functional analysis study note

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0.1 TVS

Definition 1. *X be a vector space over C, X is a topological vector space (TVS) if it is a space with topology τ, that makes addition and scalar multiplication confirm.*

example 1. *(1) $Tx_0 : x \to x$ $Tx_0(x) = x_0 + x$ for a fixed x_0 is homeomorphism*
(2) $M_\lambda : X \to X$ $M_\lambda(x) = \lambda x$ is homeomorphism of X.

Thus topological τ is invariant under translation, for $u \subset X$ is open iff $x + u$ is open for any $x \in X$. Hence, topology is determined by a neighborhood base at 0.

example 2. *(3) continuity of addition says that for all neighborhood u of $(x+y)$, exists neighborhood u_1 of x and u_2 of y such that $u_1 + u_2 \subset u$.*

Continuity of scaler multiplication says that for any neighborhood u of λx, exists neighborhood v of x and exists δ such that $\mu y \in u$ for any $y \in v$, $|\lambda - \mu| < \delta$

TVS with topology τ is translative invariant, hence determined by neighborhood base B at 0.

This is because (x, τ) is a TVS, and if $\{x + v, v \in B\}$ is a neighborhood base at 0, then $\{x + B | x \in X\}$ is a base for the X.

Definition 2. *a set $C \subset X$ is convex if for any $0 \le t \le 1$, $tx + (1-t)y \in C$ for any $x, y \in C$.*

Definition 3. *let (X, τ) be a TVS, then X is locally convex TVS (LCTVS) if $0 \in X$ has a neighborhood bases consisting of convex open set.*
(equivalent to every $x \in X$ has convex open neighborhood bases)

Definition 4. *A net $(x_\lambda)_{\lambda \in \Lambda}$ is a Cauchy net in X if for any $u \in B(0)$ is a neighborhood at 0, exists $\lambda_u \in \Lambda$ such that $x_\lambda - x_\mu \in u$ for any $\lambda, \mu > \lambda_u$*

Definition 5. *Let X be a TVS, then X is complete if every Cauchy net converges.*

Definition 6. *A complete, metrizable, locally convex TVS is called Frechet Space (T_1)*

example 3. *Banach space are Frechet space with* $d(x,y) = \|x-y\|$

example 4. $(X_n, \|\|_n)$ *a sequence of Banach space:* $X = \Pi_{n=1}^{\infty} X_n$ *with each* X_n *Banach space and with metric:*

$$d(x,y) = \sum_{n=1}^{\infty} \frac{\|x_n - y_n\|_n}{2^n(1 + \|x_n - y_n\|_n)}$$

Then d defines a TVS topo on X *is LCTVS since*
1. has metric
2. convex because TVS space is convex iff $d(x,y) \leq d(x,z) + d(z,y)$ *in this metric.*
3. complete

Theorem 1. *For any* $(x^k)_{k\in N} \subset X$ *is Cauchy iff* $(x_n^k)_{k\in N}$ *is Cauchy in* X_n *for any* n.

Proof. Fix $n \in N$ for any $\varepsilon > 0$, then exists $N > 0$ such that $d(x^k, x^l) < \varepsilon/2^n$ for any $k, l > N$.
Thus,

$$\frac{\|x_n^k - x_n^l\|_n}{2^n(1 + \|x_n^k - x_n^l\|_n)} < \varepsilon/2^n \rightarrow \frac{\|x_n^k - x_n^l\|_n}{(1 + \|x_n^k - x_n^l\|_n)} < \varepsilon$$

rearrange we have $(1-\varepsilon)\|x_n^k - x_n^l\|_n < \varepsilon$.
If $\varepsilon < 1/2$, we have $\|x_n^k - x_n^l\|_n < 2\varepsilon$

The opposite direction: Let $\varepsilon > 0$ there exists some $k_0 > 0$ such that $2^{-k_0} < \varepsilon/2$

$$d(x^k_, x^l) = \sum_{n=1}^{k_0} \frac{\|x_n^k - x_n^l\|_n}{2^n(1 + \|x_n^k - x_n^l\|_n)} + \sum_{n=k_0+1}^{\infty} \frac{\|x_n^k - x_n^l\|_n}{2^n(1 + \|x_n^k - x_n^l\|_n)}$$

Since $\frac{\|x_n^k - x_n^l\|_n}{(1 + \|x_n^k - x_n^l\|_n)} < 1$, the term

$$\sum_{n=k_0+1}^{\infty} \frac{\|x_n^k - x_n^l\|_n}{2^n(1 + \|x_n^k - x_n^l\|_n)} < \sum_{n=k_0+1}^{\infty} 1/2^n \leq \varepsilon/2$$

