Homework 1

Li Ziniu

November 11, 2018

1 Problem

Block compress sensing aims to solve such a problem:

$$\begin{gathered}\min_{x \in R^n} ||x||_{l_2,I} \\ \text{subject to} \, ||y - \Phi x||_{l_2} \leq \epsilon\end{gathered} \tag{1}$$

where, block l_2/l_1 norm is defined as:

$$||x||_{l_2,I} = \sum_{|i|=k} ||x[i]||_{l_2} \tag{2}$$

In addition, Φ satisfies block-RIP condititon.

Definition 1 (block-RIP) *Restricted isometry constant δ_k is the smallest number s.t.*

$$(1-\delta_k)||x||_{l_2}^2 \leq ||\Phi x||_{l_2}^2 \leq (1+\delta_k)||x||_{l_2}^2$$

We want to prove the following conclusion.

Theorem 1 (Noisy recovery) *With $\delta_{2k} < \sqrt{2} - 1$, the optimal solution x^* to (1) is close to a feasible solution x :*

$$\begin{gathered}||x^* - x||_{l_2} \leq \frac{2[1-(1-\sqrt{2})\delta_{2k}]}{1-(1+\sqrt{2})\delta_{2k}} e_0 + \frac{4\sqrt{1+\delta_{2k}}}{1-(1+\sqrt{2})\delta_{2k}} \epsilon \\ e_0 = k^{-1/2} ||x - x_{T_0}||_{l_2,I}\end{gathered} \tag{3}$$

In next section, we will probe theorem 1.

2 Prove

First, we will assume that there is a gap $h = x^* - x$ between x^* and x. Our goal is to find the bound of $||h||_{l_2}$.
The key trick is to split h into several disjoint group with size of k as the descending order measured by the magnitude of x. It is easy to show that

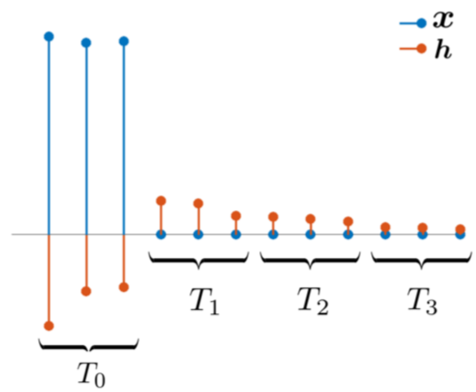

Figure 1: $h = h_{T_0} + h_{T_1} + \cdots$

$$\begin{aligned} ||h||_{l_2} &= ||h_{T_0 \cup T_1} + h_{(T_0 \cup T_1)^c}||_{l_2} \\ &\leq ||h_{T_0 \cup T_1}||_{l_2} + ||h_{(T_0 \cup T_1)^c}||_{l_2} \end{aligned} \tag{4}$$

In the following, we will show:
Step1: $||h_{(T_0 \cup T_1)^c}||_{l_2}$ is bounded by $||h_{T_0 \cup T_1}||_{l_2}$
Step2: the exact bound of $||h_{T_0 \cup T_1}||_{l_2}$.

2.1 Step1

For the first step, we note that for each $j \geq 2$,

$$||h_{T_j}||_{l_2} \leq k^{1/2} ||h_{T_j}||_\infty \leq k^{-1/2} ||h_{T_{j-1}}||_{l_2,I}$$

and thus,

$$\sum_{j \geq 2} ||h_{T_j}||_{l_2} \leq k^{-1/2} (||h_{T_1}||_{l_2,I} + ||h_{T_1}||_{l_2,I} + \cdots) = k^{-1/2} ||h_{T_0{}^c}||_{l_2,I} \tag{5}$$

In practice, this gives the estimate for

$$||h_{(T_0 \cup T_1)^c}||_{l_2} = ||\sum_{j \geq 2} h_{T_j}||_{l_2} \leq \sum_{j \geq 2} ||h_{T_j}||_{l_2} \leq k^{-1/2} ||h_{T_0^c}||_{l_2,I} \tag{6}$$

To complete step1, we need to find the relationship between $||h_{T_0^c}||_{l_2,I}$ and $||h_{T_0 \cup T_1}||_{l_2,I}$. Since x^* is the optimal solution, it is easy to see that

$$\begin{aligned} ||x||_{l_2,I} &\geq ||x + h||_{l_2,I} \\ &\geq ||x_{T_0} + h_{T_0} + x_{T_0^c} + h_{T_0^c}||_{l_2,I} \\ &\geq ||x_{T_0}||_{l_2,I} - ||h_{T_0}||_{l_2,I} + ||h_{T_0^c}||_{l_2,I} - ||x_{T_0^c}||_{l_2,I} \end{aligned} \tag{7}$$

