0.1 Egorov and Lusin

theorem 1 (Egorov's Theorem)**.** *Suppose $\{f_n\}$ is a sequence of measurable functions that converges pointwise almost everywhere to a function f on X. Then for every $\varepsilon > 0$, there exists a measurable set $E \subset X$ with $\mu(E) < \varepsilon$ such that $\{f_n\}$ converges to f uniformly on $X \setminus E$.*

lemma 1. *Under the same hypotheses as in Egorov's theorem, let $\varepsilon, \eta > 0$. Then there exists a closed subset $F \subset E$ and an integer K such that*

$$|E \setminus F| < \eta \quad \textit{and} \quad |f(x) - f_k(x)| < \varepsilon \quad \textit{for all } x \in F \textit{ and } k > K.$$

Proof. Fix $\varepsilon, \eta > 0$. By Egorov's theorem, there exists a measurable set $A \subset E$ with $\mu(A) < \eta/2$ and an integer K such that

$$\sup_{x \in E \setminus A} |f_k(x) - f(x)| < \varepsilon \quad \text{for all } k \geq K.$$

(That is, $f_k \to f$ uniformly on $E \setminus A$.)

By inner regularity of μ, there exists a closed set $F \subset E \setminus A$ such that

$$\mu\big((E \setminus A) \setminus F\big) < \eta/2.$$

Then

$$\mu(E \setminus F) \leq \mu(A) + \mu\big((E \setminus A) \setminus F\big) < \eta/2 + \eta/2 = \eta.$$

Since $F \subset E \setminus A$, the uniform estimate persists on F:

$$|f_k(x) - f(x)| < \varepsilon \quad \text{for all } x \in F \text{ and all } k \geq K.$$

This proves the lemma. □

Proof. (proof of egorov using lemma above) Fix $\varepsilon > 0$. For each $m \in \mathbb{N}$, apply Lemma with parameters $\eta_m := \varepsilon 2^{-m}$ and $\delta_m := 1/m$ to obtain a closed set $F_m \subset E$ and an integer K_m such that

$$\mu(E \setminus F_m) < \eta_m = \varepsilon 2^{-m} \quad \text{and} \quad |f_k(x) - f(x)| < \delta_m = \tfrac{1}{m} \text{ for all } x \in F_m,\ k \geq K_m.$$

Set $F := \bigcap_{m=1}^{\infty} F_m$. Then $F \subset F_m$ for every m, hence for all m and all $k \geq K_m$,

$$\sup_{x \in F} |f_k(x) - f(x)| \leq \sup_{x \in F_m} |f_k(x) - f(x)| < \tfrac{1}{m}.$$

To see uniform convergence on F, let $\delta > 0$ be arbitrary and choose m with $1/m < \delta$. Then for all $k \geq K_m$,

$$\sup_{x \in F} |f_k(x) - f(x)| < \tfrac{1}{m} < \delta,$$

which proves $f_k \to f$ uniformly on F.

Finally, since $F = \bigcap_{m\geq 1} F_m$, we have

$$E \setminus F \;=\; E \setminus \bigcap_{m\geq 1} F_m \;=\; \bigcup_{m\geq 1} (E \setminus F_m).$$

By countable subadditivity,

$$\mu(E \setminus F) \;\leq\; \sum_{m=1}^{\infty} \mu(E \setminus F_m) \;<\; \sum_{m=1}^{\infty} \varepsilon 2^{-m} \;=\; \varepsilon.$$

Thus there is a set $F \subset E$ with $\mu(E \setminus F) < \varepsilon$ on which $f_k \to f$ uniformly. This completes the proof. □

Definition 1 (Continuity relative to F)**.** *Let $F \subset \mathbb{R}^n$ and $f : F \to \mathbb{R}$. We say that f is* continuous relative to F *if for every $x \in F$ and for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$y \in F,\; |y - x| < \delta \implies |f(y) - f(x)| < \varepsilon.$$

Equivalently, f is continuous as a map from the subspace F (with the relative topology) into $\mathbb{R}$.

