functional analysis study note

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September 2025

Let X, Y be normed spaces. Define

$$B(X,Y) := \{T : X \to Y \mid T \text{ is linear and bounded}\}.$$

If $X = Y$, we denote $B(X) := B(X,X)$, which forms a Banach algebra under operator composition.

For $T \in B(X,Y)$, the operator norm is defined by

$$\|T\| := \sup_{\|x\|=1} \|T(x)\|.$$

Equivalently,

$$\|T\| = \sup_{\|x\|\le 1} \|T(x)\| = \sup_{x\neq 0} \frac{\|T(x)\|}{\|x\|} = \inf\{c > 0 : \|T(x)\| \le c\|x\| \ \forall x \in X\}.$$

Proof. Let $c > 0$ be such that $\|T(x)\| \le c\|x\|$ for all $x \in X$. In particular, if $\|x\| \le 1$, then $\|T(x)\| \le c$. Hence

$$\sup\{\|T(x)\| : \|x\| \le 1\} \le \inf\{c > 0 : \|T(x)\| \le c\|x\| \ \forall x \in X\}.$$

For the reverse inequality, take any $x \neq 0$. Then

$$\|T(x)\| = \left\|T\left(\frac{x}{\|x\|}\right)\right\| \|x\| \le \left(\sup_{\|y\|=1} \|T(y)\|\right) \|x\| = c\|x\|.$$

Thus

$$\sup\{\|T(x)\| : \|x\| \le 1\} \ge \inf\{c > 0 : \|T(x)\| \le c\|x\| \ \forall x \in X\}.$$

□

Bounded operators and the C^*-identity

Proposition 1. *Let X be a Banach space and denote*

$$B(X) = \{T : X \to X \mid T \text{ is linear and bounded}\}.$$

Equipped with operator addition, scalar multiplication, composition and the operator norm

$$\|T\| := \sup_{\|x\|=1} \|Tx\|,$$

the space $B(X)$ is a Banach algebra.

Sketch of proof. Completeness: if (T_n) is Cauchy in operator norm then for each fixed $x \in X$ the sequence $(T_n x)$ is Cauchy in X; define $Tx = \lim_n T_n x$ and check $T \in B(X)$ and $\|T_n - T\| \to 0$. Algebra property: operator composition is associative and continuous; the norm satisfies $\|ST\| \le \|S\|\|T\|$, so $B(X)$ is a Banach algebra. □

Proposition 2. *Let H be a Hilbert space. For every $T \in B(H)$ the adjoint operator $T^* \in B(H)$ exists and satisfies, for all $x \in H$,*

$$\|Tx\|^2 = \langle T^*Tx, x\rangle \le \|T^*T\| \, \|x\|^2.$$

Consequently,

$$\|T\|^2 \le \|T^*T\| \le \|T^*\| \, \|T\| = \|T\|^2,$$

and hence

$$\|T^*T\| = \|T\|^2.$$

Therefore $(B(H), +, \circ, ^*, \|\cdot\|)$ is a unital C^*-algebra (with involution $T \mapsto T^*$).

Proof. For any unit vector $x \in H$,

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle \le \|T^*T\| \, \|x\|^2 = \|T^*T\|.$$

Taking supremum over $\|x\| = 1$ yields $\|T\|^2 \le \|T^*T\|$.

On the other hand, by submultiplicativity of the operator norm,

$$\|T^*T\| \le \|T^*\| \, \|T\|.$$

and

$$\|T(x)\|^2 \le \|T^*T\| \le \|T\|^2\|x\|^2$$

so

$$\|T^*T\| \le \|T\|^2.$$

Combining the two inequalities gives $\|T^*T\| = \|T\|^2$, which is the C^*-identity. □

T^*T is a self-adjoint matrix diagonalizable.

There are value $\lambda_1, \lambda_2 \dots \lambda_n$ and an orthorgonormal basis of vectors $\{x_1, x_2 \dots x_n\}$ such that $T^*T(x_i) = \lambda_i x_i$

Then exists unitary matrix $u \in \Sigma(C)$ such that $uT^*Tu^* = D$

Note:$(T^*T(x_i)|x_i) = \lambda_i(x_i|x_i) = \lambda_i = (T(x_i)|T(x_i)) = \|T(x_i)\|^2$

Thus, $\lambda = \max\{\lambda_i\}$

Claim $\|T\| = \sqrt{\lambda}$ since $uT^*Tu^* = D$, $T^*T = u^*Du$.
$\|T^*T\| = \|u^*Du\| \leq \|u\|^2\|D\| = \|D\|$
LET $x = \sum_{i=1}^{n} \alpha_i e_i$ then

$$\|D(x)\|^2 = \|\sum_{i=1}^{n} \alpha_i e_i \lambda_i\|^2 = \sum_{i=1}^{n} \lambda_i^2 |\alpha_i|^2 \leq \lambda^2 \sum_{i=1}^{n} |\alpha_i|^2 = \lambda^2 \|x\|^2$$

thus

$$\|D\| \leq \lambda, \|T^*T\| \leq \lambda$$

also

$$\|T^*T(x_i)\| = \lambda_i \leq \|T^*T\|\|x_i\| = \|T^*T\|$$

hence, $\|T^*T\| = \lambda$, indeed, $\|T\| = \sqrt{\lambda}$ which is the biggest eigenvalue.

Definition 1. *Let $(X, \|\cdot\|)$ be a normed space. The* dual space *of X, denoted by X^*, is defined as*

$$X^* := B(X, \mathbb{C}),$$

the set of all continuous linear functionals $f : X \to \mathbb{C}$.

