INTELLIGENT ROBOTS

CHAPTER 4: PROBABILISTIC ROBOTICS

Outline

- Probability Fundamentals
- Bayes Rules and State Estimation
- Modeling Actions
- Bayes Filters

Probabilistic Robotics

Key idea:

Explicit representation of uncertainty

(using the calculus of probability theory)

- Perception = state estimation
- Action = utility optimization

Axioms of Probability

Pr*(A)* denotes probability that proposition *A* is true.

- $0 \leq \Pr(A) \leq 1$
- $\Pr(True) = 1 \qquad \Pr(False) = 0$
- $\Pr(A \vee B) = \Pr(A) + \Pr(B) - \Pr(A \wedge B)$

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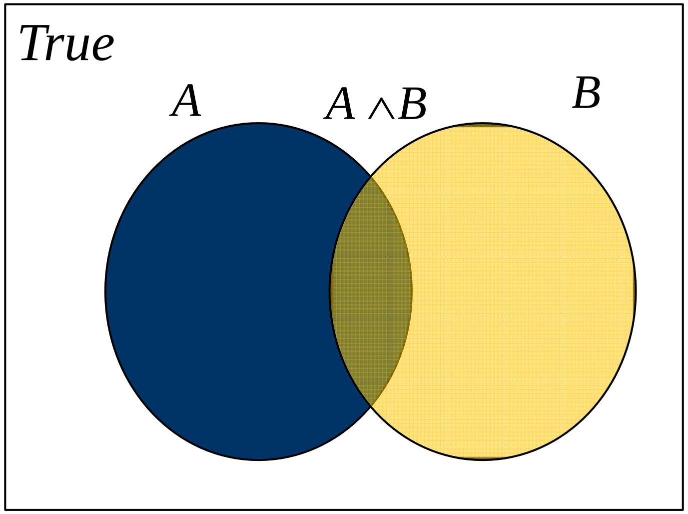


- $\Pr(A \vee B) = \Pr(A) + \Pr(B) - \Pr(A \wedge B)$

Using the Axioms

$$
\begin{aligned}
\Pr(A \vee \neg A) &= \Pr(A) + \Pr(\neg A) - \Pr(A \wedge \neg A) \\
\Pr(True) &= \Pr(A) + \Pr(\neg A) - \Pr(False) \\
1 &= \Pr(A) + \Pr(\neg A) - 0 \\
\Pr(\neg A) &= 1 - \Pr(A)
\end{aligned}
$$

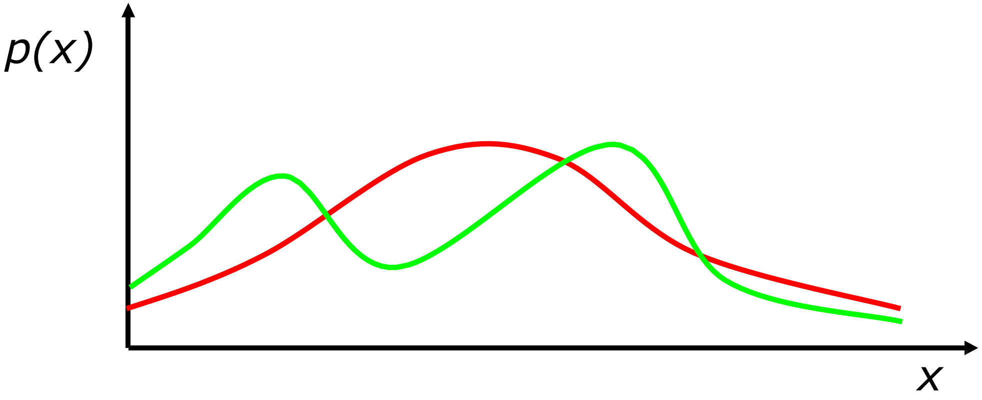
Discrete Random Variables

- *X* denotes a random variable
- *X* can take on a countable number of values in $\{x_1, x_2, \ldots, x_n\}$
- $P(X=x_i)$ or $P(x_i)$ is the probability that the random variable *X* takes on value x_i
- $P(\cdot)$ is called probability mass function
- E.g. $P(Room) = \{0.7, 0.2, 0.08, 0.02\}$

