0.1 Measurability

lemma 1. *A set E in R^n is measurable if and only if for any $\varepsilon > 0$, there exists a closed set $F \subset E$ such that $\mu^*(E - F) < \varepsilon$*

Proof. E is measurable if and only if E^C is measurable.

1. if for any $\varepsilon > 0$ there exists an open G such that $E^C \subset G$ and $\mu^*(G - E^C) < \varepsilon$, E^c measurable, then E measurable.

2. if $F = G^C$ is closed, then $F \subset E$, $\mu^*(E - F) < \varepsilon$

Note that: $G - E^c = E - F$, these two conditions are equivalent, enough to show E is measurable. □

theorem 1. *if $\{E_k\}$ is a countable collection of disjoint measurable sets, then*

$$|\bigcup_k E_k| = \sum_k |E_k|$$

Proof. $\sum |E_k| \geq |\bigcup E_k|$ is always true, need to show the opposite direction:

First, suppose each E_k is bounded. Given $\varepsilon > 0$ and we can choose closed $F_k \subset E_k$ with $|E_k - F_k| < \varepsilon 2^{-k}$.

Then,$|E_k| \leq |F_k| + \varepsilon 2^{-k}$.

Since E_k are bounded and disjoint, the F_k are compact and disjoint. Therefore,

$$|\bigcup E_k| \geq \sum |F_k| \geq (|E_k| - \varepsilon 2^{-k}) = \sum |E_k| - \varepsilon$$

Thus, we got $|\bigcup E_k| \geq \sum |E_k|$.

For the unbounded case: construct a partition S_i of R^n, and restrict each E_k in the S_i. Thus, each $E_k = \bigcup E_{k,i}$ and turn it into the bounded case. □

corollary 1. *If $\{I_k\}$ is a sequence of nonoverlapping intervals, then $|\bigcup I_k| = \sum |I_k|$*

corollary 2. *Suppose E_1 and E_2 are measurable, $E_2 \subset E_1$, and $|E_2| < \infty$ then $|E_1 - E_2| = |E_1| - |E_2|$.*

Proof. this simply followed by $|E_1| = |E_2| + |E_1 - E_2|$ the union of disjoint sets, holds when $|E_2| < \infty$ □

theorem 2. *Let $\{E_K\}$ be a sequence of measurable sets:*

(1) if E_k monotone increasing to E, then $\lim_{k\to\infty} |E_k| = |E|$

(2) IF e_K monotone decreasing to E, and $|E_k| < \infty$ for some k, then $\lim_{k\to\infty} |E_k| = |E|$

Proof. (1) Define

$$D_1 := E_1, \qquad D_k := E_k \setminus E_{k-1} \quad (k \geq 2).$$

Then $\{D_k\}_{k\geq 1}$ are pairwise disjoint and for every n,

$$E_n = \bigsqcup_{j=1}^{n} D_j, \qquad E = \bigcup_{n=1}^{\infty} E_n = \bigsqcup_{j=1}^{\infty} D_j.$$

By (finite) additivity for disjoint unions,

$$|E_n| = \sum_{j=1}^{n} |D_j| \quad \text{and} \quad |E| = \sum_{j=1}^{\infty} |D_j|.$$

Letting $n \to \infty$ and using monotone convergence of series of nonnegative terms,

$$\lim_{n\to\infty} |E_n| = \lim_{n\to\infty} \sum_{j=1}^{n} |D_j| = \sum_{j=1}^{\infty} |D_j| = |E|.$$

(2) Assume $|E_m| < \infty$ for some m. For $k \geq m$ set

$$F_k := E_m \setminus E_k.$$

Then F_k monotone increasing to $F := E_m \setminus E$ and each F_k is measurable. By part (1),

$$\lim_{k\to\infty} |F_k| = |F| = |E_m \setminus E|.$$

Since $E_k \subset E_m$ for $k \geq m$, finite additivity gives

$$|F_k| = |E_m \setminus E_k| = |E_m| - |E_k|.$$

Hence,

$$|E_m| - \lim_{k\to\infty} |E_k| = \lim_{k\to\infty} \big(|E_m| - |E_k|\big) = \lim_{k\to\infty} |F_k| = |E_m \setminus E| = |E_m| - |E|.$$

Canceling $|E_m|$ yields $\lim_{k\to\infty} |E_k| = |E|$, as required. □

theorem 3. *if E_k monotone increasing to E, then* $\lim_{k\to\infty} \mu^*(E_k) = \mu^*(E)$

Note: we don't need any measurable restriction. This is a theorem for any set equipped with outer measure.

Proof. For each k,let H_k be a measurable set such that $E_k \subset H_k$ and $|H_k| = \mu(E_k)$.

Let $V_m = \bigcap H_k$. Since the V_m are measurable and increase to $V = \bigcup V_m$.

By the theorem above, we have $\lim_{m\to\infty} |V_m| = |V|$. Since $E_m \subset V_m \subset H_m$, we have $\mu * (E_m) \leq |V_m| \leq |H_m| = \mu^*(E_m)$.

