A Local-to-Global Framework: Localization Schemes

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1 Localization Schemes and Markov Chains

Now we introduce another framework to show the local-to-global theorem. This framework, named the *localization schemes*, is highly related to the recent breakthrough of the famous Kannan-Lovász-Simonovits Conjecture, and deeply studied in Chen and Eldan [CE22] to analyze the mixing time of the Markov chains.

We fix a state space Ω equipped with a σ-algebra Σ. Usually we assume that $\Sigma = 2^{\Omega}$ when Ω is finite and $\Sigma = \mathrm{Borel}(\Omega)$ when Ω is a continuous space, and then we omit Σ. Let $\mathcal{M}(\Omega)$ be the space of all probability measures on Ω.

Definition 1.1 (Localization Process)**.** A *localization process* $(\mu_t)_{t\geq 0}$ on the state space Ω is a stochastic process satisfying

(P1) Almost surely μ_t is a probability measure on Ω for all $t \geq 0$.

(P2) For every measurable $A \subseteq \Omega$, the process $(\mu_t(A))_{t\geq 0}$ is a martingale.

(P3) For every measurable $A \subseteq \Omega$, the process $(\mu_t(A))_{t\geq 0}$ almost surely converges to either 0 or 1 as $t \to \infty$.

For convenience, we use Θ_t to denote the distribution of μ_t for every $t \geq 0$.

Definition 1.2 (Localization Scheme)**.** A *localization scheme* $\mathcal{L}$ on Ω is a mapping assigning to each probability measure $\mu \in \mathcal{M}(\Omega)$ a localization process $(\mu_t)_{t\geq 0}$ with $\mu_0 = \mu$. In this case, we say $(\mu_t)_t$ is the localization process associated with μ via the localization scheme $\mathcal{L}$.

For convenience, for every $t \geq 0$, let $\Gamma_{\mu,t}$ be the collection of all possible probability measures at time t, and $\Theta_{\mu,t}$ be the probability measure on $\Gamma_{\mu,t}$ where for every $\nu \in \Gamma_{\mu,t}$, $\Theta_{\mu,t}(\nu)$ equals the probability such that $\mu_t = \nu$ with $\mu_0 = \mu$ under $\mathcal{L}$. Usually μ is clear, and we will drop the subscript μ.

1.1 Markov dynamics associated with the localization process

In this part we associate a localization process $(\mu_t)_{t\geq 0} = \mathcal{L}(\mu)$ with a Markov dynamics reversible with respect to the distribution $\mu \in \mathcal{M}(\Omega)$.

Definition 1.3 (Markov Chains Associated with Localization Processes)**.** Let $(\mu_t)_{t\geq 0}$ be a localization process on Ω associated with μ via a localization scheme $\mathcal{L}$ and $\tau > 0$ be a stopping time. The Markov dynamics $P = P^{(\mathcal{L},\tau)}$ associated with $(\mu_t)_{t\geq 0}$ and τ is defined as

$$P(x, A) = \mathbf{E}_{\Theta_t}\left[\frac{\mu_\tau(x)\mu_\tau(A)}{\mu(x)}\right], \quad \forall x \in \Omega, A \in \Sigma.$$

Remark 1.4. An optional way to view Definition 1.3 is, let X, Y be two random variables taking values in $\Omega \times \Omega$ satisfying

$$\mathbf{Pr}\left[X \in A, Y \in B\right] = \mathbf{E}\left[\mu_\tau(A)\mu_\tau(B)\right], \quad \forall A, B \in \Sigma.$$

Then we define the kernel as

$$P(x, A) = \mathbf{Pr}\left[Y \in A \mid X = x\right].$$

Fact 1.5. *Let $P = P^{(\mathcal{L},\tau)}$ be the transition kernel defined as Definition 1.3. Then P is reversible with respect to μ.*

Proof. For every $x \in \Omega$, it almost surely holds that

$$P(x, \Omega) = \mathbf{E}_{\Theta_t}\left[\frac{\mu_\tau(x)\mu_\tau(\Omega)}{\mu(x)}\right] = \mathbf{E}_{\Theta_t}\left[\frac{\mu_\tau(x)}{\mu(x)}\right] = 1.$$

