functional analysis study note

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0.1 Hilbert Space

Definition 1. *X be a vector space an inner product is map $(|)$ that $X \times X \to G$ satsfying:*
1. $(\alpha x + \beta y|z) = \alpha(x|z) + \beta(y|z)$
2. $\overline{(x|y)} = (y|x)$
3.$(x|x) \geq 0$ and $(x|x) = 0$ if and only if $x = 0$

Definition 2. *$\|x\| = \sqrt{(x|x)}$ if $(X, \|\|)$ is complete, X is a Hilbert space.*

example 1.

$$L^2(X, m, \mu) = \{f : X \to \mathbb{C}| \int_X |f|^2 d\mu < \infty\}$$

$(f|g) = \int_X f\overline{g}d\mu$ is Hilbert space.

Fact 1. *1.*

$$\|\xi + \eta\|^2 = \|\xi\|^2 + \|\eta\|^2 + 2Re((\xi|\eta))$$

This gives us

$$\|\xi + \eta\|^2 + \|\xi - \eta\|^2 = 2(\|\xi\|^2 + \|\eta\|^2)$$

2. Cauchy Schwarz inequality (CSI)

$$|(\xi|\eta)| \leq \|\xi\|\|\eta\|$$

equality holds if and only if $\xi = c\eta$ for some $c \in \mathbb{C}$
3. Let $\eta \in H$ define $\phi_\eta(\xi) = (\xi|\eta)$.
Then, by CSI, $\phi_\eta \in H^*$ and $\|\phi_\eta\| = \|\eta\|$
4. Polarization identities. H complex:

$$(\xi|\eta) = \frac{1}{4}(\|\xi + \eta\|^2 - \|\xi - \eta\|^2) + \frac{i}{4}(\|\xi + i\eta\|^2 - \|\xi - i\eta\|^2)$$

Theorem 1 (Cauchy–Schwarz Inequality)**.** *Let $(H, (\cdot|\cdot))$ be a complex inner product space and let $\|\cdot\| := \sqrt{(\cdot|\cdot)}$ be the induced norm. For all $\xi, \eta \in H$,*

$$|(\xi|\eta)| \leq \|\xi\| \, \|\eta\|.$$

Equality holds if and only if there exists $c \in \mathbb{C}$ such that $\xi = c\eta$.

Proof. If $\eta = 0$, then both sides are zero, so equality holds and $\xi = c\eta$ for any $c \in \mathbb{C}$. Thus we may assume $\|\eta\| > 0$.

For any scalar $\lambda \in \mathbb{C}$, the positivity of the norm gives

$$0 \le \|\xi - \lambda\eta\|^2 = (\xi - \lambda\eta \mid \xi - \lambda\eta) = \|\xi\|^2 - \overline{\lambda}\,(\xi|\eta) - \lambda\,(\eta|\xi) + |\lambda|^2\|\eta\|^2.$$

Choose

$$\lambda := \frac{(\eta|\xi)}{\|\eta\|^2}.$$

Note that $(\eta|\xi) = \overline{(\xi|\eta)}$, so

$$\overline{\lambda} = \frac{\overline{(\eta|\xi)}}{\|\eta\|^2} = \frac{(\xi|\eta)}{\|\eta\|^2}.$$

Substituting λ yields

$$0 \le \|\xi\|^2 - \overline{\lambda}\,(\xi|\eta) - \lambda\,(\eta|\xi) + |\lambda|^2\|\eta\|^2.$$

The middle terms simplify as

$$\overline{\lambda}\,(\xi|\eta) = \frac{(\xi|\eta)}{\|\eta\|^2}\cdot(\xi|\eta) = \frac{|(\xi|\eta)|^2}{\|\eta\|^2},$$

and

$$\lambda\,(\eta|\xi) = \frac{(\eta|\xi)}{\|\eta\|^2}\cdot(\eta|\xi) = \frac{|(\eta|\xi)|^2}{\|\eta\|^2} = \frac{|(\xi|\eta)|^2}{\|\eta\|^2}.$$

