functional analysis study note

Zehao Li

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0.1 Krein-Milman theorem

Definition 1. *The convex ball of a set $A \subset X$ is smallest convex subset of X that contains A:*
$Convex(A) = \cap_{A \subset C, C\, convex} C$

Definition 2. *Let X be a TVS such that X^* separates. Let $K \subset X$ be a compact and convex set.*

We say $S \subset K$ is an extreme set if whenever $x, y \in K$, and if $s = tx + (1-t)y$ for some $0 < t < 1$, and $s \in S$, then $x, y \in S$.

This is equivalent to if segment $[x, y] \cap S \neq \emptyset$, then $[x, y] \subset X$.

example 1. *K is an extreme set in K*

Theorem 1. *Kreim Milman theorem: Let X be a TVS, assume x^* separates the points of X, Let $K \subset X$ be a compact convex set then*

$$K = \overline{Convex(Ext(K))}$$

Proof. We proceed in several steps.

Step 1: Every nonempty closed convex subset of K contains an extreme point.

Let $\mathcal{F}$ be the family of all nonempty closed convex subsets $F \subseteq K$ such that F is *extreme* in K, meaning:

$$\text{if } x, y \in K, \lambda \in (0,1), \text{ and } \lambda x + (1-\lambda) y \in F, \text{ then } x, y \in F.$$

Note that $K \in \mathcal{F}$, so $\mathcal{F}$ is nonempty.

Partially order $\mathcal{F}$ by reverse inclusion: $F_1 \preceq F_2$ if $F_1 \supseteq F_2$.

Let $\{F_\alpha\}_{\alpha \in A}$ be a totally ordered chain in $\mathcal{F}$. Then the intersection $F = \bigcap_\alpha F_\alpha$ is nonempty because K is compact and each F_α is a closed subset of K. Moreover, F is convex and closed, and it is also an extreme set in K: suppose $\lambda x + (1-\lambda)y \in F$ for some $x, y \in K$, $\lambda \in (0,1)$. Then for each α, $\lambda x + (1-\lambda) y \in F_\alpha$, so $x, y \in F_\alpha$ since F_α is extreme. Hence $x, y \in F$. So $F \in \mathcal{F}$ and is a lower bound for the chain.

By Zorn's Lemma, $\mathcal{F}$ has a minimal element, say M.

Claim: M is a singleton.

Suppose not. Then there exist distinct points $x_1, x_2 \in M$. Since X^* separates points, there exists $\phi \in X^*$ such that $\phi(x_1) \neq \phi(x_2)$. Without loss of generality, assume $\phi(x_1) < \phi(x_2)$.

Define:

$$c = \min_{x \in M} \phi(x).$$

Since M is compact (closed subset of compact K) and ϕ is continuous, this minimum is attained. Let:

$$F = \{x \in M : \phi(x) = c\}.$$

Then F is a nonempty proper closed convex subset of M (since $x_2 \notin F$).

We show F is extreme in K. Suppose $x, y \in K$, $\lambda \in (0,1)$, and $\lambda x + (1-\lambda)y \in F$. Then $\lambda x + (1-\lambda)y \in M$, and since M is extreme in K, we have $x, y \in M$. Now,

$$\phi(\lambda x + (1-\lambda)y) = c.$$

But $\phi(x) \geq c$, $\phi(y) \geq c$, so equality implies $\phi(x) = \phi(y) = c$. Thus $x, y \in F$. So F is extreme in K, hence $F \in \mathcal{F}$, and $F \subsetneq M$, contradicting the minimality of M.

Therefore, M must be a singleton: $M = \{x_0\}$, and $x_0 \in \mathrm{Ext}(K)$.

Thus, every nonempty closed convex subset of K contains an extreme point of K.

Step 2: $K = \overline{\text{covex}}(\mathrm{Ext}(K))$

Let $C = \overline{\text{covex}}(\mathrm{Ext}(K))$. Clearly $C \subseteq K$ since K is closed and convex.

