functional analysis study note

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0.1 Norms

Definition 1. *If A is an absorbing convex set, we define the Minkowski on A is :*

$\mu_A(x) = \inf\{t > 0 | x \in tA\}$

Note that $0 \leq \mu_A(x) \leq \infty$ because A is absorbing.

Theorem 1. *Let X be a vector space and $p : X \to R$ a semi norm, then:*

1. $p(0) = 0$

2. $|p(x) - p(y)| \leq p(x - y)$

3. $p(x) \geq 0$

4. $\{x \in X | p(x) = 0\}$ *is a subspace*

5. $B = \{x \in X | p(x) < 1\}$ *is a convex balanced absorbing set* $\mu(B) = p$.

Proof. (1) $p(0) = p(0 \cdot x) = |0|\, p(x) = 0$.

(2) By subadditivity, $p(x) \leq p(y) + p(x - y)$, hence $p(x) - p(y) \leq p(x - y)$. Exchanging x, y gives $p(y) - p(x) \leq p(x - y)$. Combine to get $|p(x) - p(y)| \leq p(x - y)$.

(3) Set $y = 0$ in (2): $|p(x) - p(0)| \leq p(x)$. Using (1) yields $|p(x)| \leq p(x)$, hence $p(x) \geq 0$.

(4) If $p(x) = p(y) = 0$, then $p(x + y) \leq p(x) + p(y) = 0$, so $p(x + y) = 0$. For $\alpha \in \mathbb{F}$, $p(\alpha x) = |\alpha| p(x) = 0$. Thus N is closed under addition and scalar multiplication.

(5) Write $B = \{x : \ p(x) < 1\}$. Convexity: For $x, y \in B$ and $0 \leq \theta \leq 1$,

$$p(\theta x + (1 - \theta)y) \leq \theta p(x) + (1 - \theta)p(y) < \theta + (1 - \theta) = 1,$$

so $\theta x + (1 - \theta)y \in B$.

Balanced: If $|\alpha| \leq 1$ and $x \in B$, then $p(\alpha x) = |\alpha| p(x) < 1$, so $\alpha x \in B$.

Absorbing: For any $x \in X$ and any $t > p(x)$, we have $p(x/t) = p(x)/t < 1$, hence $x/t \in B$, i.e., $x \in tB$.

For the Minkowski functional $\mu_B(x) := \inf\{t > 0 : \ x \in tB\}$, note that

$$x \in tB \iff \tfrac{x}{t} \in B \iff p(x/t) < 1 \iff p(x) < t.$$

Thus $\{t > 0 : \ x \in tB\} = (p(x), \infty)$ when $p(x) > 0$ (and $(0, \infty)$ when $p(x) = 0$), so $\mu_B(x) = p(x)$. □

Theorem 2. *Let X be a vector space. Let A be a convex absorbing set in X, then*

1. $\mu_A(x+y) \leq \mu_A(x) + \mu_A(y)$
2. $\mu_A(tx) = t\mu_A(x)$ *for any* $t \geq 0$
3. μ_A *is a seminorm if A is balanced.*
4. *If $B = \{x \in X | \mu_A(x) < 1\}$ and $C = \{x \in X | \mu_A(x) \leq 1\}$, then $B \subset A \subset C$, and $\mu_A = \mu_B = \mu_C$.*

Proof. 1. Let $S > \mu_A(x)$, $t > \mu_A(y)$, then we have $\frac{1}{s+t}(x+y) = \frac{s}{s+t}\frac{1}{s}x + \frac{t}{s+t}\frac{1}{t}y$

Thus, $\mu_A(x+y) \leq s+t$ so $\mu_A(x+y) \leq \mu_A(x) + \mu_A(y)$

2. Consider $H_A(x) = \{t > 0 | x \in tA\}$

We have $H_A(tx) = tH_A(x)$ finish the proof.

3.Let $\lambda \in \mathbb{C}$, $\mu_A(\lambda x) = \mu_A(|\lambda| e^{it} x) = |\lambda| \mu_A(x)$

More, we have $e^{it}A \subset A$ and $e^{-it}A \subset A$ gives us $e^{it}A = A$.

4. $H_A(x) = \{t > 0 | x \in tA\}$ is a half line with left endpoint $\mu_A(x)$. Thus, $x \in B$ then $\mu_A(x) < 1$, hence $1 \in H_A(x)$ so that $x \in A$, then $\mu_A(x) \leq 1$.

$B \subset A \subset C$ hence $H_B(x) \subset H_A(x) \subset H_C(x)$. Then, $\mu_C(x) \leq \mu_A(x) \leq \mu_B(x)$.

The opposite direction: $\mu_C(x) < s < t$ hence $s^{-1}x \in C$.

