# functional analysis study note

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**Definition 1** (Seminorm)**.** *Let $X$ be a vector space over $\mathbb{C}$. A* seminorm *is a function $\|\cdot\| : X \to [0,\infty)$ such that for all $x, y \in X$ and $\lambda \in \mathbb{C}$:*

$$\|\lambda x\| = |\lambda|\,\|x\|,$$
$$\|x+y\| \le \|x\| + \|y\|.$$

*The difference between a seminorm and a norm is that for a norm we additionally require*

$$\|x\| = 0 \iff x = 0.$$

**Definition 2** (Normed Space)**.** *If $\|\cdot\|$ is a norm on $X$, then $(X, \|\cdot\|)$ is called a* normed space*. Equipped with the metric*

$$d(x,y) = \|x-y\|,$$

*the norm induces a metric space structure on $X$.*

*A normed space $(X, \|\cdot\|)$ is called a* Banach space *if it is complete with respect to this metric.*

**Definition 3** (Banach Algebra)**.** *Let $A$ be a complex algebra. If $(A, \|\cdot\|)$ is a Banach space and*

$$\|xy\| \le \|x\|\,\|y\| \quad \textit{for all } x, y \in A,$$

*then $A$ is called a* Banach algebra*.*

*If $A$ is a unital, then $\|I\| = 1$.*

**Definition 4** (Involutive Banach Algebra)**.** *If $A$ is a Banach algebra with involution $*$, then $(A, \|\cdot\|, *)$ is called an* involutive Banach algebra*.*

*The involution $*$ satisfies:*

1. $(a^*)^* = a$,
2. $(\lambda a + b)^* = \overline{\lambda} a^* + b^*$,
3. $(ab)^* = b^* a^*$.

*In addition, if $\|x^* x\| = \|x\|^2$ for any $x$, then $A$ is called a* $C^*$-algebra*. In this case, $\|a^*\| = \|a\|$.*

**Examples:**

1. $A = M_n(\mathbb{C})$ with involution being the $T^*$ of a matrix $T \in M_n(\mathbb{C})$ and the norm is the square root of the eigenvalue of $T^*T$.

   (It is unital: $\|I_n\| = 1$). For $n = 1$, $A = \mathbb{C}$, $\|\lambda\| = |\lambda|$, $\lambda^* = \overline{\lambda}$.

2. Let $X$ be a locally compact Hausdorff space, then

   $$C_0(X) = \{f : X \to \mathbb{C} \mid f \text{ is continuous and vanishes at } \infty\}.$$

   Recall: $f$ vanishes at $\infty$ if for any $\varepsilon > 0$, there exists a compact set $K \subset X$ such that

   $$|f(x)| < \varepsilon \quad \text{for all } x \in X \setminus K.$$

   The norm and involution are

   $$\|f\|_\infty = \sup |f(x)|, \quad f^*(x) = \overline{f(x)}.$$

   Then $(C_0(X), \|\cdot\|_\infty, *)$ is a $C^*$-algebra.

   If $X$ is compact Hausdorff, then

   $$C(X) = \{f : X \to \mathbb{C} \mid f \text{ continuous}\}$$

   is a unital $C^*$-algebra with respect to the same norm $\|\cdot\|$ and $f^* = \overline{f}$.

   **Remark:**

   (a) One can show every abelian $C^*$-algebra is isomorphic to $C(X)$ or $C_0(X)$ for a locally compact Hausdorff space.
   (b) Let $X, Y$ be compact Hausdorff spaces, then $X \simeq Y$ iff $C(X)$ is isomorphic to $C(Y)$.

3. $L^1(\mathbb{R}^n)$ with Lebesgue measure:

   Involution: $f^*(x) = \overline{f(-x)}$.

   Convolution: for $f, g \in L^1(\mathbb{R}^n)$,

   $$(f * g)(t) = \int_{\mathbb{R}^n} f(x)g(t - x)\, dx.$$

   We have $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$, so $(L^1, +, *, ^*)$ is a Banach algebra, but *not* a $C^*$-algebra since $\|f^* f\| \neq \|f\|^2$.

