{Var}_{\pi_\ell}\left(f^{(\ell)}\right).
\end{aligned}$$

For entropy the proof is similar and we can just assume that $\mathbf{E}_{\pi_k}\left[f^{(k)}\right]=1$. □

Remark 3.2. Now we use the language of matrices. For every $0 \le \ell \le k \le d$, by calculation,

$$\Pi_k - \Pi_k \mathrm{P}^{\nabla}_{k\leftrightarrow\ell} = \mathbf{E}_{\tau\sim\pi_\ell}\left[\overline{\Pi_{\tau,k-\ell}}\right] - \Pi_k \mathrm{P}^{\nabla}_{k\leftrightarrow\ell}.$$

For every $\sigma, \rho \in \mathscr{C}(k)$ with $|\sigma \cap \rho| \ge \ell$, it holds that

$$\begin{aligned}
\Pi_k \mathrm{P}^{\nabla}_{k\leftrightarrow\ell}(\sigma,\rho) &= \pi_k(\sigma) \sum_{\tau\in\mathscr{C}(\ell):\tau\subseteq(\sigma\cap\rho)} \frac{1}{k(k-1)\dots(\ell+1)}\pi_{\tau,k-\ell}(\rho\setminus\tau) \\
&= \sum_{\tau\in\mathscr{C}(\ell):\tau\subseteq(\sigma\cap\rho)} \frac{\ell!}{k!}\pi_k(\sigma)\pi_{\tau,k-\ell}(\rho\setminus\tau) \\
&= \sum_{\tau\in\mathscr{C}(\ell):\tau\subseteq(\sigma\cap\rho)} \frac{\ell!}{k!}\binom{k}{\ell}\pi_\ell(\tau)\pi_{\tau,k-\ell}(\sigma\setminus\tau)\pi_{\tau,k-\ell}(\rho\setminus\tau) \\
&= \frac{1}{(k-\ell)!}\sum_{\tau\in\mathscr{C}(\ell):\tau\subseteq(\sigma\cap\rho)} \pi_\ell(\tau)\pi_{\tau,k-\ell}(\sigma\setminus\tau)\pi_{\tau,k-\ell}(\rho\setminus\tau),
\end{aligned}$$

thus leading to the identity

$$\Pi_k \mathrm{P}^{\nabla}_{k\leftrightarrow\ell} = \frac{1}{(k-\ell)!}\mathbf{E}_{\tau\sim\pi_\ell}\left[\overline{\pi_{\tau,k-\ell}\pi_{\tau,k-\ell}^{\top}}\right].$$

Plugging it into above, we know

$$\Pi_k - \Pi_k \mathrm{P}^{\nabla}_{k\leftrightarrow\ell} = \mathbf{E}_{\tau\sim\pi_\ell}\left[\overline{\Pi_{\tau,k-\ell}} - \frac{1}{(k-\ell)!}\overline{\pi_{\tau,k-\ell}\pi_{\tau,k-\ell}^{\top}}\right].$$

3.1 Variance tensorizations

To establish a Poincaré inequality via spectral independence, it's time to introduce the *tensorization of variance*. The kernel of this method is law of total covariance.

Theorem 3.3 (Law of Total Covariance)**.** *Let X, Y, Z be three random variables. Then it holds that*

$$\mathbf{Cov}(X,Y) = \mathbf{E}[\mathbf{Cov}(X,Y \mid Z)] + \mathbf{Cov}(\mathbf{E}[X \mid Z], \mathbf{E}[Y \mid Z]).$$

Proof. By law of total expectation, it holds that

$$\begin{aligned}
\mathbf{Cov}(X,Y) &= \mathbf{E}[XY] - \mathbf{E}[X]\,\mathbf{E}[Y] \\
&= \mathbf{E}[\mathbf{E}[XY \mid Z]] - \mathbf{E}[\mathbf{E}[X \mid Z]]\,\mathbf{E}[\mathbf{E}[Y \mid Z]] \\
&= \mathbf{E}[\mathbf{Cov}(X,Y \mid Z) + \mathbf{E}[X \mid Z]\,\mathbf{E}[Y \mid Z]] - \mathbf{E}[\mathbf{E}[X \mid Z]]\,\mathbf{E}[\mathbf{E}[Y \mid Z]] \\
&= \mathbf{E}[\mathbf{Cov}(X,Y \mid Z)] + \mathbf{Cov}(\mathbf{E}[X \mid Z], \mathbf{E}[Y \mid Z]).
\end{aligned}$$