Then we know $P(x, \cdot)$ is a probability measure on Ω almost surely. Also for every $A, B \in \Sigma$, it holds that

$$\begin{aligned}
\int_{x\in A} P(x, B)\ \mathrm{d}\mu(x) &= \int_{x\in A} \mathbf{E}\left[\frac{\mathrm{d}\mu_\tau(x)}{\mathrm{d}\mu(x)}\mu_\tau(B)\right]\ \mathrm{d}\mu(x) \\
&= \mathbf{E}\left[\int_{x\in\Omega} \mu_\tau(B)\ \mathrm{d}\mu_\tau(x)\right] \\
&= \mathbf{E}\left[\mu_\tau(A)\mu_\tau(B)\right] \\
&= \int_{y\in B} P(y, A)\ \mathrm{d}\mu(y).
\end{aligned}$$

Therefore we know P is reversible with respect to μ. □

To view the Markov chain more clearly, consider the following two-step transition: at the current state $x \in \Omega$,

- firstly we draw a probability measure $\nu \in \Gamma_\tau$ following probability $\Theta_\tau(\nu) \cdot \frac{\nu(x)}{\mu(x)}$;
- then we draw the next state $y \sim \nu$.

Define the transition operators $\mathcal{D}_\mu^{(t)} : \Omega \times \Gamma_t \to \mathbb{R}$ and $\mathcal{U}_\mu^{(t)} : \Gamma_t \times \Omega \to \mathbb{R}$ as

$$\mathcal{D}_\mu^{(t)}(x, \nu) = \Theta_\tau(\nu) \cdot \frac{\nu(x)}{\mu(x)}, \quad \mathcal{U}_\mu^{(t)}(\nu, x) = \nu(x), \quad \forall x \in \Omega, \nu \in \Gamma_t.$$

It is easy to see $P^{(\mathcal{L},\tau)} = \mathcal{D}_\mu^{(t)}\mathcal{U}_\mu^{(t)}$.

1.2 Functional inequalities

Recall the Dirichlet form of a random walk P with stationary distribution μ: for two functions $f, g : \Omega \to \mathbb{R}$,

$$\mathcal{E}_P(f, g) := \int_{x\in\Omega} f(x)(I - P)g(x)\ \mathrm{d}\mu(x)$$

and the spectral gap and modified log-Sobolev inequality constant of P:

$$\mathrm{Gap}(P) := \inf_{f:\Omega\to\mathbb{R}} \frac{\mathcal{E}_P(f, f)}{\mathbf{Var}_\mu\left[f\right]}, \quad \rho_{\mathrm{LS}}(P) := \inf_{f:\Omega\to\mathbb{R}_{>0}} \frac{\mathcal{E}_P(f, \log f)}{\mathbf{Ent}_\mu\left[f\right]}.$$

The following identity and the inequality illustrate the connection between the functional inequalities and the variance or entropy of the localization process.

Proposition 1.6. *Let $P = P^{(\mathcal{L},\tau)}$ be a transition kernel associated with a localization process $(\mu_t)_{t\geq 0} = \mathcal{L}(\mu)$ and $\tau > 0$. Then it holds that*

$$\mathcal{E}_P(f, f) = \mathbf{E}_{\Theta_\tau}\left[\mathbf{Var}_{\mu_\tau}\left[f\right]\right], \quad \mathcal{E}_P(f, \log f) \geq \mathbf{E}_{\Theta_\tau}\left[\mathbf{Ent}_{\mu_\tau}\left[f\right]\right].$$

for every function f supported on Ω when the Dirichlet forms are well-defined.

Proof. We prove them one by one. By calculation,

$$\begin{aligned}
\mathcal{E}_P(f,f) &= \int_{x\in\Omega} f(x)(I-P)f(x)\ \mathrm{d}\mu(x) \\
&= \int_{x\in\Omega} \left(f(x)^2 - f(x)(Pf)(x)\right)\ \mathrm{d}\mu(x) \\
&= \int_{x\in\Omega} f(x)^2\ \mathrm{d}\mu(x) - \int_{x\in\Omega} f(x)\left(\int_{y\in\Omega} f(y)\ \mathrm{d}P(x,y)\right)\ \mathrm{d}\mu(x) \\
&= \mathbf{E}_\mu\left[f^2\right] - \int_{x\in\Omega}\int_{y\in\Omega} f(x)f(y)\mathbf{E}_{\Theta_\tau}\left[\frac{\mathrm{d}\mu_\tau(x)}{\mathrm{d}\mu(x)}\ \mathrm{d}\mu_\tau(y)\right]\ \mathrm{d}\mu(x) \\
&= \mathbf{E}_\mu\left[f^2\right] - \mathbf{E}_{\Theta_\tau}\left[\int_{x\in\Omega} f(x)\left(\int_{y\in\Omega} f(y)\ \mathrm{d}\mu_\tau(y)\right)\ \mathrm{d}\mu(x)\right] \\
&= \mathbf{E}_{\Theta_\tau}\left[\mathbf{E}_{\mu_\tau}\left[f^2\right] - \mathbf{E}_{\mu_\tau}\left[f\right]^2\right] \\
&= \mathbf{E}_{\Theta_\tau}\left[\mathbf{Var}_{\mu_\tau}\left[f\right]\right]
\end{aligned}$$