Definition 2. *Let f be a function defined on a measurable set $E \subset \mathbb{R}^n$. We say that f has* property C *on E if for every $\varepsilon > 0$ there exists a closed set $F \subset E$ such that:*

1. *$|E \setminus F| < \varepsilon$,*
2. *f is continuous relative to F.*

lemma 2. *If $f : E \to \mathbb{R}$ is a simple measurable function, then f has* property C *on E; i.e., for every $\varepsilon > 0$ there exists a closed set $F \subset E$ with $\mu(E \setminus F) < \varepsilon$ such that f is continuous relative to F.*

Proof. Choose closed $F_j \subset E_j$ with $\mu(E_j \setminus F_j) < \varepsilon/N$. Then $F = \bigcup_{j=1}^{N} F_j$ is a finite union of closed sets, hence closed, and

$$\mu(E \setminus F) = \mu\Big(\bigcup_{j=1}^{N}(E_j \setminus F_j)\Big) \leq \sum_{j=1}^{N} \mu(E_j \setminus F_j) < \sum_{j=1}^{N} \frac{\varepsilon}{N} = \varepsilon.$$

To prove continuity of f relative to F, note F_j are pairwise disjoint. For fixed j, set

$$U_j := X \setminus \bigcup_{k\neq j} F_k,$$

which is open (finite union of closed sets has open complement). Then

$$F \cap U_j = \Big(\bigcup_{m=1}^{N} F_m\Big) \cap \Big(X \setminus \bigcup_{k\neq j} F_k\Big) = F_j.$$

Thus F_j is relatively open in F. Since $f \equiv a_j$ on F_j, for any $x \in F_j$ and any $\delta > 0$, choose $\eta > 0$ with $B(x, \eta) \subset U_j$. If $y \in F$ and $|y - x| < \eta$, then $y \in F \cap U_j = F_j$, so $|f(y) - f(x)| = 0 < \delta$. Hence f is continuous on F (in the relative topology). □

theorem 2. *Let f be defined and finite on a measurable set E. Then f is measurable if and only if it has property C on E*

Proof. if f is measurable, then there exist simple measurable f_k converge to f. Thus, every simple function f_k has property C.

For any fixed $\varepsilon > 0$, there exists closed $F_k \subset E$ such that $|E - F_k| < \varepsilon 2^{-k-1}$ and f_k is continuous relative to F_k.

Assuming that $|E| < \infty$, by egorov's theorem that there is a closed $F_0 \subset E$ with $|E - E_0| < \varepsilon/2$. Thus,$f_k$ uniformly converge to f on F_0.

If $F = F_0 \cap (\bigcap_k F_k)$, F is closed, and each f_k continuous relative to F. f_k uniformly converge to f on F, so we have

$$|E - F| \leq |E - F_0| + \sum_k |E - F_k| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Thus, if $|E| < \infty$, we have f has property C if f is measurable.

The opposite direction: Suppose that f has property C ON E. Then, there exists $F_k \subset E$ such taht $|E - F_k| < 1/k$ and f is continuous on F_k. Thus, we let $H = \bigcup_k F_k$. We have $H \subset E$ and $Z = E - H$ has measure 0. Then:

$$\{f(x) > a\} = \{x \in H | f(x) > a\} \cup \{x \in Z | f(x) > a\} = \bigcup_k \{x \in F_k | f(x) > a\} \cup \{x \in Z | f(x) > a\}$$

Since $|Z| = 0$, so subset of Z has measure 0.

For each $\{x \in F_k | f(x) > a\}$ is measurable since the relative continuous.

Thus, we have the function is measurable if f has the property C on E. □

0.2 convergence

Definition 3 (Convergence in measure)**.** *A sequence of measurable functions (f_n) on E is said to* converge in measure *to a measurable function f on E if for every $\varepsilon > 0$,*

$$\mu\big(\{x \in E : |f_n(x) - f(x)| > \varepsilon\}\big) \xrightarrow[n \to \infty]{} 0.$$

We write $f_n \xrightarrow{\mu} f$ on E.

theorem 3. *Let f and f_k be measurable and finite a.e. in E. If $f_k \to f$ a.e. on E and $|E| < \infty$. Then $f_k \xrightarrow{\mu} f$ on E*

