functional analysis study note

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0.1 Weak topology on Hilbert Space

Definition 1. *A net (ξ_i) converge to ξ in weak topology if and only if $(\xi_i|\eta) \to (\xi|\eta)$ for any $\eta \in H$.*

Definition 2. *Note $H = H^*$ for Hilbert space, we have*

$$V(\xi_0, \eta_1, \dots \eta_n, \varepsilon) = \{\xi \in H | |((\xi - \xi_0)|\eta_i)| < \varepsilon\}$$

for $\eta_i \in H, \varepsilon > 0$

Hence, by Alaoglu, the unit ball $B_1 = \{\xi \in H | \|\xi\| \leq 1\}$ is compact in the weak topology.

Note: if $\dim H = \infty$, then H is not locally compact in norm topology, weak topology is strictly weaker than norm topology.

0.2 Orthonormal Basis

Definition 3. *Let H be a Hilbert space, a subset $B = \{\xi_i\}$ is called ONB if $(\xi_i|\xi_j) = \delta_{ij}$*

B is ONB if B is an ONS and span(B) is dense in H.

Note ONB is not a basis if $\dim H = \infty$.

Definition 4. *Gram-Schmidt procedure: Let (η_n) be a sequence of linearly independent vector in H.*

define: $\xi_1 = \frac{\eta_1}{\|\eta_1\|}$

$\hat{\xi}_{m+1} = \eta_{m+1} - \sum_{j=1}^{m} (\eta_{m+1}|\xi_j)\xi_j$

Then, $\xi_{m+1} = \frac{\hat{\xi}_{m+1}}{\|\hat{\xi}_{m+1}\|}$

Then $\{\xi_n\}$ is an ONS such that span$\{\xi_1, \xi_2 \dots \xi_n\}$ = span$\{\eta_1, \eta_2 \dots \eta_n\}$ for any n.

Also, easy to prove $(\xi_i|\xi_j) = \delta_{ij}$

corollary 1. *Let $M \subset H$ be a finite dim subspace, then let $\{\xi_1, \dots \xi_n\}$ be an ONB of M. Then an Orthognal project $p : H \to M$ is given by $p(\xi) = \sum_{j=1}^{n} (\xi|\xi_j)\xi_j$.*

Proof. Check $(\xi - p(\xi)) \perp M$.

$$(\xi - p(\xi)|\xi_k) = (\xi|\xi_k) - \sum_{j=1}^{n}(\xi|\xi_j)(\xi_j|\xi_k) = 0$$

□

Theorem 1. *Every Hilbert Space has an ONB*

Proof. Case 1: $H = \{0\}$, $B = \emptyset$

Case 2: $H \neq \{0\}$, there exists $\xi \neq 0$.

Then $\{\frac{\xi}{\|\xi\|}\}$ is an ONS, then use Zorn lemma to get the sequence can generate whole space. □

Theorem 2. *Bessel's inequality:*

H be a Hilbert space, (ξ_α) ONS, then

$$\sum_{\alpha \in A} |(\xi|\xi_\alpha)|^2 \leq \|\xi\|^2, \forall \xi \in H$$

More, $\{\alpha \in A | (\xi|\xi_\alpha) \neq 0\}$ is at most countable.

Proof. Let $F \subset A$ be finite, then

$$0 \leq \|\xi - \sum_{\alpha \in F}(\xi|\xi_\alpha)\xi_\alpha\|^2 = \|\xi\|^2 + \sum_{\alpha \in F} |(\xi|\xi_\alpha)|^2 - 2Re(\xi| \sum_{\alpha \in F}(\xi|\xi_\alpha)\xi_\alpha)$$

$$= \|\xi\|^2 + \sum_{\alpha \in F} |(\xi|\xi_\alpha)|^2 - 2\sum_{\alpha \in F} |(\xi|\xi_\alpha)|^2$$

$$= \|\xi\|^2 - \sum_{\alpha \in F} |(\xi|\xi_\alpha)|^2$$

Since $\sum_{\alpha \in A} |(\xi|\xi_\alpha)|^2$ is bounded, so the non-zero term at most be countable. □

