functional analysis study note

Zehao Li

October 2025

0.1 Bipolar

Definition 1. *Let X be TVS topology defined by $\{p_i\}$ separate (X). Then X is LCTVS, T1, and*

$$A \subset X, A^\circ = \{\psi \in X^* || \psi(x)| \leq 1\}$$

$$B \subset X^*, B^\circ = \{x \in X || \psi(x)| \leq 1\}$$

Note: A° and B° are balanced convex sets, and

$$B^\circ = \{\bigcap_{\psi \in B} \psi^{-1}(\{z \in \mathbb{C} || z| \leq 1\})\}$$

which is closed.

Theorem 1. *X be a LCTVS T1,if A is a balanced convex neighborhood of 0, then $A^{\circ\circ} = \overline{A}$*

note: $A^{\circ\circ} = \{x \in X | \sup_{\psi \in A^\circ} |\psi(x)| \leq 1\}$

Proof. Step 1: $\overline{A} \subset A^{\circ\circ}$. For every $x \in A$ and every $\varphi \in A^\circ$ we have $|\varphi(x)| \leq 1$ by definition, hence $A \subset A^{\circ\circ}$. Since for each fixed φ the map $x \mapsto |\varphi(x)|$ is continuous, the sublevel set $\{x : |\varphi(x)| \leq 1\}$ is closed; therefore $A^{\circ\circ} = \bigcap_{\varphi \in A^\circ} \{x : |\varphi(x)| \leq 1\}$ is closed. Consequently $\overline{A} \subset A^{\circ\circ}$.

Step 2: $A^{\circ\circ} \subset \overline{A}$. Assume $x_0 \notin \overline{A}$. The set $\overline{A}$ is closed, convex, and balanced. By the (strong) Hahn–Banach separation theorem in LCTVS, there exists $\psi \in X^*$ and real numbers $\alpha < \beta$ such that

$$\operatorname{Re} \psi(x) \leq \alpha \quad (\forall x \in \overline{A}), \qquad \operatorname{Re} \psi(x_0) \geq \beta.$$

We claim that $\sup_{x \in A} |\psi(x)| \leq \sup_{x \in A} \operatorname{Re} \psi(x) \leq \alpha$. Indeed, for any $x \in A$:

- if $\mathbb{K} = \mathbb{R}$, then A being balanced implies $x, -x \in A$, so $|\psi(x)| = \max\{\psi(x), -\psi(x)\} \leq \sup_{y \in A} \operatorname{Re} \psi(y)$;
- if $\mathbb{K} = \mathbb{C}$, pick $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $\operatorname{Re}\big(\lambda \psi(x)\big) = |\psi(x)|$. Since A is balanced, $\lambda x \in A$, hence $|\psi(x)| = \operatorname{Re} \psi(\lambda x) \leq \sup_{y \in A} \operatorname{Re} \psi(y)$.

Thus $\sup_A |\psi| \le \alpha < \beta \le |\psi(x_0)|$.

Choose $t > 0$ so that

$$t \cdot \sup_{x \in A} |\psi(x)| \le 1 \quad \text{and} \quad t\,|\psi(x_0)| > 1$$

(which is possible because $|\psi(x_0)| > \sup_A |\psi|$). Set $\varphi := t\psi$. Then

$$\sup_{x \in A} |\varphi(x)| \le 1,$$

so $\varphi \in A^\circ$, while $|\varphi(x_0)| = t|\psi(x_0)| > 1$. Hence $x_0 \notin A^{\circ\circ}$.

Since every $x_0 \notin \overline{A}$ is also not in $A^{\circ\circ}$, we have $A^{\circ\circ} \subset \overline{A}$.

□

Theorem 2. *Let X be LCTVS,T1 then weakly bounded set are coincide with bounded set.*

corollary 1. *$(X, \|\|)$ be a normed space let $E \subset X$ a subset* $\sup_{x \in E} |\psi(x)| < \infty, \forall \psi \in X^*$ *then there exists a $c > 0$ such that $\|x\| \le c, \forall x \in E$*

Proof. by previous theorem, $E \subset X$ is weakly bounded if and only if each $P_\psi, \psi \in X^*$ is bounded on E, if and only if $|\psi(x))| \le c(\psi), \forall x \in E, \forall \psi \in X^*$ □

Theorem 3 (Uniform boundedness on a compact set)**.** *Let X be a topological vector space and Y a locally convex topological vector space. Let $K \subset X$ be compact and let $\Gamma \subset \mathcal{L}(X, Y)$ be a family of continuous linear maps. Assume that for every $x \in K$ the orbit $\Gamma(x) = \{\gamma(x) : \gamma \in \Gamma\}$ is bounded in Y. Then there exists a bounded set $B \subset Y$ such that $\gamma(K) \subset B$ for all $\gamma \in \Gamma$.*

Proof. Let $\{q_i\}_{i \in I}$ be a directed family of continuous seminorms generating the topology of Y. Fix $i \in I$. For $n \in \mathbb{N}$ set

$$E_{i,n} \;:=\; \bigl\{x \in K : \; \sup_{\gamma \in \Gamma} q_i\bigl(\gamma x\bigr) \le n\bigr\}.$$

Each $E_{i,n}$ is closed in the relative topology of K because

$$E_{i,n} = \bigcap_{\gamma \in \Gamma} \{x \in K : \; q_i(\gamma x) \le n\},$$

and every set $\{x \in K : \; q_i(\gamma x) \le n\}$ is closed in K (continuity of $q_i \circ \gamma$). Moreover, by pointwise boundedness, $\bigcup_{n=1}^{\infty} E_{i,n} = K$.

