functional analysis study note

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Theorem 1. *X be a TVS:*

1. Every open neighborhood of 0 contains a balanced open neighborhood of 0

2. Every convex open neighborhood of 0 contains a balanced convex open neighborhood of 0.

Proof. 1. Let V be an open neighborhood of 0. The scalar multiplication is continuous at 0, thus we have some $\delta > 0$ and u is an open neighborhood of 0 such that $\lambda u \subset V$ for any $|\lambda| < \delta$.

Let $W = \bigcup_{|\lambda|<\delta} \lambda u$ is open balanced neighborhood of 0.

2. Let V be open convex neighborhood of 0. Let $A = \bigcap_{|\lambda|=1} \lambda V \subset V$.

Since every λV is convex, we have A is also convex.

Since every λV contains 0, we have $0 \in A$, and A is nonempty.

More, there exists a balanced neighborhood W of 0 such that $W \subset V$ since $\lambda^{-1} w = w$. Since $|\lambda| = 1$ so we have $\lambda^{-1} w \subset V$ iff $w \subset \lambda V$ for any $|\lambda| = 1$.

Hence $W \subset A$, so $Int(A)$ is open convex neighborhood of 0.

Let $|\lambda| = 1, 0 \le r \le 1$ then $r\lambda A = \bigcap_{|\alpha|=1} r\lambda\alpha V = \bigcap_{|\alpha|=1} r\alpha V$. Since every $r\alpha V$ is convex and contains 0, we have $r\alpha V \subset \alpha V$.

□

Remark 1. *If X is a LCTVS, then its topology is always defined by a family of semi norm $\{\mu_V\}, V \in B$, where B is a neighborhood base at 0, consisting of open convex balanced set with exists by theorem above.*

Moreover, X is T_1 iff the family of semi norm is separating.

Fact 1. *If we have a countable family of semi norm, which are separating defined by a LCTVS topology on X, then it is metrizable by*

$$d(x, y) = \sum_{n=1}^{\infty} \frac{p_n(x - y)}{2(1 + p_n(x - y))}$$

0.1 Dual Space

(X, τ) be a TVS, then $X^* = \{\psi : X \to C | \psi \text{ is continuous linear functional}\}$

Theorem 2. *We have if $\psi \neq 0$, ψ a linear functional on TVS, then TFAE:*

1. ψ is continuous

2.$\ker(\psi)$ *is closed*
3. ψ *is bounded in a neighborhood of 0.*

Proof. $1 \to 2$ $f^{-1}(\{0\})$ is closed if f continuous.

$2 \to 3$ $f \neq 0$, then there exists $x \notin \ker f$ let $K = \{x\}$, $C = \ker f$. Then there exists a balnced open neighborhood V of 0 such that $(\ker f + V) \cap (x + V) = \emptyset$, which is $\ker f \cap (x + V) = \emptyset$.

Since $f(V)$ is balanced subset of C, suppose $f(V)$ is unbounded. Then we have $f(V) = \mathbb{C}/$ Thus there exists $y \in V$ such that $f(y) = -f(x)$, i.e. $f(x + y) = 0$. Thus, $x + y \in \ker f$ which is contradiction to $\ker f \cap x + V = \emptyset$

$3 \to 1$ Suppose there exists an open neighborhood V, such that $|f(x)| < C$ for any $x \in V$. Since $f(0) = 0$, it is continuous at 0. Then $W = \frac{\varepsilon}{C} V$ is open and $|f(x)| \leq \varepsilon$ for any $x \in W$.

□

Remark 2. *X, Y are TVS, let $T : X \to Y$ linaer then T is called bounded if $T(E)$ is bounded in Y for all E is bounded in X.*

If X, Y are normed space, then $\|T(x)\| \leq c$ for any $x \in X$ which is equivalent to T continuous.

example 1. *$(X, \|\|)$ be a normed space and dim=∞. Then X with $(\sigma(X, X^*)) \to (X, \sigma(X, X^*))$ bounded and continuous.*

Theorem 3. *Let X be a TVS, f a linear functional in X that $f \neq 0$ then f is an open mapping.*

Proof. This is equivalent to show V a balanced open neighborhood of 0, then $f(V)$ is open.

For any $x \in V$ then if $|t| < 1, tx \in tV \subset V$. Thus $\{tf(x)||t| < 1\} \subset f(V)$, gives us $\bigcup_{x \in V} \{tf(x)||t| < 1\} \subset f(V)$.

Since $f \neq 0$, so at least 1 is not $\{0\}$. Thus the union is always open.