Thus, $|V_m| = \mu^*(E)$ and $\lim_{m\to\infty} \mu^*(E_m) = |V|$.

More, $V = \bigcup V_m \supset \bigcup E_m = E$. We have $\lim_{m\to\infty} \mu^*(E_m) \geq \mu^*(E)$.

The opposite direction: Since $E_m \subset E$, $\lim_{m\to\infty} \mu^*(E_m) \leq \mu^*(E)$

□

theorem 4. *for set E:*

E is measurable if and only if $E = H - Z$ where H is type G_δ and $|Z| = 0$

E is measurable if and only if $E = H \cup Z$ where H is type F_σ and $|Z| = 0$

the proof is very similar to the proof of every closed set F is measurable. And from (1) to (2), use the theorem the complement of measurable set is measurable.

theorem 5. *(Caratheodory Definition) A set E is measurable if and only if for every set A:*

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A - E)$$

Proof. Suppose E is measurable. For any A, there exists H be G_δ set such that $A \subset H$ and $\mu^*(A) = |H|$.

Since $H = (H \cap E) \cup H(\setminus E)$ and they are 2 measurable disjoint sets: $\mu^*(A) = |H \cap E| + |H \setminus E| \geq \mu^*(A \cap E) + \mu^*(A \setminus E)$

More, by the triangle inequality: $\mu^*(A \cap E) + \mu^*(A \setminus E) \geq \mu^*(A)$;

We have $\mu^*(A \cap E) + \mu^*(A \setminus E) = \mu^*(A)$;

The opposite direction: suppose E: $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A - E)$.

If $\mu^*(E) < \infty$, there exists H is a G_δ set such that $E \subset H$ and $|H| = \mu^*(E)$.

Then, We can have $|H| = \mu^*(E) + \mu^*(H \setminus E)$ by assumption. Thus, we have $Z = H - E$ is measurable since $|Z| = \mu^*(Z) = 0$, so E is measurable.

If $\mu^*(E) = \infty$, construct a partition S_i of R^n, and restrict each E_k in the S_i. Thus, each $E_k = \bigcup E_{k,i}$ and turn it into the bounded case.

□

corollary 3. *If E is a measurable subset of A, then $\mu^*(A) = |E| + \mu^*(A \setminus E)$.*

0.2 Lipschitz

Definition 1. *if there exists a constant c, such that $|T(x) - T(x_0)| \leq c|x - x_0|$. Then we call this T is Lipschitz.*

remark: more detailed properties will be discussed in functional analysis notes.

theorem 6. *if $y = Tx$ is a Lipschitz transformation, then T maps measurable sets into measurable sets.*

theorem 7. *Let T be linear transformation of R^n and Let E be measurable. Then $|T(E)| = |\det(T)||E|$*

0.3 Nonmeasurable set

lemma 2. *Let E be a measurable subset of R with $|E| > 0$. Then the set of difference $\{d : d = x - y, x \in E, y \in E\}$ contains an interval centered at 0.*

This can be intuitively explained as If a measurable set $E \subset \mathbb{R}$ has positive Lebesgue measure, then if we shift it slightly (a little to the left or right), it will always overlap with itself. Moreover, when the shift is small enough, the overlap still has positive measure. That means all those "small" shifts can be written as $x - y$ (a point of E minus another point of E), so they belong to the difference set $E - E$. Therefore, $E - E$ contains an open interval around 0.

theorem 8. *Vitali: There exist nonmeasurable sets.*

Proof. We define $[x]$ is the equivalent class that $y \in [x]$ if $x - y = q$ where q is rational number. Then $E_x = \{x + q : q \in Q\}$.

E_x and E_y are either identical or disjoint(property of equivalent class).

$[x]$ is a partition of R (property of equivalent class).

The number of E_x is uncountable. Since each $[x]$ is countable inside because Q is countable, thus if there are only countable many $[x]$, $R = |[x]| \times [x]$ is countable. So there are uncountable many E_x.

Then, use Zermelo's axiom, let E be a consisting of exactly one element from each distinct equivalence class. Since any 2 points of E must be differ by irrational number. Thus, the set $\{d : d = x - y, x \in E, y \in E\}$ can not contain any interval.

By the lemma above, we know E is either nonmeasurable or $|E| = 0$. Since the union of the translates of E by every rational number is R, so $|E| \neq 0$, which means E is not measurable. □

corollary 4. *Any set in R with positive outer measure contains a nonmeasurable set.*

Proof. Let A satisfy $\mu^*(A) > 0$, and let E be the nonmeasurable set constructed before. For any rational number q, E_q denote the E translate by q.

Thus, E_q are disjoint and $\bigcup E_q = R$.

Thus, $A = \bigcup (A \cap E_q)$. Since $\mu^*(A) > 0$, there exists some q such that $\mu^*(A \cap E_q) \neq 0$. Thus, there exists $A \cap E_q$ is not measurable, since it is subset of E and can not contain any interval. □