0.1 Measurable function

Definition 1. *Let $E \subset \mathbb{R}^n$ be a measurable set. Let $f : E \to [-\infty, +\infty]$ be a real-valued function in the extended sense. Then f is called a* Lebesgue measurable function *on E (or simply a measurable function) if for every finite $a \in \mathbb{R}$, the set*

$$\{x \in E : f(x) > a\}$$

is a measurable subset of $\mathbb{R}^n$.

theorem 1. *Let $E \subset \mathbb{R}^n$ be a measurable set and let $f : E \to [-\infty, +\infty]$ be a real-valued function in the extended sense. Then f is measurable if and only if, for every finite $a \in \mathbb{R}$, any (and hence all) of the following conditions hold:*

1. *$\{x \in E : f(x) \geq a\}$ is measurable.*
2. *$\{x \in E : f(x) < a\}$ is measurable.*
3. *$\{x \in E : f(x) \leq a\}$ is measurable.*

Proof. Recall the defining criterion: f is measurable iff $\{f > a\}$ is measurable for every finite a.

($\Rightarrow$) Assume $\{f > a\}$ is measurable for all a. Then

$$\{f \geq a\} = \bigcap_{k=1}^{\infty} \{f > a - 1/k\}$$

is measurable (countable intersection). Also $\{f \leq a\} = E \setminus \{f > a\}$ is measurable (complement in measurable E). Finally,

$$\{f < a\} = \bigcup_{k=1}^{\infty} \{f \leq a - 1/k\}$$

is measurable (countable union).

($\Leftarrow$) Conversely, if (i) holds for all a, then

$$\{f > a\} = \bigcup_{k=1}^{\infty} \{f \geq a + 1/k\}$$

is measurable. If (iii) holds for all a, then $\{f > a\} = E \setminus \{f \leq a\}$ is measurable. If (ii) holds for all a, then

$$\{f > a\} = \bigcup_{k=1}^{\infty} \{f \not< a + 1/k\} = \bigcup_{k=1}^{\infty} \{f \geq a + 1/k\},$$

so it is measurable. In each case the defining criterion is satisfied, hence f is measurable. $\square$

corollary 1. *Let f be defined on a measurable set $E \subset \mathbb{R}^n$.*

1. *If f is measurable, then the sets*

$$\{f > -\infty\}, \quad \{f < +\infty\}, \quad \{f = +\infty\}, \quad \{a \le f \le b\}, \quad \{f = a\}, \ \textit{etc.}$$

   are all measurable.

2. *Moreover, for any f, if either $\{f = +\infty\}$ or $\{f = -\infty\}$ is measurable, then f is measurable provided that*

$$\{a < f < +\infty\}$$

   is measurable for every finite $a \in \mathbb{R}$.

remark 1. *Intuitively, a measurable function means that the measurable structure is preserved under taking preimages:*

for every measurable set B in the image space, its preimage $f^{-1}(B)$ is measurable in the domain.

recall continuous means for every open set B in the image space, its preimage $f^{-1}(B)$ is open in the domain.

theorem 2. *Let A be a dense subset of R. Then f is measurable if $\{f > a\}$ is measurable for all $a \in A$.*

Definition 2. *A property $P(x)$ is said to hold* almost everywhere *(a.e.) on a measurable set $E \subset \mathbb{R}^n$ if the set $\{x \in E : P(x) \text{ fails}\}$ has Lebesgue measure zero.*

theorem 3. *Let $E \subset \mathbb{R}^n$ be measurable. If f is measurable and $g = f$ almost everywhere, then g is measurable and*

$$|\{g(x) > a\}| = |\{f(x) > a\}|, \quad \forall a \in \mathbb{R},$$

theorem 4. *Let f, g be measurable functions. Then the set $\{f(x) > g(x)\}$ is a measurable.*

Proof. Let $\mathbb{Q}$ denote the rationals. For each $q \in \mathbb{Q}$, the sets

$$A_q := \{f(x) > q\}, \qquad B_q := \{g(x) \le q\}$$

are measurable because f and g are measurable. Observe the identity

$$\{f > g\} = \bigcup_{q \in \mathbb{Q}} \big(A_q \cap B_q\big).$$

□

theorem 5. *Let f is measurable. For any real constant $\lambda \in \mathbb{R}$, the functions $f + \lambda$ and λf are measurable.*

theorem 6. *If f and g are measurable, so is $f + g$*

theorem 7. *If f and g are measurable, so is fg. If $g \neq 0$ a.e., then $\frac{f}{g}$ is measurable*

0.2 Sequence, Limits

theorem 8. *Let $E \subset \mathbb{R}^n$ be measurable and let $\{f_k\}_{k=1}^{\infty}$ be a sequence of measurable functions on E. Then*

$$\sup_k f_k \quad \textit{and} \quad \inf_k f_k$$

are measurable on E.

Proof. For any finite $a \in \mathbb{R}$,

$$\big\{x : \sup_k f_k(x) > a\big\} = \bigcup_{k=1}^{\infty} \{x : f_k(x) > a\},$$

which is measurable as a countable union of measurable sets (each $\{f_k > a\}$ is measurable since f_k is measurable). Hence $\sup_k f_k$ is measurable by the defining criterion.