For the first part, since $(x_n^k)_{k\in N}$ is Cauchy in X_n, there exist $N(n) > 0$ such that for any $k, l > N(n)$, $\|x_n^k - x_n^l\|_n < \varepsilon/2$
Let $N = \max\{N(1), N(2) \ldots N(k_0)\}$, then we have $\|x_n^k - x_n^l\|_n < \varepsilon/2$ for any $k, l \geq N$ and $1 \leq n \leq k_0$.
Thus, we have the first part $\sum_{n=1}^{k_0} \frac{\|x_n^k - x_n^l\|_n}{2^n(1+\|x_n^k - x_n^l\|_n)} < \sum_{n=1}^{k_0} \frac{\varepsilon}{2^{n+1}(1+\|x_n^k - x_n^l\|_n)} < \varepsilon/2$.

Thus, $d(x^k, x^l_) < \varepsilon/2 + \varepsilon/2 = \varepsilon$ for any $k, l > N$.
Thus, x^k is Cauchy in X.

□

Theorem 2. *For $X = \prod_{n\in\mathbb{N}} X_n$ equipped with the product metric*

$$d(x, y) = \sum_{n=1}^{\infty} \frac{\|x_n - y_n\|_n}{2^n\big(1 + \|x_n - y_n\|_n\big)},$$

if $X_n \neq \{0\}$ for infinitely many $n \in \mathbb{N}$, then (X, d) is not a normed space (i.e. not normable).

Proof. Suppose there exists a norm $\|\cdot\|$ on X generating the same topology as d. Let $U = \{x \in X : \|x\| < 1\}$ be the open unit ball (for $\|\cdot\|$); then U is an open 0-neighborhood. Since the topologies agree, there exists $\varepsilon > 0$ such that the d-ball $B^d_\varepsilon(0) = \{x \in X : d(x, 0) < \varepsilon\}$ satisfies $B^d_\varepsilon(0) \subset U$.

Fix $k \in \mathbb{N}$ with $2^{-k} < \varepsilon/2$ and take any $x = (x_n)_{n\in\mathbb{N}} \in X$ with

$$\|x_n\|_n < \varepsilon/2 \quad (1 \le n \le k).$$

Then

$$d(x, 0) = \sum_{n=1}^{\infty} \frac{\|x_n\|_n}{2^n(1 + \|x_n\|_n)} = \sum_{n=1}^{k} \frac{\|x_n\|_n}{2^n(1 + \|x_n\|_n)} + \sum_{n=k+1}^{\infty} \frac{\|x_n\|_n}{2^n(1 + \|x_n\|_n)}$$

$$< \frac{\varepsilon}{2} + \sum_{n=k+1}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{2} + \frac{1}{2^k} < \varepsilon,$$

hence $x \in B^d_\varepsilon(0) \subset U$.

Since $X_n \neq \{0\}$ for infinitely many n, choose $m > k$ with $X_m \neq \{0\}$. Pick $0 \neq v \in X_m$ and define $y = (y_n)_{n\in\mathbb{N}} \in X$ by

$$y_n = \delta_{n,m}\, v$$

Then for any scalar λ (over $\mathbb{R}$ or $\mathbb{C}$),

$$d(\lambda y, 0) = \frac{\|\lambda v\|_m}{2^m\big(1 + \|\lambda v\|_m\big)} \le \frac{1}{2^m} < \varepsilon,$$

so $\lambda y \in B^d_\varepsilon(0) \subset U$ for all λ. But in a normed space, $\|\lambda y\| = |\lambda|\|y\|$. Since $\|\lambda y\| < 1$ for all λ, we must have $\|y\| = 0$, hence $y = 0$, contradicting $y \neq 0$. Therefore no norm can generate the topology of (X, d); that is, (X, d) is not normable. □

V a neighborhood at 0, then there exists U is symmetric neighborhood at 0 ($-U = U$) such that $u + u \subset V$.

Proof. Since $0+0=0$ there exists neighborhood v_1, v_2 at 0 such that $v_1+v_2 \subset V$.

