Definition 1 (Outer measure)**.** *Given a set function ν on a collection $\mathcal{A} \subseteq \mathcal{P}(X)$ with $\varnothing \in \mathcal{A}$ and $\nu(\varnothing) = 0$, define for $E \subseteq X$*

$$\mu^*(E) = \inf\Big\{ \sum_{i=1}^{\infty} \nu(A_i) \;:\; E \subseteq \bigcup_{i=1}^{\infty} A_i,\ A_i \in \mathcal{A} \Big\}.$$

Then μ^* is an outer measure on X

theorem 1. *For an interval I, $\mu^*(I) = v(I)$*

theorem 2. *If $E_2 \subset E_1$, then $\mu^*(E_2) \leq \mu^*(E_1)$*

Proof. Since for any cover of E_1, this cover is also a cover of E_2 □

theorem 3. *if $E = \bigcup E_k$ is a countable union of sets, then $\mu^*(E) \leq \sum \mu^*(E_k)$*

Proof. for $E \subseteq X$,

$$\mu^*(E) = \inf\Big\{ \sum_{i=1}^{\infty} \nu(A_i) \,:\, E \subseteq \bigcup_{i=1}^{\infty} A_i,\ A_i \in \mathcal{A} \Big\}$$

for some set function ν on a cover class $\mathcal{A}$ with $\nu(\varnothing) = 0$.

Fix $\varepsilon > 0$. For each $k \in \mathbb{N}$, by the definition of the inf we can choose a sequence $\{A_i^{(k)}\}_{i=1}^{\infty} \subseteq \mathcal{A}$ such that

$$E_k \subseteq \bigcup_{i=1}^{\infty} A_i^{(k)} \quad \text{and} \quad \sum_{i=1}^{\infty} \nu\big(A_i^{(k)}\big) \leq \mu^*(E_k) + \frac{\varepsilon}{2^k}.$$

Then the countable family $\{A_i^{(k)} : i, k \in \mathbb{N}\}$ covers $E = \bigcup_k E_k$, hence

$$\mu^*(E) \leq \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \nu\big(A_i^{(k)}\big) \leq \sum_{k=1}^{\infty} \Big(\mu^*(E_k) + \frac{\varepsilon}{2^k}\Big) = \sum_{k=1}^{\infty} \mu^*(E_k) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, letting $\varepsilon \to 0^+$ yields $\mu^*(E) \leq \sum_{k=1}^{\infty} \mu^*(E_k)$. □

Definition 2 (Cantor Set)**.** *The* Cantor set *C is the subset of* $[0, 1]$ *obtained by repeatedly removing the open middle third interval in each step:*

1. *Start with $C_0 = [0, 1]$.*
2. *At step $n \geq 1$, from each closed interval of C_{n-1}, remove the open middle third. Denote the resulting set by C_n.*
3. *Define*

$$C = \bigcap_{n=0}^{\infty} C_n.$$

Equivalently, C consists of those $x \in [0,1]$ that admit a base-3 expansion

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}, \qquad a_k \in \{0,2\}.$$

That is, the Cantor set contains precisely those points in $[0,1]$ whose ternary expansions involve only the digits 0 and 2.

Cantor Set and Base-3 Expansion. Every $x \in [0,1]$ can be written in ternary (base-3) expansion as

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}, \qquad a_k \in \{0,1,2\}.$$

In the construction of the Cantor set, at each step we remove the open middle third intervals, which correspond precisely to those numbers whose ternary expansions contain the digit 1 in some position.

