Proving Markov's Inequality

Markov's inequality: for any nonnegative random variable X, and for any $t > 0$,

$$\mathbf{Pr}[X \geq t] \leq \frac{\mathbf{E}[X]}{t}.$$

Let's say that X can take values $x_1 < x_2 < \ldots < x_j = t < \ldots < x_n$. First, prove the inequality.

Proof:

$$\mathbf{E}[X] = \sum_{i=1}^{n} x_i * \mathbf{Pr}[X = x_i] \geq \sum_{i=j}^{n} x_i * \mathbf{Pr}[X = x_i] \geq \sum_{i=j}^{n} t * \mathbf{Pr}[X = x_i]$$

Then, show the equivalence to the sometimes more useful form, for $s > 0$.

$$\mathbf{Pr}[X \geq s \cdot \mathbf{E}[X]\,] \leq \frac{1}{s}.$$

Proof: let $t = s\mathbf{E}[X]$.

Finally, invent a random variable and a distribution such that,

$$\mathbf{Pr}[X \geq 10 \cdot \mathbf{E}[X]\,] = \frac{1}{10}.$$

Answer: Consider Bernoulli(1, 1/10). So, getting 1 w.p 1/10 and 0 w.p 9/10. This importantly shows that Markov's inequality is tight, because we could replace 10 with t and use Bernoulli(1, 1/t), at least with $t \geq 1$.

Proving the Chebyshev Inequality.

1. For any random variable X and scalars $t, a \in \mathbb{R}$ with $t > 0$, convince yourself that

$$\mathbf{Pr}[\,|X - a| \geq t] = \mathbf{Pr}[\,(X - a)^2 \geq t^2]$$

2. Use the second form of Markov's inequality and (1) to prove *Chebyshev's Inequality*: for any random variable X with $\mathbf{E}[X] = \mu$ and var$(X) = c^2$, and any scalar $t > 0$,

$$\mathbf{Pr}[\,|X - \mu| \geq tc] \leq \frac{1}{t^2}.$$

(*Hint:* To use Markov, don't forget the definition of variance... var$(X) = \mathbf{E}[(X - \mu)^2]$.)