□

Note that when we consider $\sigma \sim \mu$ where μ is the distribution over $\Omega \subseteq [q]^n$ and let $X = Y = f(\omega), Z = \omega(\Lambda)$ for a fixed arbitrary subset $\Lambda \subseteq [n]$, it holds that

$$\mathbf{Var}_\mu(f) = \mathbf{E}_{\sigma_\Lambda\sim\mu_\Lambda}\left[\mathbf{Var}_{\mu^{\sigma_\Lambda}}(f^{\sigma_\Lambda})\right] + \mathbf{Var}_{\sigma_\Lambda\sim\mu_\Lambda}\left(\mathbf{E}_{\mu^{\sigma_\Lambda}}[f^{\sigma_\Lambda}]\right).$$

Given $1 \le \ell \le n$, adding all identities for $\Lambda \in \binom{[n]}{\ell}$, it holds that

$$\mathbf{Var}_\mu(f) = \frac{1}{\binom{n}{\ell}}\sum_{\Lambda\subseteq[n],|\Lambda|=\ell}\left(\mathbf{E}_{\sigma_\Lambda\sim\mu_\Lambda}\left[\mathbf{Var}_{\mu^{\sigma_\Lambda}}(f^{\sigma_\Lambda})\right] + \mathbf{Var}_{\sigma_\Lambda\sim\mu_\Lambda}\left(\mathbf{E}_{\mu^{\sigma_\Lambda}}[f^{\sigma_\Lambda}]\right)\right).$$

To illustrate it in the form of law of total variance, consider the pinning set $\mathcal{P}_\ell$ defined as:

$$\mathcal{P}_\ell := \left\{(\Lambda, \sigma_\Lambda) \;\middle|\; \Lambda \in \binom{[n]}{\ell}, \sigma_\Lambda \in [q]^\Lambda\right\}.$$

Consider the distribution μ_ℓ on $\mathcal{P}_\ell$ defined as

$$\mu_\ell(\Lambda, \sigma_\Lambda) := \frac{1}{\binom{[n]}{\ell}} \mathbf{Pr}_{\omega\sim\mu}\left[\omega(\Lambda) = \sigma_\Lambda\right], \quad \forall (\Lambda, \sigma_\Lambda) \in \mathcal{P}_\ell,$$

and the function $f^{(\ell)} : \mathcal{P}_\ell \to \mathbb{R}$ as

$$f^{(\ell)}(\Lambda, \sigma_\Lambda) := \mathbf{E}_{\mu^{\sigma_\Lambda}}\left[f^{\sigma_\Lambda}\right].$$

Then by direct calculation, we obtain

$$\mathbf{Var}_\mu(f) = \mathbf{Var}_{\mu_\ell}\left(f^{(\ell)}\right) + \mathbf{E}_{(\Lambda,\sigma_\Lambda)\sim\mu_\ell}\left[\mathbf{Var}_{\mu^{\sigma_\Lambda}}\left(f^{\sigma_\ell}\right)\right]. \tag{12}$$

We remark here that (12) is exactly what we have shown in Lemma 3.1.