where the identity $\mathbf{E}_\mu\left[f^2\right] = \mathbf{E}_{\Theta_\tau}\left[\mathbf{E}_{\mu_\tau}\left[f^2\right]\right]$ holds from the martingality of the process.

For the MLSI constant, by calculation, we know

$$\begin{aligned}
\mathcal{E}_P(f,\log f) &= \int_{x\in\Omega} f(x)\left((I-P)\log f\right)(x)\ \mathrm{d}\mu(x) \\
&= \int_{x\in\Omega} \left(f(x)\log f(x) - f(x)(P\log f)(x)\right)\ \mathrm{d}\mu(x) \\
&= \int_{x\in\Omega} f(x)\log f(x)\ \mathrm{d}\mu(x) - \int_{x\in\Omega} f(x)\left(\int_{y\in\Omega} \log f(y)\ \mathrm{d}P(x,y)\right)\ \mathrm{d}\mu(x) \\
&= \mathbf{E}_\mu\left[f\log f\right] - \int_{x\in\Omega}\int_{y\in\Omega} f(x)\log f(y)\mathbf{E}_{\Theta_\tau}\left[\frac{\mathrm{d}\mu_\tau(x)}{\mathrm{d}\mu(x)}\ \mathrm{d}\mu_\tau(y)\right]\ \mathrm{d}\mu(x) \\
&= \mathbf{E}_\mu\left[f\log f\right] - \mathbf{E}_{\Theta_\tau}\left[\int_{x\in\Omega} f(x)\left(\int_{y\in\Omega} \log f(y)\ \mathrm{d}\mu_\tau(y)\right)\ \mathrm{d}\mu(x)\right] \\
&= \mathbf{E}_{\Theta_\tau}\left[\mathbf{E}_{\mu_\tau}\left[f\log f\right] - \mathbf{E}_{\mu_\tau}\left[f\right]\mathbf{E}_{\mu_\tau}\left[\log f\right]\right] \\
&\geq \mathbf{E}_{\Theta_\tau}\left[\mathbf{E}_{\mu_\tau}\left[f\log f\right] - \mathbf{E}_{\mu_\tau}\left[f\right]\log\mathbf{E}_{\mu_\tau}\left[f\right]\right] \\
&= \mathbf{E}_{\Theta_\tau}\left[\mathbf{Ent}_{\mu_\tau}\left[f\right]\right]
\end{aligned}$$

where the inequality holds from the Jensen's inequality $\log\mathbf{E}_\pi\left[f\right] \geq \mathbf{E}_\pi\left[\log f\right]$ for every distribution π on Ω and every test function $f : \Omega \to \mathbb{R}_{>0}$. □

2 Linear-Tilt Localization Processes

Now we introduce a family of localization processes which lies on the core of the analysis of the mixing time. For a distribution π on Ω, we use $\mathbf{b}(\pi)$ to denote the mass center of π, *i.e.*,

$$b(\pi) = \int_{x\in\Omega} x\ \mathrm{d}\pi(x).$$

Definition 2.1 (Linear-Tilt Localization Processes)**.** For a localization process $(\mu_t)_{t\geq 0}$, we say it is a *linear-tilt localization process* if:

- **(Discrete version)** For all $t \in \mathbb{N}$ and $x \in \Omega$,

$$\mu_{t+1}(x) = \mu_t(x)\left(1 + \langle x - \mathbf{b}(\mu_t), Z_t\rangle\right) \tag{1}$$

  where Z_t is a random variable with $\mathbf{E}\left[Z_t \mid \mu_t\right] = 0$. Or,

- **(Continuous version)** For all $t \geq 0$ and $x \in \Omega$,

$$\mathrm{d}\mu_t(x) = \mu_t(x) \langle x - \mathbf{b}(\mu_t), Z_t \rangle \tag{2}$$

  where Z_t is a random variable with $\mathbf{E}[Z_t \mid \mu_t] = 0$.

