Markov's Inequality Proof and Chebyshev's Inequality Proof

Theorem 2 (**Markov's inequality**) *Let X be a non-negative random variable and suppose that $\mathbb{E}(X)$ exists. For any $t > 0$,*

$$\mathbb{P}(X > t) \leq \frac{\mathbb{E}(X)}{t}. \tag{1}$$

Proof. Since $X > 0$,

$$\begin{aligned}\mathbb{E}(X) &= \int_0^\infty x\,p(x)dx = \int_0^t x\,p(x)dx + \int_t^\infty xp(x)dx \\ &\geq \boxed{\int_t^\infty x\,p(x)dx} \geq t \underline{\int_t^\infty p(x)dx} = t\,\underline{\mathbb{P}(X > t)}.\end{aligned}$$

$\square$

Theorem 3 (**Chebyshev's inequality**) *Let $\mu = \mathbb{E}(X)$ and $\sigma^2 = \mathsf{Var}(X)$. Then,*

$$\boxed{\mathbb{P}(|X-\mu| \geq t) \leq \frac{\sigma^2}{t^2}} \quad \text{and} \quad \mathbb{P}(|Z| \geq k) \leq \frac{1}{k^2} \tag{2}$$

where $Z = (X-\mu)/\sigma$. In particular, $\mathbb{P}(|Z| > 2) \leq 1/4$ and $\mathbb{P}(|Z| > 3) \leq 1/9$.

Proof. We use Markov's inequality to conclude that

$$\mathbb{P}(|X-\mu| \geq t) = \mathbb{P}(|X-\mu|^2 \geq t^2) \leq \frac{\mathbb{E}(X-\mu)^2}{t^2} = \frac{\sigma^2}{t^2}.$$

The second part follows by setting $t = k\sigma$. $\square$

If $X_1, \ldots, X_n \sim \text{Bernoulli}(p)$ then and $\overline{X}_n = n^{-1}\sum_{i=1}^n X_i$ Then, $\mathsf{Var}(\overline{X}_n) = \mathsf{Var}(X_1)/n = p(1-p)/n$ and

$$\mathbb{P}(|\overline{X}_n - p| > \epsilon) \leq \frac{\mathsf{Var}(\overline{X}_n)}{\epsilon^2} = \frac{p(1-p)}{n\epsilon^2} \leq \frac{1}{4n\epsilon^2}$$

since $p(1-p) \leq \frac{1}{4}$ for all p.