Definition 3.4 (Variance Independence)**.** We say a distribution μ is η-variance independent if for all functions $f : \Omega \to \mathbb{R}$,

$$\left(1 - \frac{1+\eta}{n}\right)\mathbf{Var}_\mu(f) \le \mathbf{E}_{(i,s)\sim\mathcal{P}_1}\left[\mathbf{Var}_{\mu^{i\leftarrow s}}(f)\right],$$

or equivalently

$$\mathbf{Var}_{\mu_1}\left(f^{(1)}\right) \le \frac{1+\eta}{n}\mathbf{Var}_\mu(f).$$

Lemma 3.5. *Let μ be a distribution over $\Omega \subseteq [q]^n$. Suppose that μ is η-spectrally independent. Then μ is η-variance independent.*

Proof. We consider the term $\mathbf{Var}_{\mu_1}\left(f^{(1)}\right)$. By definition,

$$\begin{aligned}\mathbf{Var}_{\mu_1}\left(f^{(1)}\right) &= \left\langle f^{(1)}, f^{(1)}\right\rangle_{\mu_1} - \left\langle f^{(1)}, \mathbf{1}\right\rangle_{\mu_1}^2 \\ &= f^\top\left(\frac{1}{n}\sum_{(i,s)\in\mathcal{P}_1}\mu_i(s)\left(\mu^{i\leftarrow s}\right)\left(\mu^{i\leftarrow s}\right)^\top\right)f - \langle f, \mathbf{1}\rangle_\mu^2.\end{aligned}$$

Now we define the random walk $\mathcal{R}_{\mu,1}$ over Ω as:

$$\mathcal{R}_{\mu,1} = \frac{1}{n}\sum_{(i,s)\in\mathcal{P}_1}\frac{1}{\mu_i(s)}\left(\mathbf{1}^{i\leftarrow s}\right)\left(\mathbf{1}^{i\leftarrow s}\right)^\top \mathrm{diag}(\mu).$$

It's not hard to observe that its stationary distribution is μ. Then we know

$$\frac{\mathbf{Var}_{\mu_1}\left(f^{(1)}\right)}{\mathbf{Var}_\mu(f)} = \frac{\left\langle f, \mathcal{R}_{\mu,1}f\right\rangle_\mu - \langle f, \mathbf{1}\rangle_\mu^2}{\langle f, f\rangle_\mu - \langle f, \mathbf{1}\rangle_\mu^2} \le \lambda_2(\mathcal{R}_{\mu,1}) = \lambda_2\left(\mathrm{diag}(\mu)^{1/2}\mathcal{R}_{\mu,1}\mathrm{diag}(\mu)^{-1/2}\right).$$

Then it suffices to bound the second largest eigenvalue of $\mathcal{R}_{\mu,1}$. Although $\mathcal{R}_{\mu,1}$ is the transition matrix in $\mathbb{R}^{\Omega\times\Omega}$, it has a decomposition as

$$\mathrm{diag}(\mu)^{1/2}\mathcal{R}_{\mu,1}\mathrm{diag}(\mu)^{-1/2} = \frac{1}{n}U_{\mu,1}U_{\mu,1}^\top$$

where $U_{\mu,1} \in \mathbb{R}^{\Omega\times\mathcal{P}_1}$ has columns $\mu_i(s)^{-1/2}\mathrm{diag}(\mu)^{1/2}\,\mathbf{1}^{i\leftarrow s}$ for each $(i,s) \in \mathcal{P}_1$. Then we only need to consider the eigenvalue of the matrix $\frac{1}{n}U_{\mu,1}^\top U_{\mu,1}$. By definition, for every $(i,s), (j,t) \in \mathcal{P}_1$,

$$\left(\frac{1}{n}U_{\mu,1}^\top U_{\mu,1}\right)((i,s),(j,t)) = \frac{1}{n}\frac{\mathbf{Pr}_{\omega\sim\mu}\left[\omega(i) = s \wedge \omega(j) = t\right]}{\sqrt{\mathbf{Pr}_{\omega\sim\mu}\left[\omega(i) = s\right]}\sqrt{\mathbf{Pr}_{\omega\sim\mu}\left[\omega(i) = s\right]}}.$$

Note that, this is the symmetrized version of the random walk $Q_{\mu,1}$ with stationary distribution μ_1, *i.e.*,

$$Q_{\mu,1}((i,s),(j,t)) = \frac{1}{n}\mathbf{Pr}_{\omega\sim\mu}\left[\omega(j)=t \mid \omega(i)=s\right].$$