Let $x \in V$, by continuous of scalar multiplication, there exists $t < 1$ such that $t^{-1} \in V$. Hence $f(x) = \{sf(t^{-1}x)||s| \leq 1\}$ gives $f(V) = \bigcup_{x \in V} \{tf(x)||t| < 1\}$ □

Theorem 4. *Hahn Banach separation theorem:*

Let X be TVS, let A, B be non-empty, disjoint convex subsets in X:
1. let A be open, then exists $f \in X^*, c \in R$ such that

$$Re(f(a)) < C \leq Re(f(b)) \forall a \in A, b \in B$$

2. If A is compact, B is closed, and X is a LCTVS: Then exists $f \in X^*, C_1, C_2 \in R$ such that

$$Re(f(a)) < C_1 < C_2 < Re(f(b)) \forall a \in A, b \in B$$

Proof. To prove the Hahn-Banach separation theorem in a topological vector space (TVS) X, consider non-empty, disjoint convex subsets A and B.

1. For any A is open, define $C = A - B = \{a - b \mid a \in A, b \in B\}$, which is convex and open since A is open, and $0 \notin C$ as $A \cap B = \emptyset$.

By the Minkowski functional $p(x) = \inf\{t > 0 \mid x \in tC\}$, a sublinear functional with $p(x) > 0$ for $x \neq 0$.

Choose $x_0 \in C$ with $p(x_0) < 1$, and define a linear functional f_0 on the subspace spanned by x_0 by $f_0(tx_0) = t$, satisfying $f_0(y) \leq p(y)$. By the Hahn-Banach theorem of function, extend f_0 to a linear functional $f : X \to \mathbb{R}$ such that $f(x) \leq p(x)$ for all $x \in X$;

since C is open, f is continuous, so $f \in X^*$. For $a \in A$, $b \in B$, $f(a - b) \leq p(a - b) < 1$, implying $\operatorname{Re}(f(a)) < \sup_{a \in A} \operatorname{Re}(f(a)) = c \leq \operatorname{Re}(f(b))$, with strict inequality due to openness. The open neighborhood $\{x \in X \mid \operatorname{Re}(f(x)) < c\}$ contains A and excludes B.

2.For A is compact, B is closed, and X is locally convex, find a convex open neighborhood U of 0 such that $(A+U) \cap B = \emptyset$ by compactness: for each $a \in A$, there is U_a with $(a + U_a) \cap B = \emptyset$, and finite subcover yields $U = \bigcap U_{a_i}$.

Set $A' = A + U$, open and convex, disjoint from B; apply the first part to get $f \in X^*$ and c with $\operatorname{Re}(f(a')) < c \leq \operatorname{Re}(f(b))$.

Since A is compact, let $c_1 = \sup_{a \in A} \operatorname{Re}(f(a)) < c$ (as points in $A + U$ can approach but strictly less than c), and set $c_2 = c$, yielding $\operatorname{Re}(f(a)) < c_1 < c_2 < \operatorname{Re}(f(b))$, with open neighborhoods $\{x \mid \operatorname{Re}(f(x)) < c_1 + \epsilon\}$ and $\{x \mid \operatorname{Re}(f(x)) > c_2 - \epsilon\}$ separating A and B strictly. $\square$

Remark 3. *$x_0 \in X$, can be approximated by elements in M: $f \in X^*, f|_M = 0, f(x_0) = 0$*

Remark 4. *Let X be a locally convex topological vector space, and let $M \subseteq X$ be a subspace. Define the annihilator $M^\perp = \{f \in X^* : f|_M = 0\}$.*

If $x_0 \in X$ can be approximated by elements of M (i.e., $x_0 \in \overline{M}$), then for every $f \in X^*$ with $f|_M = 0$, we have $f(x_0) = 0$. Equivalently, $\overline{M} = (M^\perp)^\perp$, where $(M^\perp)^\perp = \{x \in X : f(x) = 0 \text{ for all } f \in M^\perp\}$. This follows from the Hahn-Banach theorem, which guarantees that the continuous linear functionals in X^* separate points, ensuring that the double annihilator of M captures exactly the closure of M in the topology of X.

Remark 5. *if X is a LCTVS (T_1) then X^* separate points.*

Proof. It suffices to show: for any $x \neq 0$ there exists $f \in X^*$ with $f(x) \neq 0$.

Since a T_1 TVS is Hausdorff and X is locally convex, there exists an absolutely convex (balanced and convex) open neighborhood U of 0 such that $x \notin U$.

Let $p := p_U$ be the Minkowski functional of U:

$$p(z) := \inf\{t > 0 : \; z \in tU\}.$$

Then p is a continuous seminorm with $U = \{z : p(z) < 1\}$, hence $p(x) \geq 1$. Define a linear functional on the one-dimensional subspace span$\{x\}$ by

$$\phi(\alpha x) := \alpha\, p(x) \qquad (\alpha \in \mathbb{C}).$$

For all α, we have $|\phi(\alpha x)| = p(\alpha x) \leq p(\alpha x)$, so ϕ is dominated by the sublinear map p.