Continuous Random Variables

- X takes on values in the continuum.
- $p(X=x)$ or $p(x)$ is a probability density function

$$\Pr(x \in (a,b)) = \int_a^b p(x)dx$$



Continuous Random Variables

Discrete case

$$\sum_{x} P(x) = 1$$

Continuous case

$$\int p(x)\, dx = 1$$

Joint and Conditional Probability

- $P(X=x \text{ and } Y=y) = P(x,y)$

- If X and Y are independent then
  $P(x,y) = P(x)\ P(y)$

- $P(x \mid y)$ is the probability of x given y

  $P(x \mid y) = P(x,y) / P(y)$ $P(x,y) = P(x \mid y)\ P(y)$

- If X and Y are independent then
  $P(x \mid y) = P(x)$

Law of Total Probability

Discrete case

$$P(x) = \sum_y P(x \mid y) P(y)$$

Continuous case

$$p(x) = \int p(x \mid y)\, p(y)\, dy$$

Marginalization

Discrete case

$$P(x) = \sum_y P(x, y)$$

Continuous case

$$p(x) = \int p(x, y)\, dy$$

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Bayes Formula

$$P(x, y) = P(x \mid y)P(y) = P(y \mid x)P(x)$$

$$\Rightarrow$$

$$\boxed{P(x|y) = \frac{P(y \mid x)\, P(x)}{P(y)} = \frac{\text{likelihood} \cdot \text{prior}}{\text{evidence}}}$$

Normalization

$$P(x \mid y) = \frac{P(y \mid x)\, P(x)}{P(y)} = \eta\, P(y \mid x)\, P(x)$$

$$\eta = P(y)^{-1} = \frac{1}{\sum_x P(y \mid x) P(x)}$$

Bayes Rule with Background Knowledge

$$P(x \mid y, z) = \frac{P(y \mid x, z)\ P(x \mid z)}{P(y \mid z)}$$

Conditional Independence

$$P(x, y \mid z) = P(x \mid z)P(y \mid z)$$

- Equivalent to $P(x \mid z) = P(x \mid z, y)$

  and $P(y \mid z) = P(y \mid z, x)$

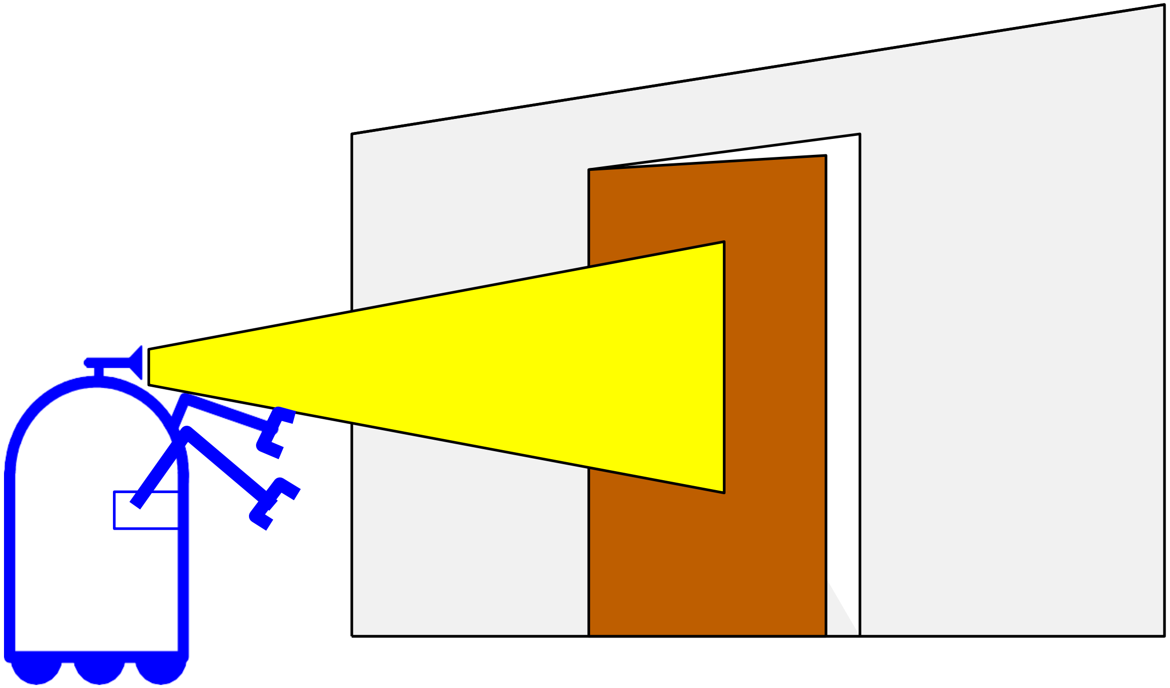
- But this does not necessarily mean

$$P(x, y) = P(x)P(y)$$

(real independence)

State Estimation

- Suppose a robot obtains measurement z
- What is *P(open|z)?*



Casual v.s. Diagnostic Reasoning

- *P(open|z)* is diagnostic
- *P(z|open)* is causal
- Often causal knowledge is easier to obtain

count frequencies!