Thus we know

$$\lambda_2(\mathcal{R}_{\mu,1}) = \lambda_2(Q_{\mu,1}) = \lambda_{\max}(Q_{\mu,1} - \mathbf{1}\mu_1^{\top}).$$

Observe that $Q_{\mu,1} - \mathbf{1}\mu_1^{\top}$ is exactly $\frac{1}{n}\widetilde{\Psi_\mu}$ where $\widetilde{\Psi_\mu}$ is defined as Remark 1.5. Then we conclude $\lambda_2(\mathcal{R}_{\mu,1}) \le \frac{1+\eta}{n}$. □

Since for the Glauber dynamics P^{GD}, we have already known for every function $f : \Omega \to \mathbb{R}$,

$$\mathcal{E}_{P^{\mathrm{GD}}}(f,f) = \mathbf{E}_{i\sim[n]}\left[\mathbf{E}_{\tau\sim\mu_{[n]\setminus\{i\}}}\left[\mathbf{Var}_{\mu^{\tau}}(f)\right]\right],$$

the spectral independence immediately implies the mixing rate of P^{GD}.

Theorem 3.6 (A Reformulation of Theorem 2.8)**.** *Let μ be a distribution over $\Omega \subseteq [q]^n$. Suppose that μ is η-spectrally independent. Then Glauber dynamics for the distribution μ has a spectral gap $\Omega\left(n^{-(1+\eta)}\right)$, and thus has the mixing time $O(n^{2+\eta})$.*

Proof. We only need to show the spectral gap of P^{GD}. Since μ is η-spectrally independent, by Lemma 3.5, for $1 \le \ell \le n$ and every $(\Lambda,\sigma_\Lambda) \in \mathcal{P}_\ell$, μ^{σ_Λ} is η-variance independent. Then it holds that

$$\begin{aligned}
\mathbf{Var}_{\mu}(f) &\le \left(1-\frac{1+\eta}{n}\right)^{-1}\mathbf{E}_{(i,s)\sim\mu_1}\left[\mathbf{Var}_{\mu^{i\leftarrow s}}(f)\right]\\
&\le \left(1-\frac{1+\eta}{n}\right)^{-1}\left(1-\frac{1+\eta}{n-1}\right)^{-1}\mathbf{E}_{(\{i,j\},\sigma_{\{i,j\}})\sim\mu_2}\left[\mathbf{Var}_{\mu^{\{i,j\}\leftarrow\sigma_{\{i,j\}}}}(f)\right]\\
&\le \cdots\\
&\le \prod_{j=0}^{\ell-1}\left(1-\frac{1+\eta}{n-j}\right)^{-1}\mathbf{E}_{(\Lambda,\sigma_\Lambda)\sim\mu_\ell}\left[\mathbf{Var}_{\mu^{\sigma_\Lambda}}(f)\right]\\
&\lesssim \exp\left((1+\eta)\sum_{j=0}^{\ell-1}\frac{1}{n-j}\right)\mathbf{E}_{(\Lambda,\sigma_\Lambda)\sim\mu_\ell}\left[\mathbf{Var}_{\mu^{\sigma_\Lambda}}(f)\right]\\
&\lesssim \left(\frac{n}{n-k}\right)^{1+\eta}\mathbf{E}_{(\Lambda,\sigma_\Lambda)\sim\mu_\ell}\left[\mathbf{Var}_{\mu^{\sigma_\Lambda}}(f)\right].
\end{aligned}$$

Let $\ell = n-1$, and we conclude the result. □

We remark here that, when $\eta > 1$, it holds that $1 - \frac{1+\eta}{n-\ell} < 0$ for $\ell = n-1$. To avoid this case, alternatively we define: for every $0 \le \ell \le n$,

$$\eta_\ell = \max_{(\Lambda,\sigma_\Lambda)\in\mathcal{P}_\ell}\lambda_{\max}\left(\Psi_\mu^{\sigma_\Lambda}\right).$$

Usually η_ℓ has the upper bound

$$\eta_\ell \le \min\{\eta, C(n-\ell)\}$$

where $0 < C < 1$. However, this is only a technical issue and is not the heart of most cases.