For convenience, we say $(Z_t)_{t\geq 0}$ is the driving factor of $(\mu_t)_{t\geq 0}$.

We will focus on the following two kinds of localization schemes: (1) the coordinate-by-coordinate localization schemes; (2) the stochastic localization schemes driven by standard Brownian motion.

2.1 The coordinate-by-coordinate localization schemes

Given a distribution μ over $\Omega \subseteq \mathbb{R}^n$, we construct a discrete-time localization process $(\mu)_{t\geq 0}$ as follows:

- Firstly we pick a permutation $k_1, \ldots, k_n$ of $[n]$ uniformly at random.
- Let $X \sim \mu$. For $t \geq 0$, we set μ_t to be the law of X conditional on $X_{k_1}, \ldots, X_{k_i}$ where $i = \min\{n, \lfloor t \rfloor\}$.

Now we show the observation that the dynamics associated with the coordinate-by-coordinate localization process is the well-known *Glauber dynamics*.

Fact 2.2. *Given a coordinate-by-coordinate localization scheme $\mathcal{L}$ over $\Omega \subseteq \mathbb{R}^n$ and an integer $\tau = n-1$, the Markov chain $P = P^{(\mathcal{L},\tau)}$ associated with $(\mu_t)_{t\geq 0} = \mathcal{L}(\mu)$ and τ is the single-site Glauber dynamics denoted by P^{GD} with stationary distribution μ.*

Proof. We verify the fact by definition. For every $x \in \Omega$ and $i \in [n]$, define $L_{x,i} := \left\{z \in \Omega \,\middle|\, \forall j \in [n] \setminus \{i\}, z_j = x_j\right\}$. It's not hard to see that it suffices to show the case $\|x - y\|_0 = 1$.

Assume that x, y only differ at the coordinate $i \in [n]$, *i.e.*, $x_i \neq y_i$ and $x_j = y_j$ for every $j \in [n] \setminus \{i\}$. Then by definition,

$$\begin{aligned}
P(x,y) &= \mathbf{E}_{\Theta_{n-1}}\left[\frac{\mu_{n-1}(x)\mu_{n-1}(y)}{\mu(x)}\right] \\
&= \sum_{j\in[n]} \frac{1}{n}\mathbf{E}\left[\frac{\mu_{n-1}(x)\mu_{n-1}(y)}{\mu(x)} \,\middle|\, k_n = j\right] \\
&= \frac{1}{n}\mathbf{E}\left[\frac{\mu_{n-1}(x)\mu_{n-1}(y)}{\mu(x)} \,\middle|\, k_n = i\right] \\
&= \frac{1}{n}\mathbf{Pr}\left[\mathrm{supp}(\mu_{n-1}) = L_{x,i}\right]\mathbf{E}\left[\frac{\mu_{n-1}(x)\mu_{n-1}(y)}{\mu(x)} \,\middle|\, k_n = i, \mathrm{supp}(\mu_{n-1}) = L_{x,i}\right] \\
&= \frac{1}{n}\frac{\mu(L_{x,i})\mu(x)\mu(y)}{\mu(x)\mu(L_{x,i})^2} \\
&= \frac{1}{n}\frac{\mu(y)}{\mu(L_{x,i})}.
\end{aligned}$$

When $\|x - y\|_0 \geq 2$, it is easy to see $P(x,y) = \mathrm{P}^{\mathrm{GD}}(x,y) = 0$. Thus we conclude the statement. □

Remark 2.3. When $\tau = n - \ell$, the corresponding Markov kernel associated with the coordinate-by-coordinate localization process and τ is the ℓ-uniform block dynamics $\mathrm{P}^{\ell-\mathrm{GD}}$.

2.1.1 The coordinate-by-coordinate localization process as a linear-tilt process

In this part we will show the coordinate-by-coordinate localization process $(\mu_t)_{t\geq 0}$ is a linear-tilt localization process. Fix a probability measure μ on $\Omega = \{-1,+1\}^n$. We pick a permutation $k_1,\dots,k_n$ of $[n]$ uniformly at random. Let $U_1,\dots,U_n$ be independent random variables uniformly distributed in $[-1,+1]$.

