functional analysis study note

Zehao Li

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0.1 Weak Topology

Definition 1. *Schwartz Space:*

$$S(R) = \{f \in C^\infty(R^n) | \|f\|_{(N,d)} < \infty\} N \in N, \alpha \in N^n$$

where $\|f\|_{(N,d)} = \sup_{x \in R^n} (1 + |x|)^N |\partial^x f(x)|$

f consists of C^∞ functions on R^n all derivative vanish at ∞ faster than any polynomials.

Definition 2. *Schwartz space: $S(R) = \{f \in C^\infty(R^n) | \|f\|_{(n,\alpha)} < \infty\}$, $n \in N, \alpha \in \mathbb{N}^n$ where $\|f\|_{(n,\alpha)} = \sup_{x \in R^n} (1 + x)^N |\partial^\alpha f(x)|$.*

f consists of C^∞ functions on R^n. All derivative vanish at ∞ faster than any polynomials.

Definition 3. *Weak and strong operator topologies.*

Let X, Y be Banach space. Consider $B(X, Y)$ the topology generated by semi norm $p_{\psi,x}(T) = \|\psi(Tx)\|$

Define a LCTVS topology on $B(X, Y)$ called the weak operator topology.

Definition 4. *a Net $(T_\lambda)_{\lambda \in \Lambda} \subset B(X, Y)$ converge to $T \in B(X, Y)$ in weak operator topology if and only if $|\psi(T_\lambda x) - \psi(T_x)| \to 0 \forall x \in X, \psi \in Y^*$*

$(B(X, Y), \{p_{\psi,x}\})$ is a LCTVS.

Definition 5. *Strong operator topology is the LCTVS topology on $B(X, Y)$ generated by the semi-norm $p_x(T) = \|T_x\|, x \in X$ a net $(T_\lambda) \subset B(X, Y) \to$ (strong operator converge) $\to T \in B(X, Y)$ if and only if $\|T_\lambda x - Tx\| \to 0 \forall x \in X$.*

Definition 6. *X be a vector space, Let Y be a space of linear functionals on X and suppose Y separates point of X. Then, define a seminorm: $p_\psi(x) = |\psi(x)|, \psi \in Y, x \in X$.*

The LCTVS topology T1 on X define by $\{p_\psi\}$ is called the weak topology on X induced by Y, denoted by $\sigma(X, Y)$

Recall the sets $V(x_0, \psi_1 \dots \psi_n, \varepsilon) = \{x \in X | |\psi_j(x - x_0)| < \varepsilon\}, \psi_j \in T, \varepsilon > 0, n \in N$ form a neighborhood base x_0 for the $\sigma(X, Y)$ topology.

Since Y is separating $\sigma(X, Y)$ is T_1 gives Hausdorff T_2.

More, each $\psi \in Y$ is continuous in $\sigma(X, Y)$ topology $V(x_0, \psi_1 \dots \psi_n, \varepsilon)$ is a neighborhood of 0.

If (X, τ) is a TVS such that $\psi \in Y$ is continuous in τ, then $\sigma(X, Y) \subset \tau$.

Thus, $\sigma(X, Y)$ is the weakest topology on X which means every $\psi \in Y$ is continuous.

$X_\lambda \to X$ in a $\sigma(X, Y)$ if and only if $\psi(x_\lambda - x) \to 0 \forall \psi \in Y$

If $Y_1 \subset Y_2$ then $\sigma(X, Y_1) \subset \sigma(X, Y_2)$.

Special case: Let (X, τ) be a TVS and X^* separates the points of X. Then, $\sigma(X, X^*)$ is called the weak topology on X. More, $\sigma(X, X^*) \subset \tau$

Remark 1. *One can characterize all TVS on X that have X^* on continuous dual.*

Theorem 1. *Let (X, τ) be a LCTVS that is T_1. Let $C \subset X$ be a convex subset. Then $\overline{C} = \overline{C}^W$ (weak topology $C^W = C^\sigma(X, X^*)$)*

Proof. Write $\sigma := \sigma(X, X^*)$ for brevity. Since σ is weaker than τ, closures satisfy

$$\overline{C}^{\sigma} \supset \overline{C}^{\tau}.$$

Thus it suffices to prove the reverse inclusion $\overline{C}^{\sigma} \subset \overline{C}^{\tau}$.

Assume, toward a contradiction, that there exists $x_0 \in X$ with $x_0 \notin \overline{C}^{\tau}$ but $x_0 \in \overline{C}^{\sigma}$. Since τ is Hausdorff and X is locally convex, $\overline{C}^{\tau}$ is a closed convex set not containing x_0.