Thus, the Cantor set C can be described equivalently as

$$C = \Big\{ x \in [0,1] \ : \ \text{there exists a ternary expansion with } a_k \in \{0,2\} \ \forall k \Big\}.$$

In other words, points in the Cantor set are exactly those whose ternary digits involve only 0 and 2. For example,

$$0, \quad \tfrac{1}{3} = 0.02222\ldots_3, \quad \tfrac{2}{3} = 0.20000\ldots_3, \quad 1 = 0.22222\ldots_3$$

are all contained in C.

theorem 4. *Let $E \subset R^n$. Then given $\varepsilon > 0$, there exists an open set G such that $E \subset G$ and $\mu^*(G) \le \mu^*(E) + \varepsilon$.*

Definition 3 (G_δ set)**.** *A subset A of a topological space X is called a G_δ* set *if it can be expressed as a countable intersection of open sets, i.e.*

$$A = \bigcap_{n=1}^{\infty} U_n, \quad U_n \textit{ open in } X.$$

Definition 4 (F_σ set)**.** *A subset B of a topological space X is called an F_σ* set *if it can be expressed as a countable union of closed sets, i.e.*

$$B = \bigcup_{n=1}^{\infty} F_n, \quad F_n \textit{ closed in } X.$$

theorem 5. *if $E \subset R^n$, there exists a set H type G_δ such that $E \subset H$ and $\mu^*(E) = \mu^*(H)$*

Proof. By theorem 4, for every $\varepsilon > 0$ there exists an open set $G \supset E$ such that

$$\mu^*(G) \leq \mu^*(E) + \varepsilon.$$

For each $k \in \mathbb{N}$, choose an open set $G_k \supset E$ with

$$\mu^*(G_k) \leq \mu^*(E) + 2^{-k}.$$

Define

$$H := \bigcap_{k=1}^{\infty} G_k.$$

Then H is a G_δ set containing E. Moreover, by monotonicity of the outer measure,

$$\mu^*(E) \leq \mu^*(H) \leq \mu^*(G_k) \leq \mu^*(E) + 2^{-k}, \quad \text{for all } k.$$

Letting $k \to \infty$ yields $\mu^*(H) = \mu^*(E)$, as required. □

Definition 5. *A subset $E \subset R^n$ is Lebesgue measurable if for any $\varepsilon > 0$, there exists an open set G such that:*
$E \subset G$ and $\mu^*(G - E) < \varepsilon$
and $|E|$ is the Lebesgue measure of E, $|E| = \mu^*(E)$, for measurable E

theorem 6. *the union $E = \bigcup E_k$ of a countable number of measurable sets is mesurable and*

$$|E| \leq \sum |E_k|$$

Proof. **Measurability.** The collection of Lebesgue measurable sets is a σ-algebra, hence closed under countable unions. Since each E_k is measurable, the union $E = \bigcup_{k=1}^{\infty} E_k$ is measurable.

Subadditivity. First consider the finite case. For $n \in \mathbb{N}$ define the increasing sequence

$$E^{(n)} := \bigcup_{k=1}^{n} E_k.$$

For each n, disjointize the union by setting

$$A_1 := E_1, \qquad A_k := E_k \setminus \bigcup_{j=1}^{k-1} E_j \quad (k \geq 2).$$

Then $\{A_k\}_{k=1}^{n}$ are pairwise disjoint, measurable, $A_k \subset E_k$, and $E^{(n)} = \bigsqcup_{k=1}^{n} A_k$. Countable additivity on disjoint unions yields

$$|E^{(n)}| = \sum_{k=1}^{n} |A_k| \leq \sum_{k=1}^{n} |E_k|.$$

Now pass to the countable union. The sets $\{E^{(n)}\}$ form an increasing sequence with $\bigcup_{n=1}^{\infty} E^{(n)} = E$. By continuity from below of measures,

$$|E| \;=\; \lim_{n\to\infty} |E^{(n)}| \;\le\; \lim_{n\to\infty} \sum_{k=1}^{n} |E_k| \;=\; \sum_{k=1}^{\infty} |E_k|.$$

This proves the desired inequality.