Let $\mu_0 = \mu$. For $i = 0,1,\dots,n$, we define

$$\mu_{i+1}(x) = \mu_i(x)\left(1 + \langle x - \mathbf{b}(\mu_i), Z_i\rangle\right), \quad \forall x \in \Omega$$

where Z_i is a $\sigma(\mu_0,\dots,\mu_i)$-measurable random variable defined as

$$Z_i := \mathbf{e}_{k_{i+1}} \times \begin{cases} \frac{1}{1+\mathbf{b}(\mu_i)_{k_{i+1}}} & \mathbf{b}(\mu_i)_{k_{i+1}} \geq U_{i+1}, \\ \frac{-1}{1-\mathbf{b}(\mu_i)_{k_{i+1}}} & \mathbf{b}(\mu_i)_{k_{i+1}} \leq U_{i+1}, \end{cases}$$

where $(\mathbf{e}_1,\dots,\mathbf{e}_n)$ is the standard basis of $\mathbb{R}^n$.

It is not hard to see $\mathbf{E}\left[Z_i \mid \mu_i\right] = 0$, and

$$\begin{aligned}
\mu_{i+1}(\Omega) &= \int_{x\in\Omega} \mathrm{d}\mu_{i+1}(x) \\
&= \int_{x\in\Omega} \left(1 + \langle x - \mathbf{b}(\mu_i), Z_i\rangle\right)\, \mathrm{d}\mu_i(x) \\
&= \mu_i(\Omega) + \left\langle \int_{x\in\Omega} (x - \mathbf{b}(\mu_i))\, \mathrm{d}\mu_i(x), Z_i \right\rangle \\
&= \mu_i(\Omega)
\end{aligned}$$

meaning that $\mu_i(\Omega) = 1$ for each $i \in [n]$. To show μ_{i+1} is a pinning of μ_i, firstly note that the marginal distribution of the coordinate k_{i+1} is

$$\mathbf{Pr}_{X\sim\mu_t}\left[X_{k_{i+1}} = 1\right] = \frac{1+\mathbf{b}(\mu_i)_{k_{i+1}}}{2}, \quad \mathbf{Pr}_{X\sim\mu_t}\left[X_{k_{i+1}} = 1\right] = \frac{1-\mathbf{b}(\mu_i)_{k_{i+1}}}{2}.$$

By the definition of Z_i, when x is not identical to the pinned value, the inner product will be -1 and the probability will vanish.

2.2 Stochastic localization schemes driven by standard Brownian motion

Now we introduce a kind of linear-tilt localization schemes named the *stochastic localization scheme* firstly constructed by Eldan [Eld13]. Fix a probability measure μ on $\Omega \subseteq \mathbb{R}^n$. Let $(B_t)_{t\geq 0}$ be the standard Brownian motion in $\mathbb{R}^n$ adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Let $(C_t)_{t\geq 0}$ be a stochastic process adapted to $(\mathcal{F}_t)_{t\geq 0}$ taking values in $n\times n$ positive semidefinite matrices. We define a measure-valued stochastic process $(\mu_t)_{t\geq 0}$ by $\frac{\mathrm{d}\mu_t}{\mathrm{d}\mu}(x) = F_t(x)$ as,

$$F_0(x) = 1, \;\; \mathrm{d}F_t(x) = F_t(x)\,\langle x - \mathbf{b}(\mu_t), C_t\, \mathrm{d}B_t\rangle, \quad \forall x \in \Omega. \tag{3}$$

Proposition 2.4. *If $\int_{t=0}^{\infty} C_t^2\, \mathrm{d}t = \infty$, then $(\mu_t)_{t\geq 0}$ is a localization process. Moreover,*

$$\frac{\mathrm{d}\mu_t}{\mathrm{d}\mu_t}(x) = F_t(x) = \frac{1}{Z_t}\exp\left(-\frac{1}{2}\langle \Sigma_t x, x\rangle + \langle \mathbf{y}_t, x\rangle\right)$$

where Z_t is a normalizing factor to ensure that $\int_{x\in\Omega} F_t(x)\, \mathrm{d}\mu(x) = 1$ and $(\Sigma_t)_{t\geq 0}, (\mathbf{y}_t)_{t\geq 0}$ are stochastic processes adapted to $\mathcal{F}_t$ in the form of

$$\mathrm{d}\mathbf{y}_t = C_t\, \mathrm{d}B_t + C_t^2 \mathbf{b}(\mu_t)\, \mathrm{d}t, \;\; \mathrm{d}\Sigma_t = C_t^2\, \mathrm{d}t.$$