By the Hahn–Banach theorem, ϕ extends to some $f \in X^*$ with $|f(z)| \leq p(z)$ for all $z \in X$. In particular $f(x) = p(x) \geq 1 \neq 0$.

For general $x \neq y$, apply the above to $x - y$ to obtain $f(x) \neq f(y)$. $\square$

Theorem 5. *Recall X is LCTVS generated by $\{p_a\}_{a\in A}$ then a net $(x_\lambda)_{\lambda\in\Lambda} \subset X$ converges to x iff $p_a(x_\lambda - x) \to 0$ for any $a \in A$*

If X, Y be LCTVS topology generatedby $\{p_a\}_{a\in A}, \{q_b\}_{b\in B}$. Let $T : X \to Y$ linear, then T is continuous iff $\forall b \in B$ there exists $a_1, a_2 \dots a_n \in A, c > 0$ such that

$$q_b(Tx) \leq c \sum_{i=1}^{n} p_{ai}(x), \forall x \in X$$

Proof. 1.Assume the inequality holds for each $b \in B$.

Since T is linear, continuity follows from continuity at 0. Consider a basic open neighborhood V of 0 in Y: $V = \bigcap_{j=1}^{m}\{y \in Y : q_{b_j}(y) < \epsilon_j\}$ for finite $b_j \in B$ and $\epsilon_j > 0$. For each j, there exist $a_1^j, \dots, a_{n_j}^j \in A$ and $c_j > 0$ such that $q_{b_j}(Tx) \leq c_j \sum_{k=1}^{n_j} p_{a_k^j}(x)$ for all $x \in X$.

To ensure $q_{b_j}(Tx) < \epsilon_j$, it suffices to have $\sum_{k=1}^{n_j} p_{a_k^j}(x) < \epsilon_j/c_j$, which holds if $p_{a_k^j}(x) < (\epsilon_j/c_j)/n_j =: \delta_k^j$ for each k.

Define $W_j = \bigcap_{k=1}^{n_j}\{x \in X : p_{a_k^j}(x) < \delta_k^j\}$. Then $T(W_j) \subset \{y : q_{b_j}(y) < \epsilon_j\}$.

Set $U = \bigcap_{j=1}^{m} W_j$, an open neighborhood of 0 in X (as a finite intersection of subbasic open sets). Thus, $T(U) \subset \bigcap_{j=1}^{m}\{y : q_{b_j}(y) < \epsilon_j\} = V$, so T is continuous.

Suppose T is continuous at 0. Fix $b \in B$ and set $V_b = \{y : q_b(y) < 1\}$. By continuity at 0 there exist $a_1, \dots, a_n$ and $\varepsilon > 0$ such that

$$U := U(a_1, \dots, a_n; \varepsilon) \subset X \quad \text{satisfies} \quad T(U) \subset V_b.$$

Let $x \in X$ and put $S := \sum_{i=1}^{n} p_{a_i}(x) \geq 0$. If $S = 0$, then for all $t > 0$ we have $tx \in U$, hence $q_b(T(tx)) < 1$, i.e. $t\, q_b(Tx) < 1$ for all $t > 0$, which forces $q_b(Tx) = 0$ and the desired estimate holds. If $S > 0$, choose $t = \varepsilon/S$. Then $tx \in U$ and $q_b(T(tx)) < 1$, so by homogeneity

$$\frac{\varepsilon}{S}\, q_b(Tx) = q_b(T(tx)) < 1 \quad \Rightarrow \quad q_b(Tx) \leq \frac{1}{\varepsilon}\, S = \frac{1}{\varepsilon}\sum_{i=1}^{n} p_{a_i}(x).$$

Setting $c := 1/\varepsilon$ yields the claimed inequality.

$\square$

example 2. *Let* $\Omega \subset \mathbb{R}^d$ *be open and let* $(K_n)_{n\in\mathbb{N}}$ *be an increasing sequence of compact sets with*

$$K_n \subset K_{n+1} \quad \textit{and} \quad \Omega = \bigcup_{n=1}^{\infty} \operatorname{Int}(K_n).$$

Define seminorms on $C(\Omega)$ *by*

$$p_n(f) := \sup_{x\in K_n} |f(x)| \qquad (n \in \mathbb{N}).$$

Then the locally convex topology on $C(\Omega)$ *generated by* $\{p_n\}_{n\in\mathbb{N}}$ *is the topology of uniform convergence on compact subsets of* Ω *(the compact-open topology). In particular,* $\big(C(\Omega), \{p_n\}\big)$ *is a Frechet space.*