- Bayes rule allows us to use causal knowledge:

$$P(open \mid z) = \frac{P(z \mid open) P(open)}{P(z)}$$

Example

- $P(z|open) = 0.6$ $\qquad P(z|\neg open) = 0.3$
- $P(open) = P(\neg open) = 0.5$

$$P(open \mid z) = \frac{P(z \mid open)P(open)}{P(z \mid open)\,p(open) + P(z \mid \neg open)\,p(\neg open)}$$

$$P(open \mid z) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{0.3}{0.3 + 0.15} = 0.67$$

- *z* raises the probability that the door is open

Combining Evidence

- Suppose our robot obtains another observation z_2
- How can we integrate this new information?
- More generally, how can we estimate $P(x| z_1...z_n)$?

Combining Evidence

$$P(x \mid z_1, \mathrm{K}, z_n) = \frac{P(z_n \mid x, z_1, \ldots, z_{n-1})\, P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \mathrm{K}, z_{n-1})}$$

Markov assumption:

z_n is independent of $z_1, \ldots, z_{n-1}$ if we know x

$$P(x \mid z_1, \ldots, z_n) = \frac{P(z_n \mid x)\, P(x \mid z_1, \ldots, z_{n-1})}{P(z_n \mid z_1, \mathrm{K}, z_{n-1})}$$

$$= \eta\, P(z_n \mid x)\, P(x \mid z_1, \ldots, z_{n-1})$$

$$= \eta_{1 \ldots n} \prod_{i=1 \ldots n} P(z_i \mid x)\, P(x)$$

Example

- $P(z_2|open) = 0.5$ $\qquad$ $P(z_2|\neg open) = 0.6$
- $P(open|z_1)=2/3$

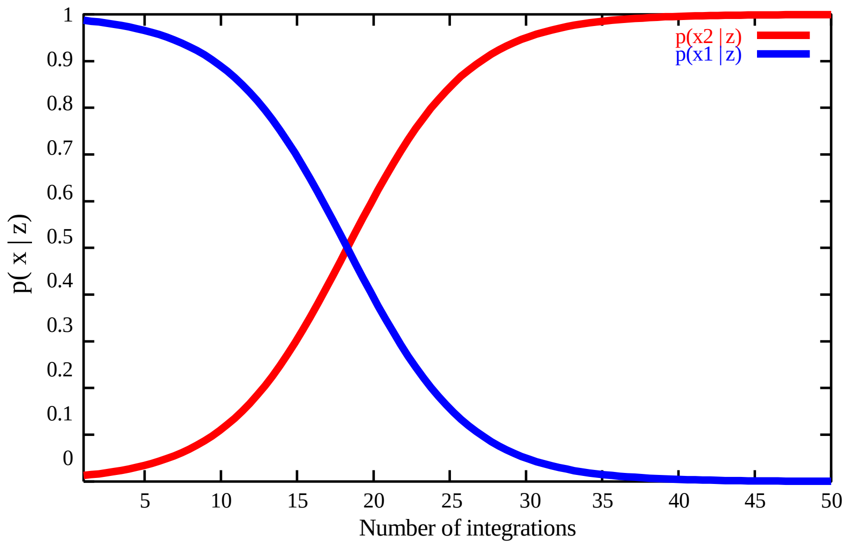
$$P(open \mid z_2, z_1) = \frac{P(z_2 \mid open)\, P(open \mid z_1)}{P(z_2 \mid open)\, P(open \mid z_1) + P(z_2 \mid \neg open)\, P(\neg open \mid z_1)}$$

$$= \frac{\frac{1}{2} \cdot \frac{2}{3}}{\frac{1}{2} \cdot \frac{2}{3} + \frac{3}{5} \cdot \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{5}} = \frac{\frac{1}{3}}{\frac{8}{15}} = \frac{5}{8} = 0.625$$

- *z2 lowers the probability that the door is open*

A Typical Pitfall

- Two possible locations x_1 and x_2
- $P(x_1)=0.99$ $P(x_2)=0.01$
- $P(z|x_2)=0.09$ $P(z|x_1)=0.07$



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Actions

- Often the world is **dynamic** since
  - **actions carried out by the robot**,
  - **actions carried out by other agents**,
  - or just the **time** passing by change of the world

- How can we **incorporate** such **actions**?