Remove the $||x_{T_0^c}||_{l_2,I}$ to the left hand,

$$\begin{aligned} ||x||_{l_2,I} - ||x_{T_0}||_{l_2,I} &\geq -||h_{T_0}||_{l_2,I} + ||h_{T_0^c}||_{l_2,I} - ||x_{T_0^c}||_{l_2,I} \\ ||x_{T_0^c}||_{l_2,I} &\geq -||h_{T_0}||_{l_2,I} + ||h_{T_0^c}||_{l_2,I} - ||x_{T_0^c}||_{l_2,I} \end{aligned} \tag{8}$$

Thus,

$$||h_{T_0^c}||_{l_2,I} \leq ||h_{T_0}||_{l_2,I} + 2||h_{T_0^c}||_{l_2,I} \tag{9}$$

Combine (6) and (9), using Cauchy-Schwarz inequality to bound $||h_{T_0}||_{l_2,I} \leq k^{1/2}||h_{T_0}||_{l_2}$,

$$\begin{aligned} ||h_{(T_0 \cup T_1)^c}||_{l_2} &\leq k^{-1/2}||h_{T_0}||_{l_2,I} + 2k^{-1/2}||h_{T_0^c}||_{l_2,I} \\ &\leq ||h_{T_0}||_{l_2} + 2||x - x_{T_0}||_{l_2,I} \\ &\leq ||h_{T_0 \cup T_1}||_{l_2} + 2e_0 \end{aligned} \tag{10}$$

where,

$$e_0 = k^{-1/2}||x - x_{I_0}||_{l_2,I}$$

Take e_0 back to (4), we can test our goal,

$$\begin{aligned} ||h||_{l_2} &\leq ||h_{T_0 \cup T_1}||_{l_2} + ||h_{(T_0 \cup T_1)^c}||_{l_2} \\ &\leq 2||h_{T_0 \cup T_1}||_{l_2} + 2e_0 \end{aligned} \tag{11}$$

Therefore, we are going to the step2 to find the bound of $||h_{T_0 \cup T_1}||_{l_2}$.

2.2 Step2

Before we start to find the bound, we firstly introduce a useful lemma.

Lemma 1 *We have*

$$|\langle \Phi x, \Phi x' \rangle| \leq \delta_{k+k'}||x||_{l_2}||x'||_{l_2}$$

for all x, x' supported on disjoint subsets T, T' with $|T| \leq k$,$|T'| \leq k'$.

To prove lemma 1, just use RIP and inner-product inequality,

$$2(1 - \delta_{k+k'})||x||_{l_2}||x'||_{l_2} \leq ||\Phi x \pm \Phi x'||_{l_2}^2 \leq 2(1 + \delta_{k+k'})||x||_{l_2}||x'||_{l_2}$$

Then,

$$|\langle \Phi x, \Phi x' \rangle| = \frac{1}{4}|||\Phi + \Phi x'||_{l_2}^2 - ||\Phi x - \Phi x'||_{l_2}^2| \leq (\delta_{k+k'})||x||_{l_2}^2||x'||_{l_2}^2$$

Now, we begin to find the bound $||h_{T_0 \cup T_1}||_{l_2}$. To do this, observe that

$$\Phi h_{T_0 \cup T_1} = \Phi h - \sum_{j \geq 2} \Phi h_{T_j}$$

Then,

$$||\Phi h_{T_0\cup T_1}||_{l_2}^2 = \langle \Phi h_{T_0\cup T_1}, \Phi h\rangle - \langle \Phi h_{T_0\cup T_1}, \sum_{j\geq 2}\Phi h_{T_j}\rangle \tag{12}$$

Look at the first term,

$$\langle \Phi h_{T_0\cup T_1}, \Phi h\rangle \leq ||\Phi h_{T_0\cup T_1}||_{l_2}||\Phi h||_{l_2} \tag{13}$$

Accord to the equality restriction,

$$||\Phi h||_{l_2} = ||\Phi(x^* - x)||_{l_2} \leq ||\Phi x^* - y||_{l_2} + ||y - \Phi x||_{l_2} \leq 2\epsilon \tag{14}$$

Apply (14) and RIP to (13)

$$\langle \Phi h_{T_0\cup T_1}, \Phi h\rangle \leq 2\epsilon\sqrt{1+\delta_{2k}}||h_{T_0\cup T_1}||_{l_2} \tag{15}$$

Return back to look at the second term in (12), for $j \leq 2$, using RIP,

$$\begin{aligned} \langle \Phi h_{T_0\cup T_1}, \Phi h_{T_j}\rangle &\leq \langle \Phi_{h_{T_0}}, \Phi h_{T_j}\rangle + \langle \Phi_{h_{T_1}}, \Phi h_{T_j}\rangle \\ &\leq \delta_{2k}||h_{T_0}||_{l_2}||h_{T_j}||_{l_2} + \delta_{2k}||h_{T_1}||_{l_2}||h_{T_j}||_{l_2} \\ &\leq \delta_{2k}(||h_{T_0}||_{l_2} + ||h_{T_1}||_{l_2})||h_{T_j}||_{l_2} \\ &\leq \sqrt{2}\delta_{2k}||h_{T_0\cup T_1}||_{l_2}||h_{T_j}||_{l_2} \end{aligned} \tag{16}$$

