functional analysis study note

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Definition 1. *if X is a Banach Space, and then X is called reflexive if $\hat{X} = X^{**}$*

example 1. $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ *then* $(L^p)^* = L^q$, $(L^p)^{**} = L^p$.

but this is not true for $p = 1$, *since* $(L^\infty)^* \neq L^1$

Definition 2. *Let X be a topological space. $A \subset X$ a subset. Then A is nowhere dense if* $(\overline{A})^\circ = \emptyset$

Theorem 1. *The Barie Category theorem:*

Let X be a complete metric space or a locally compact Hausdorff space.

1. if $\{u_n\}_{n=1}^\infty$ is a sequence of open dense sets, then $\bigcap_{n=1}^\infty u_n$ is dense in X.

2.X is not a countable union of nowhere dense sets.

Remark 1. *if a set is countable union of nowhere dense sets, it is called of 1st category or meager set.*

Proof. Proof of 1:

$A \subset X$ is dense iff $\overline{A} = X$ iff for any $W \in X$ is open and non-empty, $A \cap W \neq \emptyset$

Thus,

$$W \cap (\bigcup_{n=1}^{\infty} u_n) \neq \emptyset$$

since u_1 is open and dense, so $u_1 \cap W \neq \emptyset$ and open.

thus, exists $B(x_1, r_1) = \{y \in X | d(x, y) < r_1\}$ such that $\overline{B(x_1, r_1)} \subset u_1 \cap W$

since u_2 is dense and open, $B(x_2, r_2) = \{y \in X | d(x_2, y) < r_2\}$ such that $\overline{B(x_2, r_2)} \subset u_2 \cap B(x_1, r_1)$

Thus, exists $\overline{B(x_n, r_n)} \subset B(x_{n-1}, r_{n-1}) \cap u_n$.

If we let $0 < r_n < 2^{-n}$. If X is local compact Hausdorff space, replace the balls $B(x_n, r_n)$ be neighborhood $B(x_n, r_n)$ such that $\overline{B(x_n, r_n)}$ is compact.

Note that $\{x_n\}_{n \in \mathbb{N}}$ is Cauchy sequence in X, $x_n, x_m \in B(x_N, r_N)$ if $n, m > N$.

Thus, $x = \lim_{n \to \infty} x_n$ exists since X is complete.

(if X is locally compact Hausdorff space, then compact sets $\{\overline{B(x_n, r_n)}\}$ have the finite intersection properly, hence $\bigcap_{n=1}^\infty \overline{B(x_n, r_n)} \neq \emptyset$)

Note: $x_n \in B(x_k, r_k)$ for any $n \geq k$, we have $x \in \overline{B(x_k, r_k)}$ for any k. Given $n \in \mathbb{N}$, $\overline{B(x_n, r_n)} \subset u_n \cap B(x_{n-1, r_{n-1}}) \subset u_n \cap B(x_1, r_1) \subset u_n \cap W$

Thus, $\bigcap_{n=1}^\infty u_n \cap W \neq \emptyset$ □

Proof. Proof of 2:

Let $(A_n)_{n=1}^{\infty}$ the nowehre dense sets. Then $(\overline{A_n})^{\circ} = \emptyset$ for any n, i.e. $\overline{A_n}^C$ is dense.

$$\overline{(\overline{A_n})^C} = ((\overline{A_n})^{\circ})^C = X$$

hence,

$$\bigcap_{n=1}^{\infty} (\overline{A_n})^C \neq \emptyset$$

then

$$\bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} \overline{A_n} = (\bigcap_{n=1}^{\infty} (\overline{A_n})^C)^C \subsetneq X$$

□

Theorem 2. *The uniform boundness principle:*

X, Y be normed space and $A \subset B(X, Y)$

1.if $\sup_{T \in A} \|T(x)\| < \infty$ for all x in a non-meager set, then $\sup_{T \in A} \|T\| < \infty$

2.if X is Banach space, then if $\sup_{T \in A} \|T(x)\| < \infty$ for any $x \in X$, then $\sup_{T \in A} \|T\| < \infty$