Theorem 3. *Let $(\xi_\alpha) \in A$, be an ONS on H, then TFAE:*

1. If $(\xi|\xi_\alpha) = 0, \forall \alpha \in A$, then $\xi = 0$

2. Parseval identity:

$$\|\xi\|^2 = \sum_{\alpha \in A} |(\xi|\xi_\alpha)|^2, \forall \xi \in H$$

3. $\forall \xi \in H$, we have $\xi = \sum_{\alpha \in A}(\xi|\xi_\alpha)\xi_\alpha$ where at most countably $(\xi|\xi_\alpha) \neq 0$ and the series converges in $\|\|$.

Remark 1. *the theorem is a charaterization of when ONS is an ONB.*

Proof. beginproof We prove the equivalence in the order: (1) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

Step 1: (1) $\Rightarrow$ (3)

Assume (1) holds: the only vector orthogonal to all ξ_α is zero.

Let $\xi \in H$. By Bessel's inequality, we have

$$\sum_{\alpha \in A} |(\xi \mid \xi_\alpha)|^2 \leq \|\xi\|^2 < \infty.$$

This implies that the set $\{\alpha \in A : (\xi \mid \xi_\alpha) \neq 0\}$ is at most countable (since an uncountable sum of positive numbers diverges unless all but countably many terms are zero).

Enumerate this countable set as $\{\alpha_n\}_{n=1}^\infty$ (finite or infinite). Define the partial sum

$$s_N = \sum_{n=1}^N (\xi \mid \xi_{\alpha_n}) \xi_{\alpha_n}.$$

The sequence (s_N) is Cauchy in H because for $M > N$,

$$\|s_M - s_N\|^2 = \sum_{n=N+1}^M |(\xi \mid \xi_{\alpha_n})|^2 \to 0 \quad \text{as } N, M \to \infty,$$

by the convergence of the series in Bessel's inequality. Since H is complete, $s_N \to s$ for some $s \in H$.

Let $\eta = \xi - s$. We claim $\eta = 0$. Indeed, for any $\alpha \in A$,

$$(\eta \mid \xi_\alpha) = (\xi \mid \xi_\alpha) - (s \mid \xi_\alpha).$$

But $(s \mid \xi_\alpha) = \lim_{N\to\infty} (s_N \mid \xi_\alpha)$. If $\alpha = \alpha_n$, then $(s_N \mid \xi_{\alpha_n}) \to (\xi \mid \xi_{\alpha_n})$ as $N \to \infty$. If $\alpha \notin \{\alpha_n\}$, then $(s_N \mid \xi_\alpha) = 0$ for all N, so $(s \mid \xi_\alpha) = 0 = (\xi \mid \xi_\alpha)$. Thus $(\eta \mid \xi_\alpha) = 0$ for all $\alpha \in A$. By (1), $\eta = 0$, so $\xi = s = \sum_{\alpha \in A} (\xi \mid \xi_\alpha) \xi_\alpha$.

This proves (3).

Step 2: (3) $\Rightarrow$ (2)

Assume (3): $\xi = \sum_{\alpha \in A} (\xi \mid \xi_\alpha) \xi_\alpha$ with norm convergence.

Let $\{\alpha_n\}$ be the countable set where $(\xi \mid \xi_{\alpha_n}) \neq 0$. Then

$$\xi = \lim_{N\to\infty} \sum_{n=1}^N (\xi \mid \xi_{\alpha_n}) \xi_{\alpha_n}.$$

Taking the norm squared and using the continuity of the inner product,

$$\|\xi\|^2 = \left\| \lim_{N\to\infty} \sum_{n=1}^N (\xi \mid \xi_{\alpha_n}) \xi_{\alpha_n} \right\|^2 = \lim_{N\to\infty} \left\| \sum_{n=1}^N (\xi \mid \xi_{\alpha_n}) \xi_{\alpha_n} \right\|^2.$$

By orthonormality,

$$\left\|\sum_{n=1}^{N} c_n \xi_{\alpha_n}\right\|^2 = \sum_{n=1}^{N} |c_n|^2,$$

so

$$\|\xi\|^2 = \lim_{N\to\infty} \sum_{n=1}^{N} |(\xi \mid \xi_{\alpha_n})|^2 = \sum_{\alpha\in A} |(\xi \mid \xi_\alpha)|^2.$$

Thus Parseval's identity holds. This proves (2).