Suppose for contradiction that $C \subsetneq K$. Then there exists $x_0 \in K \setminus C$. Since C is closed and convex, and X^* separates points, by the Hahn–Banach separation theorem (geometric form), there exists $\phi \in X^*$ and $\alpha \in \mathbb{R}$ such that:

$$\phi(x_0) > \alpha \geq \phi(x) \quad \text{for all } x \in C.$$

Now consider the set:

$$F = \{x \in K : \phi(x) = \max_{y \in K} \phi(y)\}.$$

This is nonempty (by compactness of K and continuity of ϕ), closed, convex, and extreme in K (same argument as above). By Step 1, F contains an extreme point e of K. But then $e \in \mathrm{Ext}(K) \subset C$, so $\phi(e) \leq \alpha$. Yet $\phi(e) = \max_K \phi > \alpha$ (since $\phi(x_0) > \alpha$ and $x_0 \in K$), contradiction.

Therefore, $K \subseteq C$, so:

$$K = \overline{\text{covex}}(\mathrm{Ext}(K)).$$

$\square$

0.2 Distribution theory

Definition 3 (Schwartz Space $\mathcal{S}(\mathbb{R}^n)$)**.** *The **Schwartz space** $\mathcal{S}(\mathbb{R}^n)$ is the space of all smooth functions $\phi \in C^\infty(\mathbb{R}^n)$ such that for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$, the seminorms*

$$\|\phi\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta \phi(x)|$$

are finite. Elements of $\mathcal{S}(\mathbb{R}^n)$ are called rapidly decreasing functions*.*

Definition 4 (Tempered Distribution)**.** *A **tempered distribution** is a continuous linear functional on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.*

The space of all tempered distributions is denoted by

$$\mathcal{S}'(\mathbb{R}^n)$$

*and is called the **dual of the Schwartz space**.*

That is, $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ (or $\mathbb{R}$) is in $\mathcal{S}'(\mathbb{R}^n)$ if:

1. *(Linearity) $u(a\phi + b\psi) = au(\phi) + bu(\psi)$ for all $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$, $a, b \in \mathbb{C}$;*
2. *(Continuity) If $\phi_k \to \phi$ in $\mathcal{S}(\mathbb{R}^n)$ (i.e., $\|\phi_k - \phi\|_{\alpha,\beta} \to 0$ for all α, β), then $u(\phi_k) \to u(\phi)$.*

example 2.

$$\mathcal{F}f(\xi) = \int_{R^n} f(x) e^{-2\pi i \xi x} dx$$

is the Foureir transform of the function $f(\xi)$

Definition 5. *A function $\phi \in C^\infty(\mathbb{R}^n)$ is called a **test function** if it has **compact support**, i.e.,*

$$\mathrm{supp}(\phi) = \overline{\{x \in \mathbb{R}^n : \phi(x) \neq 0\}} \quad \textit{is compact.}$$

which is

$$C_c^\infty(K) = \{\phi \in C_c^\infty(R^n) | supp(f) \subset K\}$$

The space of all test functions is denoted

$$\mathcal{D}(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n).$$

Thus, each $C_c^\infty(K)$ is a Frechet space with semi-norm $\|f\|_\alpha = \sup_{x \in K} |\partial^\alpha f(x)|$

Then $f_n \to f$ converges in $C_c^\infty(K)$ if and only if $f_n, f \in K$ and $\partial^\alpha f_n \to \partial^\alpha f$ uniformly on K for all α.