Proof. 1. set

$$E_u = \{x \in X | \sup_{T \in A} \|T(x)\| \leq u\} = \bigcap_{T \in A} \{x \in X | \|T(x)\| \leq u\}$$

E_n is closed. By assumption $\bigcup_{n=1}^{\infty} E_n$ is not meager, i.e. at least one of the E_n is not nowhere dense. Thus, exists $n \in N$ s.t. $\overline{(E_n)}^{\circ} \neq \emptyset$

hence, exists $B(x_0, r)$ s.t $\overline{B(x_0, r)} \subset E_n$, hence $\overline{B(0, r)} \subset E_{2n}$:

if $\|x\| \leq r$, then $x_0 - x \in \overline{B(x_0, r)}$ hence

$$\|T(x)\| = \|T(x - x_0 + x_0)\| \leq \|T(x - x_0)\| + \|T(x_0)\| \leq 2n$$

for any $T \in A$.

Thus, $\|T\| \leq \frac{2n}{r}$ for any $T \in A$ □

Proof. 2. follows by Barie category: Since Banach space are complete and metric space proved by 1. □

Definition 3. *X, Y be topological space $f : X \to Y$ a function. if u open in X and then $f(u)$ is open in Y*

Theorem 3. *open mapping theorem:*

Let X, Y be Banach space and $T : X \to Y$ a bounded linear operator such that T is surjection, then T is open.

more, X, Y Banach space, $T : X \to Y$ bounded linear operator, T bijection then T^{-1} is continuous.

Definition 4. *Let X, Y be topological space $f : X \to Y$ a function, then f is open if $f(u)$ is open in Y for all open u in X*

Theorem 4. *Let X, Y be normed space $f : X \to Y$ linear map. Then f is open iff $f(B)$ contain a ball centered at 0 in Y where $B = \{x \in X | \|x\| < 1\}$*

Proof. if f is open, then $f(0) = 0$ is trivial.

the opposite direction: By assumption $D \subset f(B)$ where $D = \{y \in Y | \|y\| < r\}$.

Since $0 \in f(B)^\circ$, then f is linear implies $0 \in (f(B_r))^\circ$, so let

$$B_r = \{x \in X | \|x\| < r\} = rB, f(B_r) = rf(B)$$

and $x \in X \to rx$ $(r > 0)$ is a homomorphism. Thus, we have $0 \in f(B)$

Let $O \subset X$ be an open set, then for any $x \in O$, exists $rx > 0$ such that $B(x, rx) = \{w \in X | \|w - x\| < rx\} \subset O$. Since $0 \in (f(B_{rx}))^\circ$, exist ball $D_{tx} \subset f(B_{rx})$ in Y, $f(x) + D_{tx}$ is open as the transition by transition invariant is a homomorphism.

so we have $f(x) \in f(B(x, rx))^\circ$, then exists $sx > 0$ such that $D_{f(x)} = \{y \in Y | \|y - f(x)\| < sx\} \subset f(O)$.

Implies

$$\bigcup_{x \in O} D_{f(x)} \subset f(O)$$

$f(x) \in D_{f(x)}$ hence, $f(O) = \bigcup_{x \in O} D_{f(x)}$ is open. □

Theorem 5. *open mapping theorem: let X, Y be Banach space, $T \in B(X, Y)$. If T is surjection, then T open.*

Proof. Let $B_r = \{x \in X | \|x\| < r\}$

need to show $T(B_1)$ contain a ball centered at 0 in Y.

$X = \bigcup_{n=1}^{\infty} B_n$, then $Y = T(X)$ (T is surjection), we have $Y = T(X) = \bigcup_{n=1}^{\infty} T(B_n)$

by Barie category: Since Y is complete, exists a n such that $T(B_n)$ is not nowhere dense.

Since $x \in X \to kx \in X$ is a homomorphism for any $k \in \mathbb{N}$, this implies:

$T(B_1)$ can not be nowhere dense hence $(\overline{T(B_1)})^\circ \neq \emptyset$.