We have $\mu_A(s^{-1}x) \leq 1$ gives us

$$\mu_A(t^{-1}x) = \mu_A(t^{-1}ss^{-1}x) = \frac{s}{t}\mu_A(s^{-1}x) \leq \frac{s}{t} < 1$$

Thus, $t^{-1}x \in B$ gives us $\mu_B(x) \leq t$, then $\mu_B(x) \leq \mu_C(x)$. $\square$

Theorem 3. *Let X be a TVS which is T_1*

1. *let V be an open convex balanced neighborhood of 0, then $V = \{x \in X | \mu_V(x) < 1\}$*
2. *Let B be a neighborhood base at 0, consisting of open convex balanced sets, then $\{\mu_V | V \in B\}$ is a separating family of continuous semi-norm on X*

Proof. 1. Let $x \in V$, since the scalar multiplication is continuous, there exists $t < 1$ such that $t^{-1}x \in V$ i.e. $\mu_V(x) \leq t \leq 1$ thus we have $x \in \{y \in X | \mu_V(y) < 1\}$.

Since V is absorbing and convex, it gives us $\{y \in X | \mu_V(y) < 1\} \subset V$ follows (4) of previous theorem.

Thus, the double inclusion gives us the equation.

2.Let V be a open convex balanced neighborhood of 0, then μ_V is a seminorm by Minkowski;

Recall μ_V is continuous: for any net $(x_\lambda)_{\lambda \in \Lambda} \in X$, fix V, let $\varepsilon > 0$, there exists $\lambda_0 \in \Lambda$ such that for any $\lambda \geq \lambda_0$, we have $x - x_\lambda \in \varepsilon V$.

Consider $|\mu_V(x\lambda) - \mu_V(x)| \leq \mu_V(x - x\lambda) = \varepsilon \mu_V(\varepsilon^{-1}(x - x\lambda)) < 1$. Thus μ_V is continuous.

Recall separating family, $\{\mu_V | V \in B\}$ is separating: Let $x, y \in X$ and $x \neq y$, then $x - y \neq 0$. Since X is T_1 and exists neighborhood U of 0 such that $x - y \notin U$. B is a neighborhood basis at 0, there exists $V \in B$, $V \subset U$ and $x - y \notin V$.

Thus, we have $\mu_V(x-y) \geq 1$. $\square$

Theorem 4. *Let X be a vector space, Let P be a separating family of seminorm on X. Consider $V(p,n) = \{x \in X | p(x) < 1/n\}, n \in N, p \in P$. Let B be a collection of finite intersection of sets $V(p,n)$*

Then, B is a convex, balanced neighborhood at 0 for a topology τ on X which means X is a LCTVS is T_1 such that

1. every $p \in P$ is continuous

2. for any $E \subset X$ is bounded iff $p \in P$ is bounded on E

Proof. Define $A \subset X$ be open if $A \neq \emptyset$ or $A = X$ or A is union of finite intersection of translates of sets in $B(x + B)$

Note, B is consisting convex balanced sets and they are open by definition of τ.

WTS: τ is T_1 : $\{x\}$ is closed for any x (equivalent to show $\{0\}$ is closed).

Since P is separating, there exists $p \in P$ such that $p(x) > 0$ so there exists $n_0 \in N$ such that $np(x) > 1$ for any $n > n_0$.

Thus, $x \notin V(p,n)$ for this p and n. We have $0 \notin x - V(p,n) = x + V(p,n)$. Thus, $x \notin \{0\}$ for any $x \neq 0$. Hence $\{0\}$ is closed.

Then, we have (X, τ) is T_1.

(a) Next, addition is continuous on X, it is enough to show addition is continuous at 0. Then, let u be a neighborhood of 0, by the definition of τ, there exists some $p_i \in P$ such that $V(p_1, n_1) \cap V(p_2, n_2) \dots V(p_k, n_k) \subset u$.

Let $V = V(p_1, 2n_1) \cap V(p_2, 2n_2) \dots V(p_k, 2n_k)$, then $V + V \subset u$.

1. every $p \in P$ is continuous.

(b) Scalar multiplication is continuous. Let $x \in X, \alpha \in \mathbb{C}$, let u, v be the same u and v defined above.

$S > p_i(x)2n_i$ for any $1 \leq i \leq k$.

Then, we have $x \in SV(p_i, 2n_i)$ for all i.

Thus, $x \in SV$. For $0 < t \leq \frac{s}{1+|\alpha|s}$, we let $t = \frac{1}{n}$ as $n \to \infty, tV$ is open set.

Then for $y \in x + tV$ in a neighborhood of x, there exists β such $|\beta - x| < \frac{1}{s} = \delta$, then for any $\beta y - \alpha x = \beta(y - x) + (\beta - \alpha)x$.

