functional analysis study note

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Property 1. *Let X, Y be normed spaces and $T \in B(X, Y)$ (i.e., a bounded linear operator). Then:*

1. *The kernel of T, defined by*
$$\ker(T) := \{x \in X : T(x) = 0\},$$
is a closed subspace of X.

2. *There exists a unique $S \in B(X/\ker(T), Y)$ such that*
$$T = S \circ \pi,$$
where $\pi : X \to X/\ker(T)$ is the quotient map. Moreover, $\|S\| = \|T\|$.

Proof.

1. We have
$$\ker(T) = T^{-1}(\{0\}),$$
which is closed since T is continuous.

2. Define
$$S : X/\ker(T) \to Y, \qquad S(x + \ker(T)) := T(x).$$
This is well-defined: if $x + \ker(T) = x' + \ker(T)$, then $x - x' \in \ker(T)$, hence $T(x) = T(x')$. Clearly, S is linear.

   Since $T = S \circ \pi$, where $\pi : X \to X/\ker(T)$ is the quotient map, we have for any $x \in X$:
$$\|T(x)\| = \|S(\pi(x))\|.$$
Thus,
$$\|T(x)\| = \|S(\pi(x))\| \leq \|S\| \, \|\pi(x)\| \leq \|S\| \, \|x\|,$$
which shows $\|T\| \leq \|S\|$.

   On the other hand, for any $x \in X$ and $y \in \ker(T)$,
$$\|S(\pi(x))\| = \|T(x)\| = \|T(x + y)\| \leq \|T\| \, \|x + y\|.$$
Taking the infimum over $y \in \ker(T)$ gives
$$\|S(\pi(x))\| \leq \|T\| \, \|\pi(x)\|.$$
Hence $\|S\| \leq \|T\|$.

Combining the two inequalities, we obtain $\|S\| = \|T\|$. ∎

Theorem 1. *Hahn-banach theorem: The X be a real vector space $M \subset X$ a subspace $P : X \to R$ a minkowski functional and let $f : M \to R$ be a linear functional satisfying $f(x) \leq p(x)$ for any $x \in M$ then exists linear functional $F : X \to R$ such that $F|_M = f$, $F(x) \leq p(x)$ for any $x \in X$*

example: $X = C^n$ some the linear functional on C^n: $f : C^n \to C$ there are $(a_1, a_2 \ldots a_n)$ that $f(x) = (x|a)_{c^n} = \sum_{i=1}^n x_i a_i$

$X^* = B(X, \mathbb{C})$, then $(X^*, \|\|)$ is a banach space $(X^*)^* = B((X^*)^*, \|\|)$ is also a banach space with respect to $\|\|$

example 1. *Let (X, Σ, μ) be a measure space and let $1 < p < \infty$. Set q by $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$(L^p(X, \Sigma, \mu))^* \cong L^q(X, \Sigma, \mu),$$

where the isometric isomorphism is given by

$$T : L^q \to (L^p)^*, \qquad T(g) = \Phi_g, \qquad \Phi_g(f) = \int_X fg \, d\mu \quad (f \in L^p).$$

Sketch of verification:

- *By Hölder's inequality, $|\Phi_g(f)| \leq \|g\|_q \|f\|_p$, so Φ_g is bounded and $\|\Phi_g\| \leq \|g\|_q$.*
- *One can show equality $\|\Phi_g\| = \|g\|_q$ (take appropriate normalized approximating functions to attain the norm).*
- *Surjectivity: every bounded linear functional on L^p arises from a unique $g \in L^q$; this is the standard representation theorem for L^p spaces (proofs use the Hahn–Banach/Riesz representation ideas and Radon–Nikodým arguments).*

For the endpoint $p = 1$: the map

$$L^\infty(X, \Sigma, \mu) \to (L^1(X, \Sigma, \mu))^*, \qquad g \mapsto \Phi_g(f) = \int_X fg \, d\mu,$$

is an isometric embedding. Moreover, this map is surjective (hence an isometric isomorphism) provided the measure space (X, Σ, μ) is σ-finite. In general (without σ-finiteness) the dual $(L^1)^*$ may be strictly larger than L^∞.