Proof. We prove the proposition by solving (3). Consider the stochastic process $(\log F_t(x))_{t\geq 0}$. By Itô's formula,

$$\begin{aligned} \mathrm{d}\log F_t(x) &= \frac{\mathrm{d}F_t(x)}{F_t(x)} - \frac{\mathrm{d}[F(x)]_t}{2F_t(x)^2} \\ &= \langle x - \mathbf{b}(\mu_t), C_t \ \mathrm{d}B_t\rangle - \frac{1}{2}\|C_t(x-\mathbf{b}(\mu_t))\|_2^2 \ \mathrm{d}t. \end{aligned}$$

This leads to the form

$$F_t(x) = \frac{1}{Z_t}\exp\left(-\frac{1}{2}\langle \Sigma_t x, x\rangle + \langle \mathbf{y}_t, x\rangle\right)$$

where $Z_t, \Sigma_t, \mathbf{y}_t$ are described as the proposition. Also we know $\mu_t(x) \geq 0$ for every $x \in \Omega$. By definition,

$$\begin{aligned} \mathrm{d}\mu_t(\Omega) &= \ \mathrm{d}\int_{x\in\Omega} \ \mathrm{d}\mu_t(x) \\ &= \int_{x\in\Omega} F_t(x)\,\langle x - \mathbf{b}(\mu_t), C_t \ \mathrm{d}B_t\rangle \ \mathrm{d}\mu(x) \\ &= \left\langle \int_{x\in\Omega} (x-\mathbf{b}(\mu_t)) \ \mathrm{d}\mu_t(x), C_t \ \mathrm{d}B_t \right\rangle \\ &= 0. \end{aligned}$$

Then we know $\mu_t(\Omega) = 1$ for every $t \geq 0$ almost surely. Thus we know μ_t is almost surely a probability measure on Ω. The martingality comes directly from the definition, and to see the convergence of the process, note that when $\Sigma_t \to \infty$, by the form of F_t it will be a Dirac measure. □

When $C_t \equiv Q^{-1/2}$, we know the law of $\mathbf{y}_t$ by El Alaoui and Montanari [EAM22].

Theorem 2.5 ([EAM22])**.** *Fix a probability measure μ on Ω and a positive semidefinite matrix $Q \in \mathbb{R}^{n\times n}$. Let $(\mu_t)_{t\geq 0}$ be a stochastic localization process starting from μ driven by $C_t \equiv Q^{-1/2}$. Define the stochastic process $(\Sigma_t)_{t\geq 0}$, $(\mathbf{y}_t)_{t\geq 0}$ as above. Then*

$$\Sigma_t = tQ^{-1}, \ \mathbf{y}_t/t \sim \mu * \mathcal{N}(0, \Sigma_t), \quad \forall t \geq 0.$$

2.3 Variance contraction via linear-tilt localization processes

Now we show how to bound the spectral gap of the Glauber dynamics P^{GD}. The following property named the *variance conservation* is the key in our analysis.

Definition 2.6 (Variance Conservation - Discrete)**.** Given a time-discrete localization process $(\mu_t)_{t\in\mathbb{N}}$ on Ω satisfying $(\kappa_1, \kappa_2, \ldots)$-*variance conservation* up to time $t \in \mathbb{N}$, if for every test function $f : \Omega \to \mathbb{R}$,

$$\mathbf{E}\left[\mathbf{Var}_{\mu_i}[f] \;\middle|\; \mu_{i-1}\right] \geq (1-\kappa_i)\mathbf{Var}_{\mu_{i-1}}[f], \quad \forall 1 \leq i \leq t.$$