Let $u = v_1 \cap v_2 \cap (-v_1) \cap (-v_2)$

all 4 containing 0 so the cap is not empty. $\square$

Theorem 3. *Let X be a TVS, $K \subset X$ compact and $K \neq \emptyset$. $C \subset X$ closed and such that $K \cap C = \emptyset$ then exists open neighborhood V at 0 such that $(K+V) \cap (C+V) = \emptyset$.*

Proof. Apply the lemma above, there exists symmetric neighborhood u' at 0 such that $u'+u'+u'+u' \subset V$.

In particular, $0 \in u'$ we have $0+u'+u'+u' \subset v$. Let $x \in K$, then $x \in C^c$ which is open.

By the lemma above, there exists $v_x+v_x+v_x+v_x \subset C^c$

Thus, we have $x+v_x+v_x+v_x \subset C^c$, and we claim $(x+v_x+v_x) \cap (C+v_x) = \emptyset$.

Suppose there exists $y_i \in v_x$, and $c \in C$ such that $x+y_1+y_2 = y_3+c$. This implies $x+y_1+y_2-y_3 = c$. However, v_x is symmetric. Thus, we have $x+y_1+y_2-y_3 \in x+v_x+v_x+v_x \subset C^c$

There is contradiction that $x+y_1+y_2-y_3$ is in C and C^c.

So there is no such y_i, i.e. $\{x+v_x\}_{x \in K}$ is open cover of K. There exists $x_1, x_2 \dots x_n$ such that $K \subset \bigcup_{i=1}^n (x_i + v_{x_i})$ since K is compact, so the $n < \infty$

Thus, $V = v_1 \cap v_2 \cdots \cap v_{x_n}$ is an open neighborhood of 0.

Thus, $K+V \subset \bigcup_{i=1}^n (x_i+v_{x_i}) + V \subset \bigcup_{i=1}^n (x_i + v_{x_i} + v_{x_i})$

Since $x_i + v_{x_i} + v_{x_i} \cap (c+v_{x_i}) = \emptyset$ We have $(K+V) \cap (C+V) = \emptyset$ $\square$

corollary 1. *If X is a TVS that is T_1, then X is Hausdorff*

Proof. Let $x, y \in X$ and $x \neq y$. Let $K = \{x\}$ and $C = \{y\}$.

By T_1 (T_1 means if $x \neq y$ exists U_y such that $y \in U_y, x \notin U_y$) so $\{x\} = X - \bigcup_{y \neq x} U_y$ is closed set. Then use the theorem above.

$\square$

Definition 7. *Let E be a subset in TVS X, E is balanced if $\lambda E \subset E$ for any $\lambda \in \mathbb{C}$ and $|\lambda| \leq 1$*

Definition 8. *X be a TVS and $A \subset X$ is absorbing if for any $x \in X$, there exists $t > 0$ such that $x \in tA$.*

If A is absorbing then $0 \in A$.

example 5. *$X = \mathbb{C}^2$, $A = \{(z,w) | |z| \leq |w|\}$ is a balanced but not convex set*

Int(A) is nonempty since if $|z| < |w|$, then a small neighborhood in C^2 centered at (z,w) and $r < \frac{|w|-|z|}{2}$ is contained in A

However, $0 \notin$ Int(A) since the neighborhood at 0 $\{(z,w) | |z|^2 + |w|^2 < r\}$ not contained in A

Theorem 4. *Let X be a TVS, V is an open neighborhood of 0, let $t_1 < t_2 \dots t_n \to \infty$, then $X = \bigcup_{n=1}^{\infty} t_n V$.*

Proof. Take any $x \in X$ and let $A = \{t \in C | tx \in V\}$. Note that A is open since $f : C \to X$ $f(\lambda) = \lambda x$

$A = f^{-1}(V)$, more, since $1/t_n \to 0$ as $n \to \infty$, there exists N such that for any $n \geq N$, $\frac{1}{t_n} \in A$ thus we have $x \in t_n V$ for any $n \geq N$. □

Theorem 5. *Let X be a TVS and V be an open neighborhood of 0, then $t_1 > t_2 \cdots \to 0$. If V is bounded, then $\{t_n V\}_{n=1}^{\infty}$ is neighborhood basis at 0. Hence X is first countable.*

Proof. Let U be neighborhood of 0. Since V is bounded, there exists a $t_0 > 0$ such that $V \subset tU$ for any $t \geq t_0$.

Let n be large enough, then $\frac{1}{t_n} \geq t_0$. Thus we have

$$V \subset \frac{1}{t_n} U \iff t_n V \subset U$$

□

Remark 1. *Many TVS do not have bounded neighborhood with at 0. The convex balanced sets in a neighborhood base of 0 is the same stuff of seminorm on X.*