4. $(X, \Sigma, \mu)$ is a measure space:

   $$L^p(X, \Sigma, \mu) = \{f : X \to \mathbb{C} \mid f \text{ is measurable and } \|f\|_p = (\int_X |f|^p\, d\mu)^{1/p} < \infty\}.$$

   Define $f \sim g$ if $f = g$ a.e.

   If $p \geq 1$, we have the Minkowski inequality:

   $$\|f + g\|_p \leq \|f\|_p + \|g\|_p,$$

   hence $L^p(X, \Sigma, \mu)$ is a normed space (Banach space).

**Definition 5** (Semi-finite Measure). *Assume $\mu$ is* semi-finite *if for any $E \in \Sigma$ with $\mu(E) = \infty$, there exists $A \subset E$, $A \in \Sigma$, such that*

$$0 < \mu(A) < \infty.$$

**Definition 6** ($\sigma$-finite Measure). *$\mu$ is* $\sigma$-finite *if there exists a sequence $\{A_n\} \subset \Sigma$ such that*

$$\mu(A_n) < \infty \quad \text{for all } n \in \mathbb{N}, \quad \text{and} \quad \bigcup_{n=1}^{\infty} A_n = X.$$

**Definition 7** ($L^\infty$ Space).

$$L^\infty(X, \Sigma, \mu) = \{f : X \to \mathbb{C} \mid f \text{ is measurable and } \|f\|_\infty < \infty\},$$

*where the norm can be defined as*

$$\|f\|_\infty = \sup |f(x)|,$$

*or equivalently,*

$$\|f\|_\infty = \inf\{c \geq 0 \mid \mu(\{x \in X \mid |f(x)| > c\}) = 0\}.$$

*Two functions are considered equal in $L^\infty$ if and only if they are equal almost everywhere (a.e.):*

$$f = g \text{ in } L^\infty \iff f = g \text{ a.e.}$$

**Remark 1.** *Then $L^\infty((X, \Sigma, \mu), +, *, -,^* \|\cdot\|_\infty)$ is a $C^*$-algebra.*

*Note: $f \in L^\infty$ if and only if there exists a measurable function $g$ such that $g = f$ $\mu$-a.e., for example*

$$g = f \cdot \chi_E, \quad E = \{x \in X \mid |f(x)| \leq \|f\|_\infty\}.$$

*This allows every element of $L^\infty$ to be represented by a strictly bounded measurable function.*

## Normed Spaces and Equivalent Norms

**Definition 8** (Equivalent Norms). *Let $X$ be a vector space. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $X$ are called* equivalent *if there exist constants $c_1, c_2 > 0$ such that*

$$c_1\|x\|_1 \leq \|x\|_2 \leq c_2\|x\|_1, \quad \text{for all } x \in X.$$

**Remark 2.** *Recall that if $(X, \|\cdot\|)$ is a normed space, then*

$$d(x, y) = \|x - y\|$$

*defines a metric on $X$, and the associated metric space topology is called the* norm topology *on $X$.*

**Proposition 1.** *Equivalent norms generate the same topology. Conversely, if two norms generate the same topology, then they are equivalent.*

## Convergence of Series in Normed Spaces

**Definition 9** (Convergence of Series)**.** *Let $(X, \|\cdot\|)$ be a normed space, and let $\{x_n\} \subset X$. We say that the series $\sum_{n=1}^{\infty} x_n$* converges *to $x \in X$ if the sequence of partial sums*

$$S_N = \sum_{n=1}^{N} x_n$$

*satisfies*

$$\lim_{N\to\infty} S_N = x.$$

**Definition 10** (Absolute Convergence)**.** *The series $\sum_{n=1}^{\infty} x_n$ converges* absolutely *if*

$$\sum_{n=1}^{\infty} \|x_n\| < \infty.$$

**Theorem 1.** *Let $(X, \|\cdot\|)$ be a normed space. Then $X$ is complete (i.e., a Banach space) if and only if every absolutely convergent series in $X$ converges.*