Proposition 2.7. *Let $(\mu_t)_{t\in\mathbb{N}}$ be a time-discrete localization process on Ω satisfying $(\kappa_1, \kappa_2, \ldots)$-variance conservation up to time $t \in \mathbb{N}$. Let P be the random walk associated with $(\mu_t)_{t\in\mathbb{N}}$ and time t. Then its spectral gap* $\mathrm{Gap}(P)$ *satisfies*

$$\mathrm{Gap}(P) \geq \prod_{i=1}^{t}(1-\kappa_i).$$

Proof. By Proposition 1.6, it suffices to show for every test function $f : \Omega \to \mathbb{R}^n$,

$$\frac{\mathbf{E}_{\Theta_t}\left[\mathbf{Var}_{\mu_t}[f]\right]}{\mathbf{Var}_{\mu}[f]} \geq \prod_{i=1}^{t}(1-\kappa_i).$$

Note that $\mu_0 = \mu$. Then by direct calculation,

$$
\begin{aligned}
\frac{\mathbf{E}_{\Theta_t}\left[\mathbf{Var}_{\mu_t}[f]\right]}{\mathbf{Var}_{\mu}[f]} &= \mathbf{E}_{\Theta_t}\left[\frac{\mathbf{Var}_{\mu_t}[f]}{\mathbf{Var}_{\mu_0}[f]}\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[\dots\mathbf{E}\left[\frac{\mathbf{Var}_{\mu_t}[f]}{\mathbf{Var}_{\mu_{t-1}}[f]} \;\middle|\; \mu_{t-1}\right]\dots\right]\frac{\mathbf{Var}_{\mu_1}[f]}{\mathbf{Var}_{\mu_0}[f]} \;\middle|\; \mu_0\right] \\
&\geq \prod_{i=1}^{t}(1-\kappa_i)
\end{aligned}
$$

where the last inequality holds from Definition 2.6. □

Now it's time for us to show the variance contraction for a linear-tilt localization process $(\mu_t)_{t\in\mathbb{N}}$. The first step is to show the form of the evolution of its variance.

Lemma 2.8. *Let $(\mu_t)_{t\in\mathbb{N}}$ be a time-discrete linear-tilt localization process and $(Z_t)_{t\in\mathbb{N}}$ be its driving factor. Then for every test function $f:\Omega\to\mathbb{R}$ and $t\in\mathbb{N}$,*

$$
\mathbf{E}\left[\mathbf{Var}_{\mu_{t+1}}[f] \;\middle|\; \mu_t\right] = \mathbf{Var}_{\mu_t}[f] - \langle V_t, C_t V_t\rangle
$$

where

$$
V_t := \int_{x\in\Omega}(x-\mathbf{b}(\mu_t))f(x)\;\mathrm{d}\mu_t(x),\;\; C_t := \mathbf{Cov}\left(Z_t \mid \mu_t\right).
$$

Proof. Fix a test function $f:\Omega\to\mathbb{R}$. By direct calculation,

$$
\begin{aligned}
\mathbf{E}\left[\mathbf{Var}_{\mu_{t+1}}[f] \;\middle|\; \mu_t\right] &= \mathbf{E}\left[\int_\Omega f(x)^2\;\mathrm{d}\mu_{t+1}(x) - \left(\int_\Omega f(x)\;\mathrm{d}\mu_{t+1}(x)\right)^2 \;\middle|\; \mu_t\right] \\
&= \int_\Omega f(x)^2\;\mathrm{d}\mu_t(x) - \mathbf{E}\left[\left(\int_\Omega f(x)\left(1+\langle x-\mathbf{b}(\mu_t), Z_t\rangle\right)\;\mathrm{d}\mu_t(x)\right)^2 \;\middle|\; \mu_t\right] \\
&= \int_\Omega f(x)^2\;\mathrm{d}\mu_t(x) - \left(\int_\Omega f(x)\;\mathrm{d}\mu_t(x)\right)^2 - \mathbf{E}\left[\left(\int_\Omega f(x)\langle x-\mathbf{b}(\mu_t), Z_t\rangle\;\mathrm{d}\mu_t(x)\right)^2 \;\middle|\; \mu_t\right] \\
&= \mathbf{Var}_{\mu_t}[f] - \mathbf{E}\left[\langle V_t, Z_t\rangle^2 \;\middle|\; \mu_t\right] \\
&= \mathbf{Var}_{\mu_t}[f] - V_t^\top\mathbf{E}\left[Z_t^\top Z_t \;\middle|\; \mu_t\right]V_t \\
&= \mathbf{Var}_{\mu_t}[f] - V_t^\top C_t V_t.
\end{aligned}
$$