The last inequality in (16) follows $\sqrt{a}+\sqrt{b} \leq \sqrt{2ab}$. Now apply (15) and (16) to (12),

$$||\Phi h_{T_0\cup T_1}||_{l_2}^2 \leq ||h_{T_0\cup T_1}||_{l_2}(2\epsilon\sqrt{1+\delta_{2k}} + \sqrt{2}\delta_{2k}\sum_{j\geq 2}||h_{T_j}||_{l_2})$$

Using (5) to further bound $\sum_{j\geq 2}||h_{T_j}||_{l_2}$,

$$||\Phi h_{T_0\cup T_1}||_{l_2}^2 \leq ||h_{T_0\cup T_1}||_{l_2}(2\epsilon\sqrt{1+\delta_{2k}} + \sqrt{2}\delta_{2k}k^{-1/2}||h_{T_0{}^c}||_{l_2,I}) \tag{17}$$

For another thing, the low bound of (17) can be found using RIP,

$$(1-\delta_{2k})||h_{T_0\cup T_1}||_{l_2}^2 \leq ||\Phi h_{T_0\cup T_1}||_{l_2}^2$$

Dividing $||h_{T_0\cup T_1}||_{l_2}$ on the two hand,

$$||h_{T_0\cup T_1}||_{l_2} \leq \frac{2\epsilon\sqrt{1+\delta_{2k}}}{1-\delta_{2k}} + \frac{\sqrt{2}\delta_{2k}k^{-1/2}||h_{T_0{}^c}||_{l_2,I}}{1-\delta_{2k}} \tag{18}$$

To further bound $||h_{T_0{}^c}||_{l_2,I}$, use (9),

$$\begin{aligned} ||h_{T_0\cup T_1}||_{l_2} &\leq \frac{2\epsilon\sqrt{1+\delta_{2k}}}{1-\delta_{2k}} + \frac{\sqrt{2}\delta_{2k}k^{-1/2}(||h_{T_0}||_{l_2,I} + 2||h_{T_0{}^c}||_{l_2,I})}{1-\delta_{2k}} \\ &\leq \frac{2\epsilon\sqrt{1+\delta_{2k}}}{1-\delta_{2k}} + \frac{\sqrt{2}\delta_{2k}(||h_{T_0}||_{l_2} + 2||h_{T_0{}^c}||_{l_2})}{1-\delta_{2k}} \\ &\leq \frac{2\epsilon\sqrt{1+\delta_{2k}}}{1-\delta_{2k}} + \frac{\sqrt{2}\delta_{2k}(||h_{T_0\cup T_1}||_{l_2} + 2||h_{T_0{}^c}||_{l_2})}{1-\delta_{2k}} \\ &= \frac{2\epsilon\sqrt{1+\delta_{2k}}}{1-\delta_{2k}} + \frac{\sqrt{2}\delta_{2k}(||h_{T_0\cup T_1}||_{l_2} + 2e_0)}{1-\delta_{2k}} \end{aligned} \tag{19}$$

The second inequality in (19) is suggested by Cauchy-Schwarz inequality. By algebra, from (19), we conclude that

$$||h_{T_0\cup T_1}||_{l_2}\leq \frac{2\epsilon\sqrt{1+\delta_{2k}}}{1-(1+\sqrt{2})\delta_{2k}}+\frac{2\sqrt{2}\delta_{2k}e_0}{1-(1+\sqrt{2})\delta_{2k}} \tag{20}$$

Finally, we complete the step2. Return (20) to (11),

$$\begin{aligned} ||h||_{l_2} &\leq 2||h_{T_0\cup T_1}||_{l_2}+2e_0 \\ &\leq \frac{4\sqrt{1+\delta_{2k}}}{1-(1+\sqrt{2})\delta_{2k}}\epsilon+\frac{4\sqrt{2}\delta_{2k}}{1-(1+\sqrt{2})\delta_{2k}}e_0+2e_0 \\ &= \frac{4\sqrt{1+\delta_{2k}}}{1-(1+\sqrt{2})\delta_{2k}}\epsilon+\frac{2[1-(1-\sqrt{2})\delta_{2k}]}{1-(1+\sqrt{2})\delta_{2k}}e_0 \end{aligned} \tag{21}$$

Now we complete the prove of theorem 1. To ensure the denominator is positive, we get the premise $\delta_{2k}\leq\sqrt{2}-1$.