Since K is compact Hausdorff, it is a *Baire space.* Hence for this fixed i there exists $n_i \in \mathbb{N}$ and a nonempty relatively open set $U_i \subset K$ such that $U_i \subset E_{i,n_i}$, i.e.,

$$\sup_{\gamma \in \Gamma} q_i\bigl(\gamma x\bigr) \;\le\; n_i \qquad (\forall x \in U_i).$$

Pick any $y \in U_i$. Consider the continuous map

$$\Phi : K \times [0,1] \to K, \qquad \Phi(x,t) = (1-t)\,y + t\,x.$$

Since U_i is open in K and $\Phi(K \times \{0\}) = \{y\} \subset U_i$, by the *tube lemma* there exists $\delta \in (0, 1]$ such that

$$\Phi\big(K \times [0, \delta]\big) \ \subset \ U_i.$$

Fix such a δ. Then for every $x \in K$ the point

$$z = (1 - \delta)\, y + \delta\, x \in U_i.$$

Now for any $\gamma \in \Gamma$, by linearity and the seminorm properties,

$$q_i\big(\gamma z\big) = q_i\big((1 - \delta)\,\gamma y + \delta\,\gamma x\big) \le (1 - \delta)\, q_i(\gamma y) + \delta\, q_i(\gamma x).$$

Since $y, z \in U_i$ we have $q_i(\gamma y) \le n_i$ and $q_i(\gamma z) \le n_i$. Hence

$$\delta\, q_i(\gamma x) \ \le \ q_i(\gamma z) + (1 - \delta)\, q_i(\gamma y) \ \le \ n_i + (1 - \delta)\, n_i \ \le \ 2n_i,$$

and therefore

$$q_i(\gamma x) \ \le \ \frac{2}{\delta}\, n_i \qquad (\forall x \in K, \ \forall \gamma \in \Gamma).$$

Thus, for the fixed i, the set $\bigcup_{\gamma \in \Gamma} \gamma(K)$ is bounded with respect to q_i. Since $i \in I$ was arbitrary and $\{q_i\}_{i \in I}$ generates the topology of Y, it follows that

$$B \ := \ \bigcup_{\gamma \in \Gamma} \gamma(K)$$

is a bounded subset of Y. Equivalently, there exists a bounded set $B \subset Y$ with $\gamma(K) \subset B$ for all $\gamma \in \Gamma$. □

Theorem 4. *Weakly bounded set if and only if bounded set*

Proof. 1. let τ be the topology on the LCTVS X (T1). Since $\sigma(X, X^*) \subset \tau$, we have τ bounded is $\sigma(X, X^*)$ bounded

2. the opposite direction: Let $E \subset X$ be the weakly bounded. Let U be a τ neighborhood of 0. Since X is LCTVS, there exists a V is open convex balanced and $\overline{V} \subset U$.

Let

$$K = V^\circ = \{\psi \in X^* || \psi(x)| \le 1\}$$

. By bipolar theorem $K^\circ = V^{\circ\circ} = \overline{V}$.

E is weakly bounded, hence $\forall \psi \in X^*$, there exists $c(\psi) > 0$ such that $|\psi(x)| \le c(\psi)$ for any $x \in E$.

By Alaoglu, we have K is weak* compact, convex. Then, by uniform boundedness on a compact set, we have

K compact, $Y \subset \mathbb{C}, \Gamma = E$, $x(\psi) = \psi(x)$ on the topology $(X^*, \sigma(X^*, X))$.

Let $\Gamma(\psi) = \{x(\psi) | x \in E\} = \{\psi(x) | x \in E\}$ are bounded because E is weakly bounded.

Thus, $|\psi(x)| \le c$ for any $x \in E, \psi \in K$, giving us $c^{-1}x \in K^\circ$ for any $x \in E$ $k^\circ = V^{\circ\circ} = \overline{V} \subset U$

Hence, $x \in cu$ for any $x \in E$. $E \subset tU$ for any $t > c$, which is equivalent to E is bounded. □

Definition 2. *Let K be a convex set in a vector space X. A point $x \in K$ is an* extreme point *of K if it cannot be expressed as a convex combination of points in K distinct from itself. In other words, x is extreme if whenever*

$$x = \lambda y + (1 - \lambda) z$$

for some $y, z \in K$ and $\lambda \in (0, 1)$, it follows that $y = z = x$.

Equivalently, $x \in \mathrm{ext}(K)$ if and only if $K \setminus \{x\}$ is convex (or, if the only way to write x as a convex combination is the trivial one).

example 1. *1.*

$$K = \{x^2 + y^2 \leq 1\}$$

the extreme set of K is $\{x^2 + y^2 = 1\}$

example 2. *2.*

$$K = \{(x, y)|x = 0\}$$

the extreme set of K is $\emptyset$

example 3. *3.*

$$K = \{(x, y)|x < 0\} \cup \{(0, 0)\}$$

the extreme set of K is $\{(0, 0)\}$

example 4. *4.*

$$K = \{f \in L_1([0, 1])|\|f\| \leq 1\}$$

the extreme set of K is $\emptyset$

Proof. 1.suppose $f \in L_1([0, 1])$ and $\int_0^1 f(t)dt = 1$, then there exists $x \in [0, 1]$ such that $\int_0^1 f(t)dt = \frac{1}{2}$

let $h(t) = 2f(t)$ for $0 \leq t \leq x$,

$g(t) = 2f(t)$ for $x \leq t \leq 1$

We have $f = \frac{1}{2}(h + g)$ and $\|h\|_1 = \|g\|_1 = 1$

2. For the case $\|f\|_1 < 1$, it is in the unit ball so that is obvious f not an extreme point of K.

□