3.1.1 Optimal spectral gap for sparse graphical models

For graphical models with constant degree, it is well-known that the mixing time of a single-site Markov chain is at least $\Omega_\Delta(n\log n)$ in Hayes and Sinclair [HS05]. To achieve an optimal mixing rate, we show how to improve the result in Theorem 3.6.

Definition 3.7 (Graphic Markov Property)**.** For a distribution μ, we say it has the *graphic Markov property* if there exists a graph $G = (V(G), E(G))$ such that μ is a Markov distribution with respect to G, *i.e.*, for every partition

$$V(G) = A \sqcup \Lambda \sqcup B$$

such that A is isolated with B by Λ, it holds that for every pinning σ_Λ on Λ, the distribution $\mu^{\sigma_\Lambda}_{A \sqcup B}$ is the product probability measure as $\mu^{\sigma_\Lambda}_{A \sqcup B} = \mu^{\sigma_\Lambda}_A \otimes \mu^{\sigma_\Lambda}_B$.

Also the following shattering lemma is of great importance.

Lemma 3.8 (Shattering Lemma for Sparse Graph)**.** *Let $G = (V(G), E(G))$ be an n-vertex graph of maximum degree Δ. Then for every positive integer $\ell > 0$,*

$$\mathbf{Pr}_S \left[|S_v| = \ell\right] \le (2e\Delta\theta)^{\ell-1}$$

where S is a uniformly random subset of $V(G)$ of size $\lceil \theta n \rceil$, and S_v is the unique maximal connected component of $G[S]$ containing v.

Theorem 3.9. *Let μ be a distribution over $\Omega \subseteq [q]^n$. Suppose that μ is η-spectrally independent and graphic Markov. Then the Glauber dynamics for μ has spectral gap* $\mathrm{Gap}\left(P^{\mathrm{GD}}\right) \ge \Omega(1/Cn)$ *for some constant $C = C(\eta, \Delta) > 0$.*

Proof. Let $\ell = (1-\theta)n$ for some parameter $0 \le \theta \le 1$. By Lemma 3.8, we have

$$\begin{aligned}
\mathbf{Var}_\mu(f) &\le \theta^{-(1+\eta)} \mathbf{E}_{S \sim \binom{n}{\ell}} \left[\mathbf{E}_{\tau \sim \mu_{V \setminus S}} \left[\mathbf{Var}_{\mu^\tau}(f)\right]\right] \\
&\le \theta^{-(1+\eta)} \mathbf{E}_{S \sim \binom{n}{\ell}} \left[\mathbf{E}_{\tau \sim \mu_{V \setminus S}} \left[\sum_{U \text{ is the maximal connected component in } S} \mathbf{Var}_{\mu^\tau_U}(f)\right]\right] \\
&\le \theta^{-(1+\eta)} \sum_{v \in V} \mathbf{E}_{\tau \sim \mu_{-v}} \left[\mathbf{Var}_{\mu^\tau_v}(f)\right] \mathbf{E}_{S \sim \binom{n}{\ell}} \left[C_{|S_v|}\right] \\
&\le \theta^{-(1+\eta)} n \mathbf{E}_{v \sim V} \left[\mathbf{E}_{\tau \sim \mu_{-v}} \left[\mathbf{Var}_{\mu^\tau_v}(f)\right]\right] \sum_{k=1}^{\infty} (2e\Delta\theta)^{k-1} C_\ell.
\end{aligned}$$

when $\theta \le O(1/\Delta)$, it holds that $\mathbf{Var}_\mu(f) \le C(\eta, \Delta) n \cdot \mathbf{E}_{v \sim V}\left[\mathbf{E}_{\tau \sim \mu_{-v}}\left[\mathbf{Var}_{\mu^\tau_v}(f)\right]\right]$, thus leading to the result. □

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