Remark. If, moreover, the sets E_k are pairwise disjoint, then the above disjointization gives $|E| = \sum_{k=1}^{\infty} |E_k|$ (countable additivity). $\square$

lemma 1. *If $\{A_i\}_{i=1}^{N}$ is a finite collection of nonoverlapping intervals, then $\bigcup A_i$ is measurable and $|\bigcup A_k| = \sum |A_k|$*

Proof. this lemma is directly followed by remark $\square$

lemma 2. *(followed lemma 1) if $d(E_1, E_2) > 0$, then $\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2)$*

Proof. The subadditivity of the outer measure gives $\mu^*(E_1 \cup E_2) \le \mu^*(E_1) + \mu^*(E_2)$. It remains to show the reverse inequality.

Let $\delta := d(E_1, E_2) > 0$ and fix $\varepsilon > 0$. Choose a countable cover of $E_1 \cup E_2$ by rectangles (or boxes) $\{Q_i\}_{i=1}^{\infty}$ with

$$E_1 \cup E_2 \subset \bigcup_{i=1}^{\infty} Q_i \qquad \text{and} \qquad \sum_{i=1}^{\infty} |Q_i| \le \mu^*(E_1 \cup E_2) + \varepsilon.$$

Pick $\eta \in (0, \delta/3)$ and set the disjoint open neighborhoods

$$U := \{x : \operatorname{dist}(x, E_1) < \eta\}, \qquad V := \{x : \operatorname{dist}(x, E_2) < \eta\}.$$

Then $U \cap V = \varnothing$, $E_1 \subset U$ and $E_2 \subset V$. Since $E_1 \subset U \cap \bigcup_i Q_i = \bigcup_i (Q_i \cap U)$, the family $\{Q_i \cap U\}_{i=1}^{\infty}$ covers E_1; each $Q_i \cap U$ is open (hence Lebesgue measurable), and for any $\zeta > 0$ we may cover $Q_i \cap U$ by a finite union of rectangles with total volume within $\zeta 2^{-i}$ of $|Q_i \cap U|$. Summing over i and letting $\zeta \downarrow 0$, we obtain

$$\mu^*(E_1) \le \sum_{i=1}^{\infty} |Q_i \cap U|.$$

An identical argument yields

$$\mu^*(E_2) \le \sum_{i=1}^{\infty} |Q_i \cap V|.$$

Because U and V are disjoint, for each i the subsets $Q_i \cap U$ and $Q_i \cap V$ are disjoint, hence

$$|Q_i \cap U| + |Q_i \cap V| \le |Q_i|.$$

Adding over i gives

$$\mu^*(E_1) + \mu^*(E_2) \ \le \ \sum_{i=1}^{\infty} \big(|Q_i \cap U| + |Q_i \cap V|\big) \ \le \ \sum_{i=1}^{\infty} |Q_i| \ \le \ \mu^*(E_1 \cup E_2) + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude $\mu^*(E_1)+\mu^*(E_2) \le \mu^*(E_1 \cup E_2)$. Together with subadditivity, this proves the equality. $\square$

theorem 7. *Every closed set F is measurable.*

Proof. Fix $\varepsilon > 0$, there exists an open set $G \supset F$ with

$$\mu^*(G) \le \mu^*(F) + \varepsilon.$$

Since $G \setminus F$ is open, it is a (countable) disjoint union of open intervals, say $G \setminus F = \bigsqcup_{k=1}^{\infty} I_k$, and hence by lemma

$$\mu^*(G \setminus F) = |\bigcup I_k| = \sum_{k=1}^{\infty} |I_k| \le \varepsilon.$$

$\square$

theorem 8. *the complement of a measurable set is measurable*

theorem 9. *the intersection $E = \bigcap E_k$ of countable number measurable sets is measurable.*

theorem 10. *if E_1 and E_2 are measurable, then $E_1 - E_2$ is measurable.*

Fact 1. *The collection of measurable subsets of R^n is a $\sigma-$ algebra. Thus, every Borel set is measurable*