example 2. *Let X be a locally compact Hausdorff space and let $C_0(X)$ denote the Banach space of complex continuous functions vanishing at infinity equipped with the sup norm $\| \cdot \|_\infty$. Let $M(X)$ denote the space of finite (signed or complex) regular Borel measures (Radon measures) on X, equipped with the total variation norm $\|\mu\| := |\mu|(X)$. Then the Riesz–Markov theorem states that*

$$(C_0(X), \| \cdot \|_\infty)^* \cong (M(X), \| \cdot \|),$$

via the isometric isomorphism

$$\mu \in M(X) \longmapsto \Phi_\mu \in C_0(X)^*, \qquad \Phi_\mu(f) = \int_X f \, d\mu \quad (f \in C_0(X)).$$

Here "regular" (Radon) means the measure is inner regular on open sets and outer regular on Borel sets, etc.; the Riesz–Markov correspondence identifies continuous linear functionals on $C_0(X)$ with regular Borel measures and preserves the norms.

Remark 1. *If X is a second-countable locally compact Hausdorff space (for instance a separable metric locally compact space), then many measurability/regularity pathologies are avoided: finite Borel measures that are finite on compact sets are regular (Radon), so the above identification applies in the usual concrete sense.*

Let X be a complex vector space and $f : X \to \mathbb{C}$ a complex linear functional. Define the real linear functional

$$u(x) := \mathrm{Re}(f(x)), \quad x \in X.$$

Proposition 1. *We have the decomposition*

$$f(x) = u(x) - i\,u(ix), \qquad x \in X.$$

Conversely, given any real linear functional u on X, the map

$$f(x) := u(x) - i\,u(ix), \qquad x \in X,$$

is complex linear. Moreover, if u is continuous, then f is continuous and

$$\|f\| = \|u\|.$$

Proof. If f is complex linear, then clearly f is also real linear. For $x \in X$,

$$f(ix) = if(x) \quad \Rightarrow \quad \mathrm{Im}(f(x)) = -u(ix).$$

Thus

$$f(x) = u(x) + i\,\mathrm{Im}(f(x)) = u(x) - iu(ix).$$

Conversely, suppose u is real linear and set $f(x) := u(x) - i\,u(ix)$. For $x, y \in X$,

$$f(x+y) = u(x+y) - iu(i(x+y)) = f(x) + f(y).$$

For $a + ib \in \mathbb{C}$,

$$f((a+ib)x) = u(ax + ibx) - iu(i(ax + ibx)).$$

Using real linearity of u,

$$= au(x) + bu(ix) - i\big(au(ix) - bu(x)\big) = (a+ib)\big(u(x) - iu(ix)\big) = (a+ib)f(x).$$

So f is complex linear.

For the norms, note that

$$|u(x)| = |\mathrm{Re}(f(x))| \le |f(x)| \quad \Rightarrow \quad \|u\| \le \|f\|.$$

On the other hand, for any $x \neq 0$, choose $\theta \in \mathbb{R}$ so that $e^{-i\theta} f(x) \in \mathbb{R}$. Then

$$|f(x)| = |e^{-i\theta} f(x)| = \big|u(e^{-i\theta} x)\big| \le \|u\|\,\|e^{-i\theta} x\| = \|u\|\|x\|.$$

So $\|f\| \le \|u\|$. Together, $\|f\| = \|u\|$. $\square$

Definition 1. *Let X be a real vector space. A* Minkowski functional *on X is a function $p : X \to \mathbb{R}$ such that:*

1. *(Subadditivity)* $\quad p(x+y) \leq p(x) + p(y), \quad \forall x, y \in X.$
2. *(Positive homogeneity)* $\quad p(\lambda x) = \lambda p(x), \quad \forall x \in X,\ \lambda \geq 0.$