Typical Actions

- The robot **turns its wheels** to move
- The robot **uses its manipulator** to grasp an object
- Plants grow over **time**…

- Actions are never carried out with **absolute certainty**
- In contrast to measurements, actions generally **increase the uncertainty**

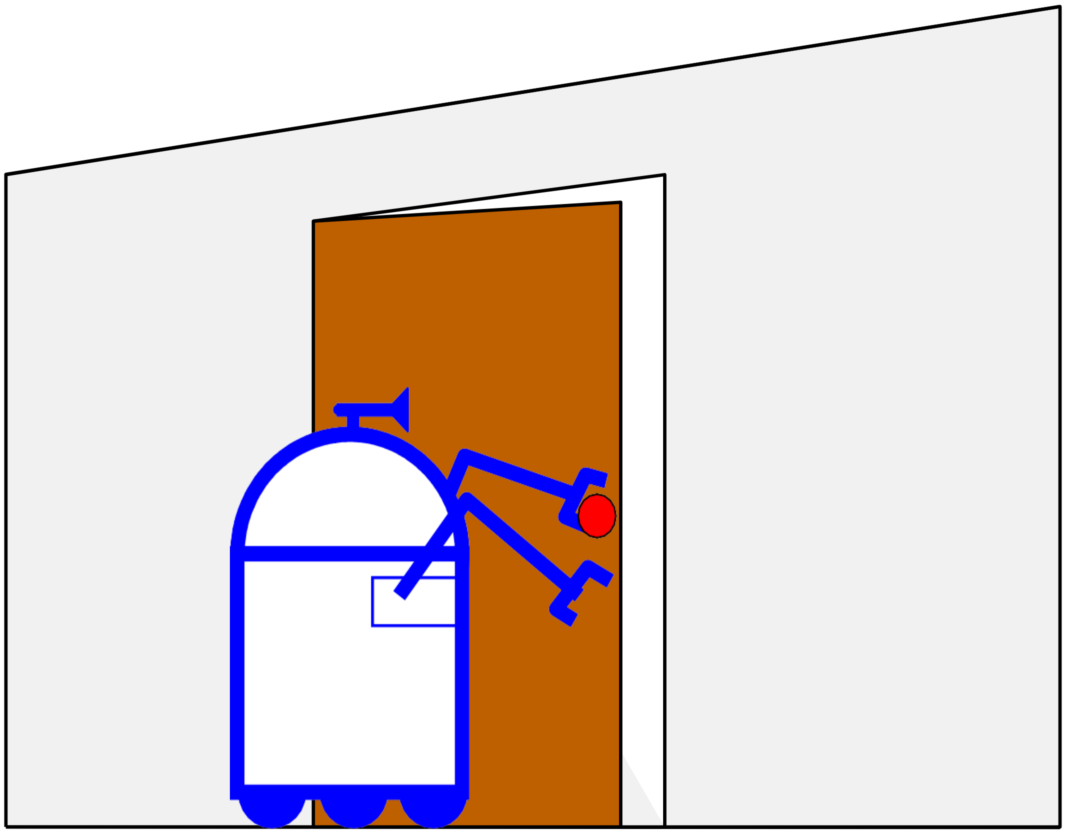
Modeling Actions

- To incorporate the outcome of an action ***u*** into the current "belief", we use the conditional pdf