Let $C_c^\infty(\Omega) = \bigcup_{K \subset \Omega} C_c^\infty(K)$, we have $C_c^\infty(\Omega)$ with induced limit topology, induced from $C_c^\infty(K)$

The topology is characterized by

1. $(f_j), f \subset C_c^\infty(\Omega)$ then $f_j \to f$ if exists $K \subset \Omega$ compact, such that $supp(f_j) \subset K$ and $f_n \to f$ converges in $C_c^\infty(K)$

2.If X a LCTVS, and $T : C_c^\infty(\Omega) \to X$ is a linear map, then T is continuous if $T|_{C_c^\infty(K)} : C_c^\infty(K) \to X$ is continuous for any compact $K \subset \Omega$

3.A linear map $T : C_c^\infty(\Omega) \to C_c^\infty(\hat{\Omega})$. If for any compact $K \subset \Omega$, there exists compact $\hat{K} \subset \hat{\Omega}$ such that $T(C_c^\infty(K)) \subset T(C_c^\infty(\hat{K}))$, and $T|_{C_c^\infty(K)} : C_c^\infty(K) \to C_c^\infty(\hat{K})$ is continuous.

This induced topology is Frechet.

$D(\Omega)$ be the space of test function. A distribution T is an element in D^* which is $D'(\Omega)$.

$D'(\Omega)$ with weak topology is a LCTVS

Definition 6 (Radon Measure)**.** *recall the Radon measure:*

Let $\Omega \subset \mathbb{R}^n$ be an open set. A Radon measure *on Ω is a Borel measure μ on Ω such that:*

1. $\mu(K) < \infty$ *for every compact set $K \subset \Omega$* (locally finite)*,*
2. $\mu(U) = \sup\{\mu(K) : K \subset U,\ K \textit{ compact}\}$ *for every open set $U \subset \Omega$* (inner regular)*,*
3. $\mu(B) = \inf\{\mu(U) : B \subset U,\ U \textit{ open}\}$ *for every Borel set $B \subset \Omega$* (outer regular)*.*

example 3. *1. Let μ be a Radon measure on Ω, $\psi \to \int_\Omega \psi d\mu$:*

$$|\int_\Omega \psi d\mu| \leq \mu(\Omega)\|\psi\|_\infty$$

defines an element in D'.

example 4. *Let $\alpha \in N^n$ $x_0 \in \Omega$ then $\psi \in D(\Omega) \to D^\alpha\psi(x_0)$ also defines an element in D'*

Fact 1. *Some operation extended from function to distribution:*

Let $\psi \in D(\Omega)$, then $f \in C^{|\alpha|}(\Omega), \psi \in D(\Omega)$, then

$$\int_\Omega (\partial^\alpha f)\psi dx = \int_\Omega f(\partial^\alpha \psi)dx$$

By partial integral, note that all boundary term vanishes since the support is compact.

If $T \in D'(\Omega)$, we define $D^\alpha = (\partial^\alpha)$ by $D^\alpha T(\psi) = (-1)^{|\alpha|}T(\partial^\alpha\psi)$. Then, $D^\alpha T \in D'(\Omega)$

example 5. *$f \in C(R^n)$, then define distribution $T_f(\psi) = \int_N f\psi dx$.*

The weak derivative is defined by $(D^\alpha f)\psi = (-1)^{|\alpha} \int_\Omega f(\partial^\alpha\psi)dx$

$D^\alpha f$ is a distribution.

Let $P(x_1, \ldots x_n)$

$$P(D) = \sum_{|\alpha|\leq k} a_\alpha D^\alpha$$

Let $T \in D'$ *define* $P(D)T = T(\sum_{|\alpha|\leq K}(-1)^{|\alpha|}D^\alpha\psi)$ *for* $\psi \in D'$

Suppose $f \in C(R^n)$*, then* $u \in C^k(R^n)$ *is a strict solution if* $P(D)u = f$*.*

Let $T \in D'$ *is called weak solution if* $P(D)T = f$ *in some distributions.*

So $T(\sum_{|\alpha|\leq K}(-1)^{|\alpha|}D^\alpha\psi) = \int_\Omega f\psi dx$ *has weak solution.*

There are regularity theorem in PDE that gives condition on different operator D that weakly allow one to deduce that a weak solution is strict solution.