Also,

$$|\lambda|^2\|\eta\|^2 = \left|\frac{(\eta|\xi)}{\|\eta\|^2}\right|^2 \|\eta\|^2 = \frac{|(\eta|\xi)|^2}{\|\eta\|^4}\cdot\|\eta\|^2 = \frac{|(\xi|\eta)|^2}{\|\eta\|^2}.$$

Hence,

$$0 \le \|\xi\|^2 - \frac{|(\xi|\eta)|^2}{\|\eta\|^2} - \frac{|(\xi|\eta)|^2}{\|\eta\|^2} + \frac{|(\xi|\eta)|^2}{\|\eta\|^2} = \|\xi\|^2 - \frac{|(\xi|\eta)|^2}{\|\eta\|^2}.$$

Multiplying through by $\|\eta\|^2 > 0$ gives

$$0 \le \|\xi\|^2\,\|\eta\|^2 - |(\xi|\eta)|^2,$$

or

$$|(\xi|\eta)|^2 \le \|\xi\|^2\,\|\eta\|^2.$$

Taking square roots (both sides non-negative) yields the desired inequality:

$$|(\xi|\eta)| \le \|\xi\|\,\|\eta\|.$$

Equality case. Equality holds if and only if $\|\xi - \lambda\eta\|^2 = 0$, i.e.,

$$\xi - \lambda\eta = 0 \quad \Rightarrow \quad \xi = \lambda\eta.$$

With $\lambda = \frac{(\eta|\xi)}{\|\eta\|^2}$, set $c := \lambda$. If $\eta = 0$, equality holds for any c (as treated initially). Thus equality is equivalent to $\xi = c\eta$ for some $c \in \mathbb{C}$. $\square$

Theorem 2. *Let H, K be 2 Hilbert space. H and K are isomorphic if there exists $f : H \to K$ linear onto map such that $(f\xi|f\eta)_K = (\xi|\eta)_H$ for any $\xi, \eta \in H$.*

Note: $\|f\xi\| = \|\xi\|$ for any $\xi \in H$, hence $\|f\| = 1$.

If $T : H \to K$ is a linear map with $(T\xi|T\eta) = (\xi|\eta)_H$ for any $\xi, \eta \in H$, then T is isometry. If T is also onto, then T is unitary.

Remark 1. *$\|T\xi\| = \|\xi\|$ is equivalent to $(T\xi|T\eta)_K = (\xi|\eta)_H$*

If $T : H \to H$ is an isometry, then it is automatically a unitary if $\dim H < \infty$, since $\|T\xi\| = \|\xi\|$ for any $\xi \in H$. Thus T is injection and $\dim T = \dim H$ gives us T is also surjection.

Theorem 3 (Wold–von Neumann decomposition)**.** *Let H be a Hilbert space and let $V : H \to H$ be an* isometry*, i.e. $V^*V = I$. Set*

$$W := \ker V^*, \qquad H_1 := \overline{\bigoplus_{n\geq 0} V^n W}, \qquad H_0 := \bigcap_{n\geq 0} V^n H.$$

Then $H = H_0 \oplus H_1$ (orthogonal direct sum), and:

1. *$V|_{H_0}$ is unitary on H_0;*
2. *$V|_{H_1}$ is unitarily equivalent to the unilateral shift of multiplicity* $\dim W$

Theorem 4. *If $\xi_n \to \xi, \eta_n \to \eta$ then $(\xi|\eta_n) \to (\xi|\eta)$.*

Proof. Since $\eta_n \to \eta$, we have $\|\eta_n - \eta\| \to 0$. By the Cauchy–Schwarz inequality,

$$\big|(\xi \mid \eta_n) - (\xi \mid \eta)\big| = \big|(\xi \mid \eta_n - \eta)\big| \leq \|\xi\| \, \|\eta_n - \eta\| \xrightarrow[n\to\infty]{} 0.$$

Hence $(\xi \mid \eta_n) \to (\xi \mid \eta)$. $\square$

Definition 3. *H be a Hilbert space, we say ξ is orthogonal to η if $(\xi|\eta) = 0$.*

If $A \subset H$ a subset then

$$A^\perp = \{\xi \in H | (\xi|\eta) = 0, \forall \eta \in A\}$$

which is the orthogonal complement of A.