$$P(x|u,x')$$

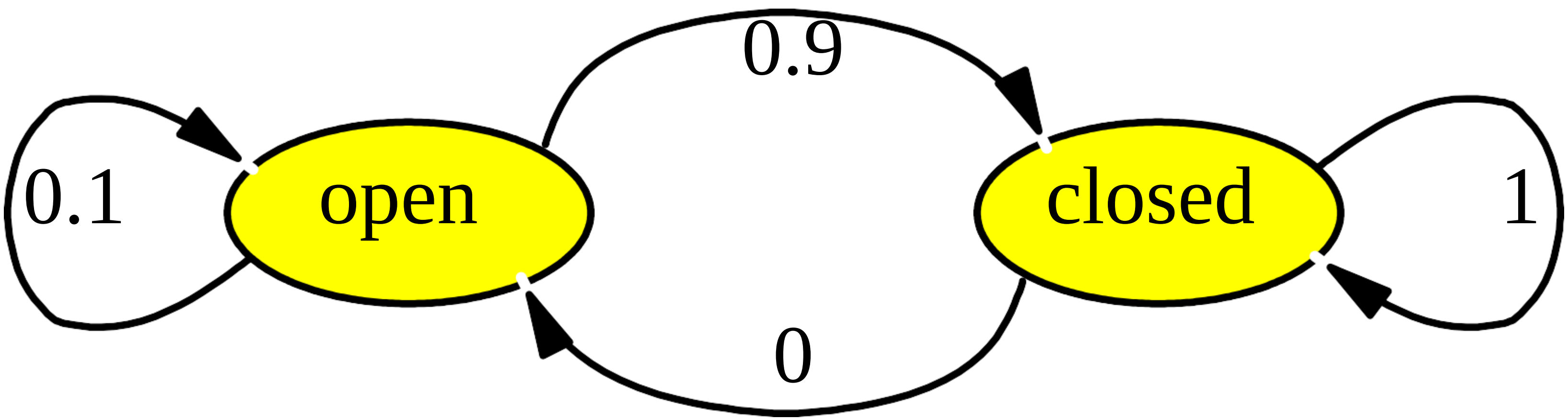
- This term specifies the pdf that executing ***u*** changes the state from ***x'*** *to* ***x***.

Example: Closing the Door



State Transitions

$P(x|u,x')$ for u = "close door":



If the door is open, the action "close door" succeeds in 90% of all cases

Integrating the Outcome of Actions

Continuous case:

$$P(x \mid u) = \int P(x \mid u, x')P(x')dx'$$

Discrete case:

$$P(x \mid u) = \sum P(x \mid u, x')P(x')$$

Example: The Resulting Belief

$$P(closed \mid u) = \sum P(closed \mid u, x')P(x')$$

$$= P(closed \mid u, open)P(open) + P(closed \mid u, closed)P(closed)$$

= 9/10*5/8+1/1*3/8 = 15/16

$$P(open \mid u) = \sum P(open \mid u, x')P(x')$$

$$= P(open \mid u, open)P(open) + P(open \mid u, closed)P(closed)$$

= 1/10*5/8+0/1*3/8 = 1/16

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Bayes Filters: Framework

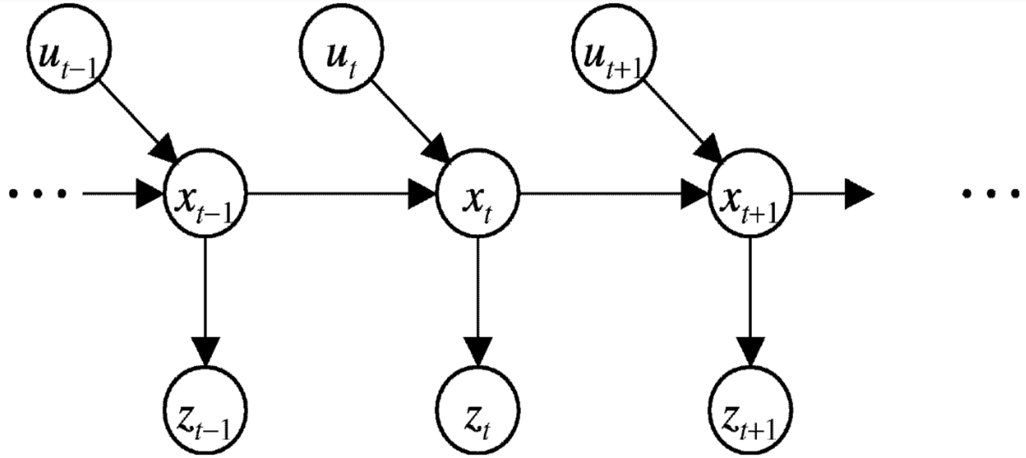
- **Given:**
  - Stream of observations *z* and action data *u:*

$$d_t = \{u_1, z_1, \ldots, u_t, z_t\}$$

  - Sensor model *P(z|x)*
  - Action model *P(x|u,x')*
  - Prior probability of the system state *P(x)*
- **Wanted:**
  - Estimate of the state *X* of a dynamical system
  - The posterior of the state is also called **Belief**:

$$Bel(x_t) = P(x_t \mid u_1, z_1, \ldots, u_t, z_t)$$

Markov Assumption



$$p(z_t \mid x_{0:t}, z_{1:t}, u_{1:t}) = p(z_t \mid x_t)$$
$$p(x_t \mid x_{1:t-1}, z_{1:t}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$$