Since $\beta \in V, \beta(y - x) \in V$, $(\beta - \alpha)x \in |\beta - \alpha|sV$ is balanced.

By $||\beta| - |\alpha|| \leq |\beta - \alpha| < \frac{1}{s}$, we have $|\beta| < \frac{1}{s} + |\alpha|$, gives us $|\beta|t < s$. Thus, $|\beta|tV \subset V$.

Also, $|\beta - \alpha|s < 1$ gives us $|\beta - \alpha|sV \subset V$ since balanced.

Thus, we have $\beta y - \alpha x \subset V + V \subset U$.

2.$E \subset X$ is bounded iff every $p \in P$ is bounded on E.

(a) Suppose we have E is bounded. Let $V = V(p, 1)$ for any $p \in P$. Then, $V(p, 1)$ is a neighborhood of 0 and exists $k > 0$ such that $E \subset kV(p, 1)$.

Thus, $p(x) < k$ for any $x \in E$. Since p is arbitrary chosen, we have every $p \in P$ is bounded on E.

(b) Let $U \in \mathcal{B}$ be arbitrary, say

$$U = \bigcap_{i=1}^{m} U_{p_i,n_i} \quad \text{with} \quad U_{p_i,n_i} := \{x \in X : p_i(x) < 1/n_i\},$$

where $p_i \in P$ and $n_i \in \mathbb{N}$. Set $M_i := \sup_{x \in E} p_i(x) < \infty$ (finite by assumption), and choose

$$t > \max_{1 \leq i \leq m} n_i M_i.$$

Then for every $x \in E$ and each i we have $p_i(x) \leq M_i < t/n_i$, hence $x \in t\,U_{p_i,n_i} = \{y \in X : p_i(y) < t/n_i\}$. Therefore $x \in \bigcap_{i=1}^{m} t\,U_{p_i,n_i} = tU$. Since U was arbitrary, E is bounded.

□

Remark 1. *Let (x, τ) be a LCTVS, suppose B is a neighborhood base at 0, consisting of open convex balanced sets. Then $\{\mu_V\}, V \in B$ is a family of continuous semi norm and separating iff X is T_1.*

More, we can use this family to define a LCTVS topology $\hat{\tau}$ on X that the topology is the same: $\tau = \hat{\tau}$.

Since

1. every $p \in P$ is continuous in τ gives us all set $V(p, n)$ are open. Thus $\hat{\tau} \subset \tau$

2. If $V \in B$, $p = \mu_V$, then $V(p, 1) = \{x \in X | \mu_V(x) < 1\}$. Thus $\tau \subset \hat{\tau}$

Fact 1. *Every LCTVS actually has a neighborhood at 0 consisting of open convex balanced sets. Thus, its topology always define a semi-norm.*

Recall a theorem: X is a TVS, K is compact and nonempty, C is closed, and $K \cap C = \emptyset$, then there exists open neighborhood V at 0 such that $(K + V) \cap (C + V) = \emptyset$. Hence $\overline{(K + V)} \cap (C + V) = \emptyset$.

corollary 1. *Let X be a TVS, B is a neighborhood base at 0, then each $V \in B$ contain a $u \in B$ such that $\overline{u} \in V$.*

Proof. let $K = \{0\}$, $C = V^c$ is closed. Then there exists $u \in B$ open such that $\overline{(\{0\} + u)} \cap (V^c + u) = \emptyset$

Thus, $\overline{u} \cap V^c = \emptyset$, which is $\overline{u} \subset V$. □

Theorem 5. *Let X be a TVS then*

1. if $A \subset X$, then $\overline{A} = \bigcap_{V \text{ is open neighborhood at } 0} (A + V)$

2. $A \subset X$, $B \subset X$ then $\overline{A} + \overline{B} \subset \overline{(A + B)}$

3.Let $Y \subset X$ is a subspace, then $\overline{Y}$ is also a subspace

4.If $C \subset X$ is convex, then $\overline{C}$ and $Int(C)$ are convex.

5. If $B \subset X$ is balanced, then $\overline{B}$ is balanced. $Int(B)$ is balanced if $0 \in Int(B)$

6. if E is balanced, then $\overline{E}$ is balanced.

Proof. Let X be a topological vector space (TVS).

(1) We show $\overline{A} = \bigcap\{A + V : \; V \text{ is an open 0-neighborhood}\}$.

If $x \in \overline{A}$, then for every open neighborhood U of x we have $U \cap A \neq \emptyset$. Every neighborhood of x is of the form $x + V$ with V an open 0-neighborhood, hence $(x + V) \cap A \neq \emptyset$. Thus there exist $a \in A$ and $v \in V$ with $x + v = a$, i.e. $x = a - v \in A + (-V)$. As $\{-V : \; V\}$ runs over the same collection of 0-neighborhoods, we get $x \in \bigcap_V (A + V)$.