Theorem 5. *Let $\xi_1, \xi_2 \ldots \xi_n \in H$ and $\xi_i \perp \xi_j$ if $i \neq j$. Then $\|\sum_{i=1}^n \xi_i\|^2 = \sum_{i=1}^n \|\xi_i\|^2$*

Theorem 6. *Let H be Hilbert, $A \subset H$ colsed convex and nonempty. Let $\xi \in H$ then there exists a unique $\xi_0 \in A$ such that $\|\xi - \xi_0\| = \inf\{\|\xi - \eta\| | \eta \in A\} = dist(\xi, A)$*

Proof. Set $d := \inf_{\eta\in A} \|\xi - \eta\|$. Choose a sequence $(\eta_n) \subset A$ with $\|\xi - \eta_n\| \downarrow d$. We first prove (η_n) is Cauchy.

By convexity, for any n, m the midpoint $\zeta := \frac{\eta_n+\eta_m}{2}$ lies in A, hence $\|\xi-\zeta\| \geq d$. The parallelogram identity gives

$$\|\xi - \eta_n\|^2 + \|\xi - \eta_m\|^2 = 2\Big\|\xi - \frac{\eta_n + \eta_m}{2}\Big\|^2 + \frac{1}{2}\|\eta_n - \eta_m\|^2 \geq 2d^2 + \frac{1}{2}\|\eta_n - \eta_m\|^2.$$

Therefore,

$$\frac{1}{2}\|\eta_n - \eta_m\|^2 \leq \|\xi - \eta_n\|^2 + \|\xi - \eta_m\|^2 - 2d^2 \xrightarrow[n,m\to\infty]{} 0,$$

so (η_n) is Cauchy. Since A is closed and H is complete, there exists $\xi_0 \in A$ with $\eta_n \to \xi_0$. By continuity of the norm,

$$\|\xi - \xi_0\| = \lim_{n\to\infty} \|\xi - \eta_n\| = d,$$

so the infimum is attained at ξ_0.

For uniqueness, suppose $\xi_0, \xi_1 \in A$ both satisfy $\|\xi - \xi_0\| = \|\xi - \xi_1\| = d$. With $\zeta := \frac{\xi_0+\xi_1}{2} \in A$ and the parallelogram identity,

$$2d^2 = \|\xi - \xi_0\|^2 + \|\xi - \xi_1\|^2 = 2\|\xi - \zeta\|^2 + \frac{1}{2}\|\xi_0 - \xi_1\|^2 \geq 2d^2 + \frac{1}{2}\|\xi_0 - \xi_1\|^2.$$

Hence $\|\xi_0 - \xi_1\| = 0$, i.e. $\xi_0 = \xi_1$. This proves existence and uniqueness. □

corollary 1. *The map $\phi : H \to H^*, \eta \to \psi_\eta$, then $\psi_\eta(\xi) = (\xi|\eta)$ is a conjugate linear isometry.*

Remark 2. *The conjugate Hilbert space $\overline{H} = H$ as a set with translation invariance and multiplication $\lambda\xi = \overline{\lambda}\xi$ and $(\overline{\xi}|\overline{\eta})_{\overline{H}} = (\xi|\eta)_H$, then the map $\eta \in \overline{H} \to \psi_\eta \in H^*$ is a linear isometry*

corollary 2. *Every Hilbert space is reflexive $H \cong H^{**}$*