By the Hahn–Banach separation theorem (point vs. closed convex set in a Hausdorff LCTVS), there exists a continuous linear functional $f \in X^*$ and a real number α such that

$$\sup_{x \in C} \operatorname{Re} f(x) \le \alpha < \operatorname{Re} f(x_0). \tag{1}$$

(For complex scalars we take the real part.)

Consider the set

$$U := \{x \in X : \operatorname{Re} f(x) < \alpha\}.$$

Because $f \in X^*$ is (by definition) continuous for the weak topology, the map $x \mapsto \operatorname{Re} f(x)$ is σ-continuous, hence U is σ-open.

By (1), we have $x_0 \in U$ and $U \cap C = \varnothing$ (since $\operatorname{Re} f(x) \le \alpha$ for all $x \in C$). Therefore U is a σ-open neighborhood of x_0 disjoint from C, which implies $x_0 \notin \overline{C}^{\sigma}$, a contradiction.

Hence no such x_0 exists, and we conclude $\overline{C}^{\sigma} \subset \overline{C}^{\tau}$. Combining with the obvious reverse inclusion yields the desired equality $\overline{C}^{\tau} = \overline{C}^{\sigma}$. □

Definition 7. *Let (X, τ) be a TVS with dual X^*. Let $x \in X$ separates the points of X^*, then $\sigma(X^*, X)$ with $p_x(\psi) = |\psi(x)|$ called the weak* topology on X^*, which makes X^* into a LCTVS T1*

Remark 2 (Basic neighborhoods in the weak* topology)**.** *Let (X,τ) be a TVS with dual X^*. The weak* topology $\sigma(X^*,X)$ on X^* is generated by the seminorms $p_x(\psi)=|\psi(x)|$ for $x\in X$. A basic 0–neighborhood is*

$$V(x_1,\dots,x_n;\varepsilon)=\big\{\psi\in X^*:\ |\psi(x_i)|<\varepsilon,\ i=1,\dots,n\big\},$$

with $x_1,\dots,x_n\in X$ and $\varepsilon>0$. Equivalently, writing

$$p_{x_1,\dots,x_n}(\psi):=\max_{1\le i\le n}|\psi(x_i)|,$$

we have the "ball"

$$B_{x_1,\dots,x_n}(0,\varepsilon)=\{\psi\in X^*:\ p_{x_1,\dots,x_n}(\psi)<\varepsilon\}.$$

example 1. *$(X,\|\|)$ normed space, $(X^*,\|\|)$ normed space, then there are 2 topo on X^*:*

$$\sigma(X^*,X)\subset\sigma(X^*,X^{**})$$

and $X\subset X^{**}$

Let X be a vector space and $\psi_1,\psi_2\dots\psi_n$ are linear functional on X such that $\bigcap_{i=1}^n\ker\psi_i\subset\ker\psi$, then $\psi=c_1\psi_1+c_2\psi_2+\cdots+c_n\psi_n$

Let X be a vector space and $\psi_1,\psi_2,\dots,\psi_n$ be linear functionals on X. If $\bigcap_{i=1}^n\ker\psi_i\subseteq\ker\psi$, then there exist scalars $c_1,\dots,c_n$ such that

$$\psi=c_1\psi_1+c_2\psi_2+\cdots+c_n\psi_n.$$

Proof. Without loss of generality, assume all $\psi_i\neq0$. For each i, choose $x_i\in X$ such that $\psi_i(x_i)\neq0$, and define

$$z_i=\frac{x_i}{\psi_i(x_i)}\quad\Rightarrow\quad\psi_i(z_i)=1.$$

Define the map $\Phi:X\to\mathbb{K}^n$ by $\Phi(x)=(\psi_1(x),\dots,\psi_n(x))$, and let $V=\mathrm{Im}(\Phi)$.

Define $f:V\to\mathbb{K}$ by $f(\Phi(x))=\psi(x)$. This is well-defined: if $\Phi(x)=\Phi(y)$, then $x-y\in\bigcap\ker\psi_i\subseteq\ker\psi$, so $\psi(x)=\psi(y)$.