Underlying Assumptions

- Static world
- Independent noise
- Perfect model, no approximation errors

Bayes Filter

z = observation
u = action
x = state

$$\boxed{Bel(x_t)} = P(x_t \mid u_1, z_1, \dots, u_t, z_t)$$

Bayes $$= \eta \; P(z_t \mid x_t, u_1, z_1, \dots, u_t) \, P(x_t \mid u_1, z_1, \dots, u_t)$$

Markov $$= \eta \; P(z_t \mid x_t) \, P(x_t \mid u_1, z_1, \dots, u_t)$$

Total prob. $$= \eta \; P(z_t \mid x_t) \int P(x_t \mid u_1, z_1, \dots, u_t, x_{t-1}) \; P(x_{t-1} \mid u_1, z_1, \dots, u_t) \, dx_{t-1}$$

Markov $$= \eta \; P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \, P(x_{t-1} \mid u_1, z_1, \dots, u_t) \, dx_{t-1}$$

Markov $$= \eta \, P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \, P(x_{t-1} \mid u_1, z_1, \dots, z_{t-1}) \, dx_{t-1}$$

$$\boxed{= \eta \; P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \, Bel(x_{t-1}) \, dx_{t-1}}$$

Algorithm

Algorithm **Bayes_filter**(*Bel(x),d*):

$\eta=0$

If *d* is a perceptual data item *z* then

    For all *x* do

        $Bel'(x) = P(z\,|x)Bel(x)$

        $\eta = \eta + Bel'(x)$

    For all *x* do

        $Bel'(x) = \eta^{-1} Bel'(x)$

Else if *d* is an action data item *u* then

    For all *x* do

        $Bel'(x) = \int P(x\,|\,u, x')\ Bel(x')\ dx'$

Return *Bel'(x)*

$$Bel(x_t) = \eta\ P(z_t\,|\,x_t) \int P(x_t\,|\,u_t, x_{t-1})\ Bel(x_{t-1})\ dx_{t-1}$$

Bayes Filters

$$Bel(x_t) = \eta \; P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \; Bel(x_{t-1}) \; dx_{t-1}$$

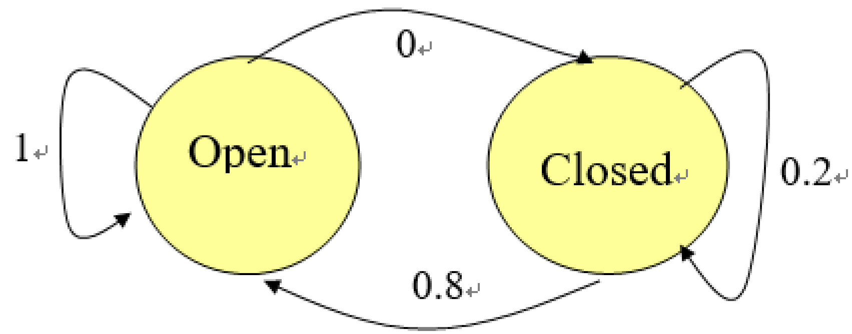
- Kalman filters
- Particle filters
- Hidden Markov models
- Dynamic Bayesian networks
- Partially Observable Markov Decision Processes (POMDPs)

Summary

- Bayes rule allows us to compute probabilities that are hard to assess otherwise.
- Under the Markov assumption, recursive Bayesian updating can be used to efficiently combine evidence.
- Bayes filters are a probabilistic tool for estimating the state of dynamic systems.

Homework 3

Problem1： The state transition for the action "open the door" is as shown in Fig. 1. If the door is closed, the action "open the door" succeeds in 80% of all cases. Assume the probabilities of the closed door and open door are 50% respectively.



Calculate the probability of

$P(open|u)$ for u = "open door":

Homework 3

Problem 2: A robot cleaner is roaming within an apartment with four rooms. The map of the apartment is given as follows. The probability of the robot going through each door is 0.1. Please answer the following questions:

(1) What is the Markov model for the robot roaming?

(2) what is the probability of the robot staying at each room?

(3) what is the probability o f the robot going through the door between (1) and (4) when the robot is going through a door?