Remark 2. *It follows immediately that $p(0) = 0$. Indeed,*

$$0 = p(0) = p(x + (-x)) \leq p(x) + p(-x).$$

By positive homogeneity, $p(-x) = p((-1)x) = (-1)p(x)$ does not apply since $\lambda \geq 0$ is required. However, from subadditivity,

$$0 = p(0) \leq p(x) + p(-x) \quad \Rightarrow \quad p(-x) \geq -p(x).$$

Combining this with subadditivity in the other direction yields

$$p(-x) \leq p(x).$$

Hence $|p(-x)| \leq p(x)$, so in particular $p(-x) \leq 2p(x)$. This shows that p behaves like a seminorm, except that symmetry is not required.

Theorem 2. *theorem: Norms and semi-norms are minkowski functional:*

Proof. we use Zorn's lemma, let $M \subset X$ be a subspace and let fix $x \in \frac{X}{M}$. we extend f to $M + Rx$ by functional g such that $g(z) \leq p(z)$ for any $z \in M + Rx$ ($g(y) = f(y)$ if $y \in M$)

Since $z = y + \lambda x$ we want have $g(z) = f(y) + \lambda\alpha, \alpha \in \mathbb{R}$

$$f(y_1) + f(y_2) = f(y_1 + y_2) \leq p(y_1 + y_2) \leq p(y_1 - x) + p(x + y_2)$$

$$f(y_1) - p(y_1 - x) \leq p(y_2 + x) - f(y_2)$$

Thus, $\sup\{f(y) - p(y - x) | y \in M\} \leq \inf\{p(y + x) - f(y) | y \in M\}$.

suppose exists some α between them, then we set $g(y + \lambda x) = f(y) + \lambda x$ then $g|_M = f$. let $\lambda > 0$

$$g(y+\lambda x) = g(\lambda(\frac{y}{\lambda}+x)) = \lambda(f(\frac{y}{\lambda})+\alpha) \leq p(\frac{y}{\lambda}+\alpha)+f(\frac{y}{\lambda}) \leq \lambda(p(\frac{y}{\lambda}+x) = p(y+\lambda x)$$

similar for $\lambda < 0$

Then consider $\mathbb{F} = \{(F, Y) | F$ is extension of f, i.e. $M \subset Y \& F|_M = f, F(z) \leq p(z)\}$

Then $(F_1, Y_1) \preceq (F_2, Y_2)$ iff $Y_1 \subset Y_2$ and $F_2|_{Y_1} = F_1$

$\mathbb{F} \neq \emptyset$ since $(f, M) \in \mathbb{F}$ if we have a linearly ordered subfamily of $\mathbb{F}$, say $\{F_\alpha\}$

by zorn lemma, exists maximal and $Y = X$, otherwise exists some $x \in X - Y$, then construct new Y including x, then new Y is larger contradiction to the Y is maximal.

case if $p : X \to \mathbb{R}$ is a semi-norm, then condition $f \leq p$ on M is equiped to the $|f| \leq p$ on M $\square$

Theorem 3. *Hahn Banach complex version: Let X be a complex vector space with $P : X \to \mathbb{R}$ a semi-norm, $M \subset X$ a subspace $f : M \to \mathbb{C}$ a complex linear functional such that $|f(x)| \le P(x)$ for any $x \in M$*

then, exists $F : X \to \mathbb{C}$ complex linear functional such that $F|_M = f$, $|F(x)| \le P(x)$ for any $x \in X$

Proof. let $u = Re(f)$, then u is $\mathbb{R}$ linear functional on X, $|u(x)| \le |f(x)| \le P(x)$ for any $x \in M$.

Thus, by Hahn Banach, there exists a real-linear functional $u : X \to \mathbb{R}$ such that $u(x) \le P(x)$ for any $x \in X$.