The opposite direction: suppose $x \notin \overline{A}$. Then there exists an open neighborhood U of x with $U \cap A = \emptyset$. Write $U = x + V$ with V an open 0-neighborhood. If $x \in A + V$, say $x = a + v$ with $a \in A$, $v \in V$, then $a = x - v \in x + (-V)$. Taking a symmetric open 0-neighborhood W with $W \subset V \cap (-V)$ (which exists in a TVS), we would get $a \in x + W \subset U$, contradicting $U \cap A = \emptyset$. Hence $x \notin A + V$. Therefore $x \notin \bigcap_V (A + V)$.

(2) Let $a \in \overline{A}$ and $b \in \overline{B}$. We will show $a + b \in \overline{A + B}$ by neighborhood chasing.

Let W be an arbitrary open neighborhood of $a + b$. By continuity of addition $+ : X \times X \to X$ at (a, b), there exist open 0-neighborhoods U, V in X such that

$$(a + U) + (b + V) \subset W.$$

Because $a \in \overline{A}$ and $b \in \overline{B}$, we have

$$A \cap (a + U) \neq \emptyset \quad \text{and} \quad B \cap (b + V) \neq \emptyset.$$

Define

$$\mathcal{V} := \big(A \cap (a + U)\big) + \big(B \cap (b + V)\big).$$

Then $\mathcal{V} \subset (a + U) + (b + V) \subset W$ and also $\mathcal{V} \subset A + B$. Moreover $\mathcal{V} \neq \emptyset$ (pick $a' \in A \cap (a + U)$ and $b' \in B \cap (b + V)$, then $a' + b' \in \mathcal{V}$). Hence $W \cap (A + B) \neq \emptyset$.

Since W was an arbitrary neighborhood of $a + b$, it follows that $a + b \in \overline{A + B}$. As $a \in \overline{A}$ and $b \in \overline{B}$ were arbitrary, we conclude

$$\overline{A} + \overline{B} \subset \overline{A + B}.$$

(3) $\alpha\overline{Y} + \beta\overline{Y} \subset (\alpha + \beta)\overline{Y}$ for $\alpha, \beta \neq 0$. Thus, $\alpha\overline{Y} = \overline{\alpha Y}, \beta\overline{Y} = \overline{\beta Y}$, so $(\alpha + \beta)\overline{Y} = \overline{(\alpha + \beta)Y}$

(4) Let C be convex. For $x, y \in \overline{C}$ and $t \in [0, 1]$, choose nets (or sequences) $x_i, y_i \in C$ with $x_i \to x$, $y_i \to y$. By convexity, $tx_i + (1 - t)y_i \in C$, and by continuity of the operations, $tx_i + (1 - t)y_i \to tx + (1 - t)y$, so $tx + (1 - t)y \in \overline{C}$. Thus $\overline{C}$ is convex.

For the interior, let $x, y \in \mathrm{Int}(C)$ and $t \in [0, 1]$. There exist open 0-neighborhoods V_x, V_y with $x + V_x \subset C$ and $y + V_y \subset C$. Set $V := V_x \cap V_y$, which is a 0-neighborhood. For any $v \in V$, $x + v \in C$ and $y + v \in C$, hence

$$tx + (1 - t)y + v = t(x + v) + (1 - t)(y + v) \in C$$

by convexity of C. Thus $tx+(1-t)y+V \subset C$, proving $tx+(1-t)y \in \mathrm{Int}(C)$. So $\mathrm{Int}(C)$ is convex.

(5) If B is balanced and $|\lambda| \leq 1$, the scalar map $m_\lambda : x \mapsto \lambda x$ is continuous and $m_\lambda(B) \subset B$. Hence

$$\lambda\,\overline{B} = m_\lambda(\overline{B}) \subset \overline{m_\lambda(B)} \subset \overline{B},$$

so $\overline{B}$ is balanced.

If moreover $x \in \mathrm{Int}(B)$ and $|\lambda| \leq 1$, then there exists a 0-neighborhood V with $x + V \subset B$. Since m_λ is continuous and $m_\lambda(B) \subset B$, we have

$$\lambda x + \lambda V = m_\lambda(x + V) \subset m_\lambda(B) \subset B.$$

As λV is a 0-neighborhood, this shows $\lambda x \in \mathrm{Int}(B)$. Thus $\mathrm{Int}(B)$ is balanced (the argument does not actually require $0 \in \mathrm{Int}(B)$, but this assumption guarantees $\mathrm{Int}(B) \neq \emptyset$).

(6) This is the first statement of (5): if E is balanced, then the same argument with m_λ gives $\lambda\,\overline{E} \subset \overline{E}$ for all $|\lambda| \leq 1$. Hence $\overline{E}$ is balanced.

□