Similarly,

$$\big\{x : \inf_k f_k(x) > a\big\} = \bigcap_{k=1}^{\infty} \{x : f_k(x) > a\},$$

a countable intersection of measurable sets, hence measurable. Therefore $\inf_k f_k$ is measurable. □

theorem 9. *If $\{f_k\}$ is a sequence of measurable functions, then* $\limsup_{k\to\infty} f_k$ *and* $\liminf_{k\to\infty} f_k$ *are measurable. More, if* $\lim_{k\to\infty} f_k$ *exists a.e., it is measurable.*

Proof. Since

$$\limsup_{k\to\infty} f_k = \inf_j \{\sup_{k\geq j} f_k\}, \quad \liminf_{k\to\infty} f_k = \sup_j \{\inf_{k\geq j} f_k\}$$

By the theorem above, the sup and inf of sequence of measurable functions are measurable. □

0.3 Simple function approximation

Definition 3. *The* characteristic function *of a set $A \subset X$ is defined by*

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

Definition 4. *A* simple function *is a measurable function that takes only finitely many distinct values. Equivalently, $f : X \to \mathbb{R}$ is simple if it can be written as*

$$f(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x),$$

where $a_i \in \mathbb{R}$ and each E_i is a measurable set.

theorem 10.

1. *Every function f can be written as the limit of a sequence $\{f_k\}$ of simple functions.*

2. *If $f \geq 0$, the sequence can be chosen to increase to f, that is, chosen such that $f_k \leq f_{k+1}$ for every k.*

3. *If the function f in either (i) or (ii) is measurable, then the f_k can be chosen to be measurable.*

The idea of the proof is very simple:

- For a general function f, we first truncate it so it stays within $[-k, k]$.
- Then we divide the vertical axis into small intervals of length 2^{-k} and replace f by a step function that is constant on the regions where f falls in each interval.
- As k increases, the step functions get closer to f, and in the nonnegative case we can choose them so that they increase to f.
- If f is measurable, these step functions can be chosen measurable as well.

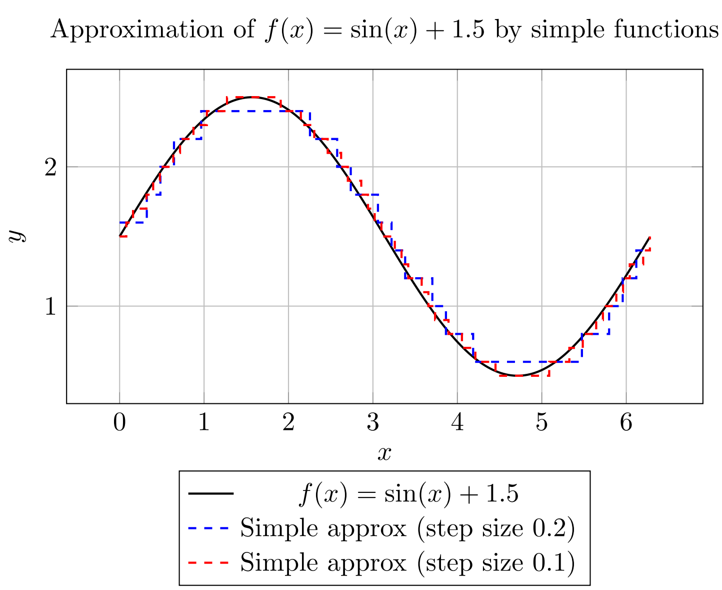

Figure 1: Approximating a function by step (simple) functions

With the figure above, we can see how simple function approximation works, and we need to construct the proof for its legality.

Proof. We start at (2). Suppose $f \geq 0$, For each $k = 1, 2, \ldots$ subdivide the values of f that fall in $[0, k]$ by partitioning $[0, k]$ into sub-intervals $[(j-1)2^{-k}, j2^{-k}]$. Let:

$$f_k(x) = \begin{cases} \dfrac{j-1}{2^k}, & \text{if } \dfrac{j-1}{2^k} \leq f(x) < \dfrac{j}{2^k}, \quad j = 1, \ldots, k2^k, \\ k, & \text{if } f(x) \geq k. \end{cases}$$

Each f_k is a simple function defined everywhere in the domain of f. Clearly $f_k \leq f_{k+1}$. Then each sub-interval is divided in half.

$f_k \to f$ since $0 \leq f - f_k \leq 2^{-k}$ for sufficiently large k. This proves (2)

Then, from (2) to prove (1):

define f^+ is the positive part of f i.e.$f^+(x) = f(x)$ if $f(x) \geq 0$,0 else;

f^- is the negative part of f i.e. $f^-(x) = -f(x)$ if $f(x) < 0$, 0 else.

Thus, the increasing sequences $f'_k \to f^+$, $f''_k \to f^-_k$ as simple function approximation. Then we have $f'_k - f''_k = f^+ - f^- = f$. This is (1).

for (3), the simple function is measurable as f_k. And condition in (1) or in (2) are equivalent. $\square$