*Proof.* **Forward direction:** Suppose $\sum_{n=1}^{\infty} \|x_n\| < \infty$, and let $S_N = \sum_{n=1}^{N} x_n$. For any $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for all $N > M > N_0$,

$$\|S_N - S_M\| = \left\| \sum_{n=M+1}^{N} x_n \right\| \le \sum_{n=M+1}^{N} \|x_n\| < \varepsilon.$$

Thus, $\{S_N\}$ is a Cauchy sequence and converges in $X$ since $X$ is complete.

**Converse direction:** Suppose $\{x_n\} \subset X$ is a Cauchy sequence. Fix $\varepsilon = 2^{-k}$; then there exists $N_k > 0$ such that for all $n > m > N_k$,

$$\|x_n - x_m\| < 2^{-k}.$$

Hence, we can choose a subsequence $x_{n_1}, x_{n_2}, \ldots$ such that

$$\|x_{n_{w+1}} - x_{n_w}\| < 2^{-w}.$$

Let $y_w = x_{n_{w+1}} - x_{n_w}$. Then

$$\sum_{w=1}^{\infty} \|y_w\| \le \sum_{w=1}^{\infty} 2^{-w} < \infty,$$

so $\sum_{w=1}^{\infty} y_w$ converges absolutely.

Moreover, for partial sums:

$$y_1 + y_2 + \cdots + y_w = (x_{n_2} - x_{n_1}) + \cdots + (x_{n_{w+1}} - x_{n_w}) = x_{n_{w+1}} - x_{n_1} \to y,$$

since absolute convergence implies convergence of partial sums.

Thus, the subsequence $x_{n_w}$ converges, and since $\{x_n\}$ is Cauchy, the whole sequence converges in $X$. □

## Bounded Linear Operators

**Definition 11** (Bounded Linear Operator)**.** *Let $X$ and $Y$ be normed spaces, and let $T : X \to Y$ be a linear map. We say that $T$ is* bounded *if there exists a constant $c > 0$ such that*

$$\|T(x)\| \le c\,\|x\|, \quad \textit{for all } x \in X.$$

**Theorem 2.** *For a linear operator $T : X \to Y$ between normed spaces, the following statements are equivalent:*

1. *$T$ is continuous on $X$.*
2. *$T$ is continuous at $0$.*
3. *$T$ is bounded.*

*Proof.* **1 $\implies$ 2:** Obvious.

**2 $\implies$ 3:** Let $V = \{y \in Y \mid \|y\| < 1\}$. Since $T(0) = 0$ and $T$ is continuous at 0, the set

$$T^{-1}(V)$$

is an open neighborhood of 0 in $X$.

Hence, there exists $\delta > 0$ such that

$$B_\delta(0) := \{x \in X \mid \|x\| \le \delta\} \subset T^{-1}(V),$$

which implies that $T(B_\delta(0)) \subset V$.

For any $x \neq 0$, write

$$\|T(x)\| = \left\| T\Big(\frac{\delta x}{\|x\|} \cdot \frac{\|x\|}{\delta}\Big) \right\| \le \frac{1}{\delta}\,\|x\| \left\| T\Big(\frac{\delta x}{\|x\|}\Big) \right\| \le \frac{1}{\delta}\|x\|,$$

since $\frac{\delta x}{\|x\|} \in B_\delta(0) \subset T^{-1}(V)$ and thus $\|T(\frac{\delta x}{\|x\|})\| \le 1$.

**3 $\implies$ 1:** If $T$ is bounded, say $\|T(x)\| \le C\|x\|$, then for any $x_1, x_2 \in X$,

$$\|T(x_1) - T(x_2)\| = \|T(x_1 - x_2)\| \le C\|x_1 - x_2\|,$$

so $T$ is Lipschitz continuous, hence continuous. ∎