Proof. Fix $\varepsilon > 0$ and let $\eta > 0$ be arbitrary. By Lemma 4.18, there exist a closed set $F \subset E$ and an integer K such that $\mu(E \setminus F) < \eta$ and

$$|f_k(x) - f(x)| < \varepsilon \quad \text{for all } x \in F \text{ and all } k \geq K.$$

Hence, for every $k \geq K$,

$$\{x \in E : \ |f_k(x) - f(x)| > \varepsilon\} \ \subset \ E \setminus F,$$

so

$$\mu\big(\{x \in E : \ |f_k(x) - f(x)| > \varepsilon\}\big) \ \leq \ \mu(E \setminus F) \ < \ \eta.$$

Since $\eta > 0$ is arbitrary, it follows that $\mu(\{|f_k - f| > \varepsilon\}) \to 0$ as $k \to \infty$. Therefore $f_k \xrightarrow{\mu} f$ on E. □

Proof. if $f_k \xrightarrow{\mu} f$ on E, there is a subsequence f_{k_j} such that $f_{k_k} \to f$ a.e. in E □

theorem 4 (Cauchy criterion for convergence in measure)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space and $E \in \mathcal{A}$ with $\mu(E) < \infty$. For measurable functions $f_k : E \to \mathbb{R}$ and $f : E \to \mathbb{R}$, the following are equivalent:*

1. *$f_k \xrightarrow{\mu} f$ on E; i.e., for every $\varepsilon > 0$, $\mu(\{|f_k - f| > \varepsilon\}) \to 0$ as $k \to \infty$.*
2. *For every $\varepsilon > 0$, $\lim\limits_{k,l \to \infty} \mu(\{|f_k - f_l| > \varepsilon\}) = 0$.*

Proof. (1) ⇒ (2): Fix $\varepsilon > 0$. By the triangle inequality,

$$\{|f_k - f_l| > \varepsilon\} \subset \{|f_k - f| > \varepsilon/2\} \cup \{|f_l - f| > \varepsilon/2\}.$$

Hence

$$\mu(\{|f_k - f_l| > \varepsilon\}) \leq \mu(\{|f_k - f| > \varepsilon/2\}) + \mu(\{|f_l - f| > \varepsilon/2\}) \xrightarrow[k,l \to \infty]{} 0.$$

(2) ⇒ (1): Assume for each $\varepsilon > 0$, $\mu(\{|f_k - f_l| > \varepsilon\}) \to 0$ as $k, l \to \infty$. Choose a subsequence (f_{k_j}) so that

$$\mu(\{|f_{k_{j+1}} - f_{k_j}| > 2^{-j}\}) < 2^{-j} \qquad (j \in \mathbb{N}).$$

Let $A_j := \{|f_{k_{j+1}} - f_{k_j}| > 2^{-j}\}$. Then $\sum_j \mu(A_j) < \infty$, so $\mu(\limsup_{j \to \infty} A_j) = 0$. Thus (f_{k_j}) is Cauchy a.e., hence converges a.e. on E to some measurable f. By the a.e. ⇒ in–measure result on finite–measure sets, $f_{k_j} \xrightarrow{\mu} f$.

Finally, to pass from the subsequence to the whole sequence, fix $\varepsilon, \delta > 0$. Choose j large so that $\mu(\{|f_{k_j} - f| > \varepsilon/3\}) < \delta/2$. By the Cauchy-in-measure assumption, choose N such that for all $n \geq N$, $\mu(\{|f_n - f_{k_j}| > \varepsilon/3\}) < \delta/2$. Then for $n \geq N$,

$$\{|f_n - f| > \varepsilon\} \subset \{|f_n - f_{k_j}| > \varepsilon/3\} \cup \{|f_{k_j} - f| > \varepsilon/3\},$$

so

$$\mu(\{|f_n - f| > \varepsilon\}) \leq \delta/2 + \delta/2 = \delta.$$

Since δ is arbitrary, $\mu(\{|f_n - f| > \varepsilon\}) \to 0$; hence $f_n \xrightarrow{\mu} f$. □