□

Proposition 2.9. *Let $(\mu_t)_{t\in\mathbb{N}}$ be a time-discrete linear-tilt localization process and $(Z_t)_{t\in\mathbb{N}}$ be its driving factor. Then $(\mu_t)_{t\in\mathbb{N}}$ satisfies $(\kappa_1,\kappa_2,\dots)$-variance conservation where*

$$
\kappa_{t+1} = 1 - \left\|C_t^{1/2}\mathbf{Cov}\left(\mu_t\right)C_t^{1/2}\right\|_{\mathrm{OP}},\quad \forall t\in\mathbb{N}.
$$

Proof. Firstly it is not hard to see that it suffices to show the case $\mathbf{E}_{\mu}[f] = \mathbf{E}_{\mu_t}[f] = 0$. By Lemma 2.8, we only need to bound the term $\langle V_t, C_t V_t\rangle$. By definition,

$$\begin{aligned}\langle V_t, C_t V_t\rangle &= \left\|C_t^{1/2} V_t\right\|_2^2 \\ &= \sup_{\theta:\|\theta\|_2=1} \left\langle C_t^{1/2} V_t, \theta\right\rangle^2 \\ &= \sup_{\theta:\|\theta\|_2=1} \left(\int_\Omega \langle C_t(x-\mathbf{b}(\mu_t)), \theta\rangle f(x)\, \mathrm{d}\mu_t(x)\right)^2 \\ &\le \sup_{\theta:\|\theta\|_2=1} \mathbf{Var}_{\mu_t}[f] \int_\Omega \langle C_t(x-\mathbf{b}(\mu_t)), \theta\rangle^2 f(x)\, \mathrm{d}\mu_t(x) \\ &= \left\|C_t^{1/2}\mathbf{Cov}(\mu_t)\, C_t^{1/2}\right\|_{\mathrm{OP}} \mathbf{Var}_{\mu_t}[f]\end{aligned}$$

where the inequality holds by the Cauchy-Schwarz inequality. □

2.3.1 Variance conservation via the coordinate-by-coordinate localization process

Now we show the main result of rapid mixing via the spectral independence by Anari, Liu and Oveis Gharan [ALOG20].

Lemma 2.10. *Fix a disbribution μ on $\Omega \subseteq \{-1,+1\}^n$. Let $(\mu_t)_{t\in\mathbb{N}}$ be a coordinate-by-coordinate localization process starting from μ. Then $(\mu_t)_{t\in\mathbb{N}}$ satisfies $(\kappa_1, \kappa_2, \dots)$-variance conservation up to n such that*

$$\kappa_{t+1} = 1 - \frac{\|\mathbf{Cor}(\mu_t)\|_{\mathrm{OP}}}{n-t}, \quad \forall 0 \le t < n$$

where $\mathbf{Cor}(\mu_t) = \mathrm{diag}(\mathbf{Cov}(\mu_t))^{-1/2}\, \mathbf{Cov}(\mu_t)\, \mathrm{diag}(\mathbf{Cov}(\mu_t))^{-1/2}$.

Proof. By Proposition 2.9, it suffices to show

$$C_t^{1/2}\mathbf{Cov}(\mu_t)\, C_t^{1/2} = \frac{\mathbf{Cor}(\mu_t)}{n-t}.$$

By direct calculation, for every unpinned $i \in [n]$,

$$\begin{aligned}C_t(i,i) &= \mathbf{Cov}(Z_t \mid \mu_t)_{i,i} \\ &= \frac{1}{n-t}\frac{1}{1-\mathbf{b}(\mu_t)_i^2} \\ &= \frac{1}{(n-t)\mathbf{Cov}(\mu_t)_{i,i}}.\end{aligned}$$

Then the identity holds. □

Since we have already know $\left\|\Psi_{\mu_t}\right\|_{\mathrm{OP}} = \|\mathbf{Cor}(\mu_t)\|_{\mathrm{OP}}$, we can establish the result of [ALOG20].

Lemma 2.11 (A Reformulation of the Main Result in [ALOG20])**.** *Given an $(\eta_0, \dots, \eta_n)$-spectrally independent Gibbs distribution μ of some hardcore model over the state space $\Omega \subseteq \{-1,+1\}^n$, the spectral gap of the ℓ-uniform block dynamics is at least*

$$\mathrm{Gap}(\mathrm{P}^{\ell\text{-GD}}) \ge \prod_{t=0}^{n-\ell-1}\left(1 - \frac{\eta_t}{n-t}\right).$$

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