Extend f to a linear functional F on $\mathbb{K}^n$, which has the form $F(a_1,\dots,a_n)=\sum c_ia_i$. Then for any $x\in X$,

$$\psi(x)=f(\Phi(x))=F(\psi_1(x),\dots,\psi_n(x))=\sum_{i=1}^n c_i\psi_i(x).$$

Hence $\psi=\sum c_i\psi_i$. $\square$

Theorem 2. *If (X,τ) is a locally convex T_1 topological vector space, then*

$$(X^*,\sigma(X^*,X))^*\cong X.$$

Proof. 1. $X \subset X^{**}$

2. opposite direction: for any $f \in X^{**}$, there exists $c > 0, x_1, x_2 \dots x_n \in X$ such that

$$|f(\psi)| \leq c \sum_{i=1}^{n} |x_i(\psi)| = c \sum_{i=1}^{n} |\psi(x_i)|$$

Hence $\bigcap_{i=1}^{n} \ker(x_i) \subset \ker f$

By lemma above, we have there exists $c_1 \dots c_n$ such that $f = c_1x_1+c_2x_2 \dots c_nx_n$.

Thus $f \in X$ for any $f \in X^{**}$. So $X \supset X^{**}$ □

Theorem 3 (Alaoglu)**.** *Let X be a normed vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let*

$$B^* = \{f \in X^* : \|f\| \leq 1\}$$

be the closed unit ball in the dual space X^*. Then B^* is compact in the weak topology $\sigma(X^*, X)$.*

Proof. For each $x \in X$, define

$$K_x = \{z \in \mathbb{K} : |z| \leq \|x\|\}.$$

Each K_x is a closed bounded interval (if $\mathbb{K} = \mathbb{R}$) or disk (if $\mathbb{K} = \mathbb{C}$), hence compact in $\mathbb{K}$. Consider the product space

$$\mathcal{K} = \prod_{x \in X} K_x.$$

By Tychonoff's Theorem, $\mathcal{K}$ is compact in the product topology. Note that the product topology on $\mathcal{K}$ is precisely the topology of pointwise convergence.

Define the map

$$\Phi : B^* \to \mathcal{K}, \quad \Phi(f) = (f(x))_{x \in X}.$$

For $f \in B^*$, we have $\|f\| \leq 1$, so

$$|f(x)| \leq \|f\| \cdot \|x\| \leq \|x\| \quad \Rightarrow \quad f(x) \in K_x \quad \forall x \in X.$$

Thus $\Phi(f) \in \mathcal{K}$, so $\Phi(B^*) \subseteq \mathcal{K}$.

Let $g = (g_x)_{x \in X} \in \mathcal{K}$ be a limit point of $\Phi(B^*)$ in the product topology. Then there exists a net $\{f_\alpha\} \subset B^*$ such that

$$f_\alpha(x) \to g_x \quad \text{for all } x \in X.$$

Define $f : X \to \mathbb{K}$ by $f(x) = g_x$. We verify:

(a) f is linear: For $x, y \in X$ and $\lambda \in \mathbb{K}$,

$$f_\alpha(x + y) = f_\alpha(x) + f_\alpha(y) \to f(x) + f(y),$$

but also $f_\alpha(x + y) \to f(x + y)$, so $f(x + y) = f(x) + f(y)$. Similarly, $f(\lambda x) = \lambda f(x)$.

(b) f is bounded with $\|f\| \leq 1$: For any $x \in X$,

$$|f_\alpha(x)| \leq \|x\| \quad (\text{since } \|f_\alpha\| \leq 1),$$

so

$$|f(x)| = \lim_\alpha |f_\alpha(x)| \leq \|x\|.$$

Thus $\|f\| \leq 1$, so $f \in B^*$.

Hence $g = \Phi(f) \in \Phi(B^*)$, so $\Phi(B^*)$ is closed in $\mathcal{K}$.

Since $\mathcal{K}$ is compact and $\Phi(B^*) \subseteq \mathcal{K}$ is closed, $\Phi(B^*)$ is compact in the product topology.

The weak* topology $\sigma(X^*, X)$ on X^* is the topology of pointwise convergence on X. The product topology on $\mathcal{K}$ is also pointwise convergence. Thus the map

$$\Phi : (B^*, \sigma(X^*, X)) \to (\Phi(B^*), \text{subspace topology})$$

is a homeomorphism (it is clearly continuous, bijective, and its inverse is continuous by definition of the subspace topology). Since $\Phi(B^*)$ is compact, B^* is weak* compact. □

corollary 1. *Let X be a normed space, then $(X^*) = \{\psi \in X^* | \|\psi\| \leq 1\}$ is compact in the weak* topology.*

Proof. Consider $V = \{x \in X | \|x\| \leq 1\}$ then $K = (X^*)$. Then apply Alaoglu theorem to it. □