第3章 一维问题

3.3 高斯积分

Gauss Quadrature

[Fish]-4.6

3.3 Gauss Quadrature

- Gauss quadrature is one of the most efficient techniques for functions that are polynomials or nearly polynomials

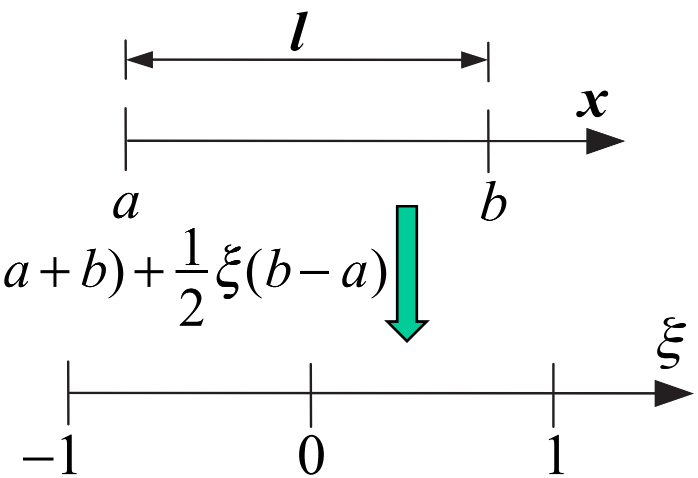
$$I = \int_a^b f(x)\,\mathrm{d}x = ?$$

$$\mathrm{d}x = \frac{1}{2}(b-a)\,\mathrm{d}\xi \;= J\,\mathrm{d}\xi$$

$$J = \frac{1}{2}(b-a) = \frac{1}{2}l$$

$$I = J\int_{-1}^{1} f(\xi)\,\mathrm{d}\xi \;= J\hat{I}$$

$$\hat{I} = \int_{-1}^{1} f(\xi)\,\mathrm{d}\xi$$



$$x = \frac{1}{2}(a+b) + \frac{1}{2}\xi(b-a)$$

- Approximate the integral by

$$\hat{I} = W_1 f(\xi_1) + W_2 f(\xi_2) + \cdots = \overbrace{\begin{bmatrix} W_1 & W_2 & \cdots & W_n \end{bmatrix}}^{\boldsymbol{W}^{\mathrm{T}}} \overbrace{\begin{bmatrix} f(\xi_1) \\ f(\xi_2) \\ \vdots \\ f(\xi_n) \end{bmatrix}}^{\boldsymbol{f}}$$

$$= \boldsymbol{W}^{\mathrm{T}}\boldsymbol{f}$$

W_i — weights

ξ_i — integration points ?

3.3 Gauss Quadrature

- Gauss Quadrature: Choose the weights and integration points that yield an exact integral of a polynomial of a given order

$$f(\xi)=\alpha_1+\alpha_2\xi+\cdots+\alpha_{p+1}\xi^p=\underbrace{\begin{bmatrix}1 & \xi & \cdots & \xi^p\end{bmatrix}}_{\boldsymbol{p}}\underbrace{\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_{p+1}\end{bmatrix}}_{\boldsymbol{\alpha}}$$

$$=\boldsymbol{p}(\xi)\boldsymbol{\alpha}$$

$$\begin{aligned}f(\xi_1)&=\alpha_1+\alpha_2\xi_1+\cdots+\alpha_{p+1}\xi_1^p\\ f(\xi_2)&=\alpha_1+\alpha_2\xi_2+\cdots+\alpha_{p+1}\xi_2^p\\ &\vdots\\ f(\xi_n)&=\alpha_1+\alpha_2\xi_n+\cdots+\alpha_{p+1}\xi_n^p\end{aligned}\qquad \underbrace{\begin{bmatrix}f(\xi_1)\\ f(\xi_2)\\ \vdots\\ f(\xi_n)\end{bmatrix}}_{\boldsymbol{f}}=\underbrace{\begin{bmatrix}1 & \xi_1 & \cdots & \xi_1^p\\ 1 & \xi_2 & \cdots & \xi_2^p\\ \vdots & \vdots & \vdots & \vdots\\ 1 & \xi_n & \cdots & \xi_n^p\end{bmatrix}}_{\boldsymbol{M}}\underbrace{\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_{p+1}\end{bmatrix}}_{\boldsymbol{\alpha}}$$

$$\boldsymbol{f}=\boldsymbol{M}\boldsymbol{\alpha}$$

3.3 Gauss Quadrature

$$\hat{I} = \int_{-1}^{1} f(\xi)\mathrm{d}\xi \xrightarrow{\text{G. Q.}} = \boldsymbol{W}^{\mathrm{T}}\boldsymbol{f} = \boldsymbol{W}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{\alpha}$$

$$= \int_{-1}^{1} \begin{bmatrix} 1 & \xi & \cdots & \xi^{p} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{p+1} \end{bmatrix} \mathrm{d}\xi$$

$$= \underbrace{\begin{bmatrix} 2 & 0 & \frac{2}{3} & \cdots \end{bmatrix}}_{\hat{\boldsymbol{P}}} \boldsymbol{\alpha}$$

$$= \hat{\boldsymbol{P}}\boldsymbol{\alpha}$$

$$\boldsymbol{M}^{\mathrm{T}}\boldsymbol{W} = \hat{\boldsymbol{P}}^{\mathrm{T}}$$

A system of nonlinear algebraic equations for the unknown matrices $\boldsymbol{M}$ and $\boldsymbol{W}$

$\boldsymbol{M}$ — n parameters
$\boldsymbol{W}$ — n parameters
$\boldsymbol{f}(\xi)$ — polynomial of order p defined by p+1 parameters

- $p+1 = 2n$: An n-point Gauss formula can integrate a ($2n-1$)-order polynomial exactly
- The number of integration points need to integrate a polynomial of order p exactly is given by $n \geq (p+1)/2$
- In one dimension the Gauss quadrature formulas are optimal, but in multi-dimensional cases, the Gauss quadrature rules are no longer necessarily optimal

3.3 Gauss Quadrature

Position of Gauss points and corresponding weights

$$\sum_{i=1}^{n} W_i = ?$$

n_{gp}	Location, ξ_i	Weights, W_i	
1	0.0	2.0	
2	$\pm 1/\sqrt{3} = \pm 0.5773502692$	1.0	
3	± 0.7745966692 $\pm\sqrt{3/5}$	0.555 555 5556	$5/9$
	0.0	0.888 888 8889	$8/9$
4	± 0.8611363116	0.347 854 8451	
	± 0.3399810436	0.652 145 1549	
5	± 0.9061798459	0.236 926 8851	
	± 0.5384693101	0.478 628 6705	
	0.0	0.568 888 8889	
6	± 0.9324695142	0.171 324 4924	
	± 0.6612093865	0.360 761 5730	
	± 0.2386191861	0.467 913 9346	

3.3 Gauss Quadrature

```
% get gauss points in the parent element domain [-1, 1]
% and the corresponding weights
function [w,gp] = gauss(ngp)

if ngp == 1
   gp = 0;
   w = 2;
elseif ngp == 2
   gp = [-0.57735027, 0.57735027];
   w = [1, 1];
elseif ngp == 3
   gp = [-0.7745966692, 0.7745966692, 0.0];
   w = [0.5555555556, 0.5555555556, 0.8888888889];
end
```