Set $F(x) = u(x) - iu(ix)$, then $F|_M = f$ and we want to show $F(x) = u(x) - iu(ix) \le P(x)$

Recall that $F(x) = e^{it}|F(x)|$, i.e.

$$|F(X)| = F(e^{it}x) = u(e^{-it}x) \le P(e^{-it}x) = |e^{-it}|P(x) = P(x)$$

□

Theorem 4. *Let X be a normed space:*

1. if $x \neq 0$, $x \in X$ then exists $f \in X^*$, $\|f\| = 1$ and $f(x) = \|x\|$

2.The bounded linear functional of X separate the points of X

3.Let $M \subset X$ be a closed subspace, let $x \notin M$, then exists $f \in X^*$ such that $f(x) \neq 0$ $f|_M = 0$, Moreover, $\delta = dist(x, M)$, Then exists $f \in X^*$ $f(x) = \delta$, $f|_M = 0, \|f\| \le 1$

4.if $x \in M$, define $\hat{x} : X^* \to \mathbb{C}$ continuous linear functional by $\hat{x}(f) = f(x)$, $f \in X^*$ then $\hat{x} = (X^*)^*$ and $\|\hat{x}\| = \|x\|$, then map $x \in X \to \hat{x} \in (X^*)^*$ is an isometric isomorphism of normed space (may no be onto).

Proof. proof of 1: let $x \neq 0$, $x \in X$ $M = \mathbb{C}x$, $P(x) = \|x\|$.

define $f(\lambda x) = \lambda\|x\|$, $\lambda \in \mathbb{C}$. Note that $|f(\lambda x)| = |\lambda|\|x\| = \|\lambda x\|$

By Hahn Banach, exists an $F : X \to \mathbb{C}$ linear functional such that $F|_M = f$ i.e. $F(x) = f(x) = \|x\|$ and $|F(y)| \le \|y\|$ for any $y \in X$

hence, F is continuous and $||F|| \le 1$

$|F(x)| = \|x\| \le \|F\|\|x\|$, so $\|F\| \ge 1$

we have $\|F\| = 1$ □

Proof. proof of 2: let $x, y \in X$ and $x \neq y$ then exists $f \in X^*$ such that $f(x - y) = \|x - y\|$ by 1. □

Proof. proof of 3:

$\delta = dist(x, M) = \inf\{\|x - y\| | y \in M\}$

Consider $M + \mathbb{C}x$ define $f(y + \lambda x) = \lambda\delta$ for any $y \in M, \lambda \in \mathbb{C}$

$$\|y+\lambda x\| = \|\lambda(\frac{y}{\lambda}+x)\| = |\lambda|\|\frac{y}{\lambda}+x\| \ge |\lambda| \inf\{\|z+x\| | z \in M\} = |\lambda|\delta = |f(y+\lambda x)|$$

by Hahn Banach, there exist $F : X \to \mathbb{C}$ linear functional such that $F|_M = f$ and $|F(Z)| \le \|z\|$ for any $z \in X$.

Hence $\|F\| \le 1$ and $F(y) = 0$ for any $y \in M$ and $F(x) = f(x) = \delta$ □

Proof. proof of 4:

$|\hat{x}(f)| = |f(x)| \leq \|f\|\|x\|$ for any $f \in X^*$.

thus, $\|\hat{x}\| = \sup_{\|f\|=1} |f(x)| \leq \|x\|$

By 1, exists $f \in X^*$ such that $f(x) = \|x\|, \|f\| = 1$ hence $|\hat{x}(f)| = |f(x)| = \|x\| \leq \|\hat{x}\|\|f\| = \|\hat{x}\|$

We have $\|\hat{x}\| = \|x\|$

Then, $x \in X \to \hat{x} \in \hat{X} \subset (X^*)^*$ is a isometric isomorphism onto. □

Remark 3. *by the previous theorem, every normed space X is naturally isomorphic to subspace of $(X^*)^*$*

Remark 4. *if X is not a Banach Space, then $\hat{X} \subsetneq X^{**}$*