The algebraic dual *of X is the set of all linear functionals $f : X \to \mathbb{C}$, without the continuity requirement.*

Note 1. *The dual space X^* is itself a Banach space, equipped with the operator norm*

$$\|f\| = \sup\{\, |f(x)| : x \in B_1(0) \subset X \,\},$$

where $B_1(0)$ denotes the closed unit ball in X.

Definition 2. *Let X, Y be normed spaces. A linear operator $T : X \to Y$ is called an* isomorphism *if*

- *T is bijective,*
- *both T and T^{-1} are continuous (equivalently, bounded).*

Definition 3. *A linear operator $T : X \to Y$ is called an* isometry *if*

$$\|T(x)\| = \|x\| \quad \textit{for all } x \in X.$$

An isometry is always injective, but it may fail to be surjective.

Definition 4. *If T is a surjective isometry, then T is called a* unitary operator. *In this case, T^{-1} exists on Y and satisfies*

$$\|T^{-1}(y)\| = \|T^{-1}(T(x))\| = \|x\| = \|T(x)\| = \|y\|, \quad \forall\, y \in Y.$$

Remark 1. *In infinite-dimensional spaces, there exist non-unitary isometries (i.e., isometries that are not surjective).*

Definition 5 (Quotient Space)**.** *Let X be a normed space and $M \subset X$ a closed subspace. The* quotient space *X/M is defined as the set of cosets*

$$X/M := \{\, x + M : x \in X \,\}.$$

We can turn X/M into a normed space by defining the quotient norm *as*

$$\|x + M\| := \inf\{\|x + y\| : y \in M\}, \quad x \in X.$$

Notation 1. *For $x \in X$, we denote its coset by $\hat{x} := x + M$. The map*

$$\pi : X \to X/M, \quad \pi(x) = x + M$$

is called the quotient homomorphism.

Fact 1.

1. *$(X/M, \|\cdot\|)$ is a normed space, where*

$$\|x + M\| := \inf\{\|x + y\| : y \in M\}.$$

Indeed, if $\|x + M\| = 0$, then there exists a sequence $\{y_n\} \subset M$ such that $\|x + y_n\| \to 0$.

2. *If M is a proper subspace, then $\pi : X \to X/M$ is a bounded linear operator with $\|\pi\| = 1$.*

Proof. For any $y \in M$,

$$\|\pi(x)\| = \|x + M\| = \inf_{y \in M} \|x + y\| \le \|x + y\|.$$

In particular, for $y = 0$, we get $\|\pi(x)\| \le \|x\|$, hence $\|\pi\| \le 1$.

Now suppose $\|\pi\| < 1$. Then there exists $r < 1$ such that $\|\pi(x)\| < r$ for all $x \in X$. By definition, there exists a sequence $\{y_n\} \subset M$ such that

$$\|x + y_n\| \to \|x + M\| = \|\pi(x)\| < r.$$

Thus some $y \in M$ satisfies $\|x + y\| < r$. But then $\pi(x) = \pi(x + y)$ lies in the ball B_r in X/M. If $X/M \neq \{0\}$, this contradicts the assumption $\|\pi\| < 1$. Hence $\|\pi\| = 1$. □

3. *If X is a Banach space, then X/M is also a Banach space.*

Proof. Let $(\hat{x}_n)$ be a Cauchy sequence in X/M. Choose a subsequence $(\hat{x}_{n_k})$ such that

$$\|\hat{x}_{n_{k+1}} - \hat{x}_{n_k}\| < 2^{-k}, \quad k = 1, 2, \ldots$$

Define

$$y_k := \hat{x}_{n_{k+1}} - \hat{x}_{n_k}, \qquad \|y_k\| < 2^{-k}.$$

Then

$$\sum_{k=1}^{\infty} \|y_k\| < \infty,$$

so the series $\sum y_k$ converges absolutely in X/M.

Lifting to X, we can represent $y_k = \hat{z}_k$ with $z_k \in X$ and $\|z_k\| < 2^{-k}$. Since X is complete, $\sum z_k$ converges in X. Denote its sum by z.

Therefore,

$$\hat{x}_{n_{k+1}} - \hat{x}_{n_1} = \sum_{j=1}^{k} y_j \to \hat{z} \quad \text{in } X/M.$$

Hence $(\hat{x}_n)$ converges in X/M, proving that X/M is complete. □

Property 1. *Let X be a vector space with a seminorm $\|\cdot\|$, and let X carry the topology induced by $\|\cdot\|$. Then*

$$M := \{x \in X : \|x\| = 0\}$$

is a closed subspace of X, and the quotient space X/M with the norm

$$\|x + M\| := \|x\|, \quad x \in X,$$

is a normed space.

Proof. First, M is a subspace:

- If $x \in M$ and $\lambda \in \mathbb{K}$, then $\|\lambda x\| = |\lambda|\|x\| = 0$, hence $\lambda x \in M$.
- If $x, y \in M$, then $\|x + y\| \leq \|x\| + \|y\| = 0$, so $x + y \in M$.

To see that M is closed: suppose $x_n \in M$ and $x_n \to x$ in X (with respect to $\|\cdot\|$). Then

$$|\|x_n\| - \|x\|| \leq \|x_n - x\| \to 0,$$

but $\|x_n\| = 0$ for all n, so $\|x\| = 0$ and $x \in M$. Thus M is closed.

Finally, define on X/M:

$$\|x + M\| := \|x\|.$$

This is well-defined: if $x + M = y + M$, then $x - y \in M$, hence $\|x - y\| = 0$, which implies $\|x\| = \|y\|$.

Thus $\|\cdot\|$ on X/M is a norm, and $(X/M, \|\cdot\|)$ is a normed space. □