Then, exists $y_0 \in Y, r > 0$ such that $B(y_o, 4r) \subset \overline{T(B_1)}$

we show a ball of radius $\frac{r}{2}$ centered at 0 in Y is contained in $T(B_1)$ since $y_0 \in \overline{T(B_1)}$, so exists $y_1 \in T(B_1), y_1 = T(x_1)$. $x_1 \in B_1$ with $\|y_1 - y_0\| < 2r$, then $B(y_1, 2r) \subset \overline{T(B_1)}$

Let $\|y\| < 2r$, then

$$y = T(x_1) - (y_1 - y_0) \in T(x) - \overline{T(B_1)} = T(x_1) + \overline{T(B_1)} \subset \overline{T(x_1 + B_1)} \subset \overline{T(B_2)}$$

Thus, we have $\frac{y}{2} \in \overline{T(B_1)}$, so we have found a $T > 0$ such that $\|y\| < r$, $y \in \overline{T(B_1)}$.

Then need to show $B_{\frac{r}{2}} \subset T(B_1)$

Since $\|y\| < r$, we have $y \in \overline{T(B_1)}$, so if $\|y\| < \frac{r}{2}$, then $y \in \overline{T(B_{\frac{1}{2}})}$. Hence, $\|y\| < r(\frac{1}{2})^n$, then $y \in \overline{T(B_{\frac{1}{2^n}})}$

Then, WTS $\{y \in Y | \|y\| < r/2\} \subset T(B_1)$

Since $\|y\| < r/2$, there exists $x_1 \in B_1/2$ such that $\|y - Tx_1\| < r/4$.

Then exists $x_2 \in B_{(1/2^2)}$ such that $\|y - Tx_1 - Tx_2\| < r/2^3$.

By induction, exists $X_n \in B_{(1/2^n)}$ such that $\|y - \sum_{i=1}^n Tx_i\| < r/2^n$ and $\|x_n\| < 1/2^n$.

Since $\sum_{i=1}^\infty x_i$ absolutely converge, that $\sum_{i=1}^N x_i \to x$ in X since X is Banach space.

Hence, exists $x \in X$ such that $x = \sum_{i=1}^\infty x_i$, $\|x\| \le \sum_{i=1}^\infty \|x_i\| < \sum_{i=1}^\infty 1/2^i = 1$

Thus, $x \in B_1$ and $\|y - \sum_{i=1}^n Tx_i\| = \|y - T(\sum_{i=1}^n x_i)\| < 1/2^n$. As $n \to \infty$, $T(\sum_{i=1}^\infty x_i) = Tx$. So $\|Tx - y\| \to 0$, i.e. $y = Tx \in T(B_1)$. □

corollary 1. *Let X, Y be Banach Space, $T \in B(X, Y)$, T is bijection then T^{-1} is continuous*

Theorem 6. *Let $T : X \to Y$ a linear map X, Y normed space then the graph of T is defined in a subspace of $X \times Y$ by $y(T) = \{(x, y) \in X \times Y | y \in T_x\}$ a linear subspace of $X \times Y$.*

T is closed if $y(T)$ is closed subspace of $X \times Y$.

Note if T is continuous, $y(T)$ is closed. $(x_n \to x, y_n \to y)$ then $T(x) = y$.

Theorem 7. *Closed Graph theorem*

Let X, Y be the Banach space, $T : X \to Y$ a closed linear map. Then T is continuous.

Proof. $\Pi_1 : X \times Y \to X, \Pi_2 : X \times Y \to Y$ be the $\Pi_1 : ((x, y)) = x, \Pi_2((x, y)) = y$.

Restrict Π_1, Π_2 to $y(T)$ which is closed subspace of $X \times Y$.

Hence, $y(T)$ is also a Banach space.

Then $\|\Pi_1(X, Tx)\| = \|x\| \le \|x\| + \|T_x\|$,

$\|\Pi_2(X, Tx)\| = \|Tx\| \le \|x\| + \|T_x\|$

Thus, Π_1, Π_2 are bounded linear operator. Π_1 is bijection from $y(T) \to X$ so Π_1^{-1} is continuous.

Then $T_x = \Pi_2 \circ \Pi_1^{-1}(x)$ is continuous for any $x \in X$. □

Remark 2. *WTS a given T is continuous, it is often easier to use closed graph theorem than check continuous.*