Step 3: (2) $\Rightarrow$ (1)

Assume (2): $\|\xi\|^2 = \sum_{\alpha\in A} |(\xi \mid \xi_\alpha)|^2$ for all $\xi \in H$.

Suppose $(\xi \mid \xi_\alpha) = 0$ for all $\alpha \in A$. Then the right-hand side of Parseval's identity is 0, so

$$\|\xi\|^2 = 0 \implies \xi = 0.$$

This proves (1).

Thus, (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3), as required. □

Theorem 4. *A Hilbert Space H is separable if and only if H has a countable ONB if and only if every ONB is countable.*

Proof.

$a \to b$: Suppose H is separable. Then there exists a countable dense subset $\{x_n\}_{n=1}^{\infty}$. WLOG (x_n) is linearly independent

Apply the Gram–Schmidt to the sequence $\{x_n\}$.

This yields a countable orthonormal system $\{\xi_n\}_{n=1}^{\infty}$ such that

$$\mathrm{span}\{\xi_n\}_{n=1}^{N} = \mathrm{span}\{x_n\}_{n=1}^{N}$$

For any $N \in \mathbb{N}$. Thus, $\mathrm{span}\{\xi_n\} \perp = \mathrm{span}\{x_n\}^{\perp} = \{0\}$

Thus, $\{\xi_n\}$ is a countable ONB for H.

$b \to c$: Let H be a Hilbert space with one countable orthonormal basis. Since all ONB have the same cardinality for H, we have every ONB in H is countable

1. $dim(H) < \infty$ then an ONB is a basis, so the claim above is true.

2. $dim(H) = \infty$, let $\{\xi_\alpha\}_{\alpha\in A}$ and $\{\eta_\beta\}_{\beta\in B}$ be 2 ONB.

Let $A_\beta = \{\alpha \in A | (\xi_\alpha|\eta_\beta) \neq 0\}$, since $\|\eta_\beta\|^2 = \sum_{a\in A} |(\xi_\alpha|\eta_\beta)|^2$, we have that A_β is countable.

Claim $A = \bigcup_\beta A_\beta$

1. $A \supset \bigcup_\beta A_\beta$ is obvious.

2. Want to show $A \subset \bigcup_\beta A_\beta$. Let $\alpha \in A$ then $\|\xi_\alpha\|^2 = \sum_{\beta\in B} |(\xi_\alpha|\eta_\beta)|^2$ so there exists $\beta \in B$ such that $(\xi_\alpha|\eta_\beta) \neq 0$, i.e.e $\alpha \in A_\beta$.

If A_β is finite, add countable many element from A, we assume each A_β is countably infinite.

Suppose $B = \bigcup_{s\in S} B_s$ where B_s is countably infinite and

$$|B| = |S \times N| = |N| \text{ if } |S| < \infty, \quad |B| = |S \times N| = |S| \text{ if } |S| = \infty$$

since a countable set can be written as a countable infinite union of disjoint countable sets, we have $|B| = |S|$ implies $B = \bigcup_{s\in S} B_s = \bigcup_{\beta\in B} B_\beta$

Then the map $B_\beta \to A_\beta$ is bijection. Thus, the map $B \to A$ is well defined and $|A| = |B|$

$c \to a$: Suppose every orthonormal basis of H is countable. Then in particular, there exists a countable orthonormal basis $\{\xi_n\}_{n=1}^{\infty}$

then $\mathbb{Q} + i\mathbb{Q}$ span of $\{\xi_n\}_{n=1}^{\infty}$ is a countable dense subset.

Therefore, H is separable.

$\square$