INTELLIGENT ROBOTS

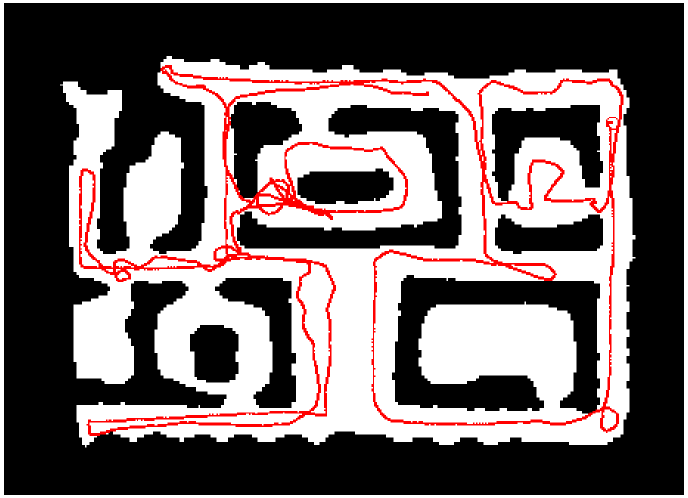
CHAPTER 5: PROBABILISTIC MOTION MODELS

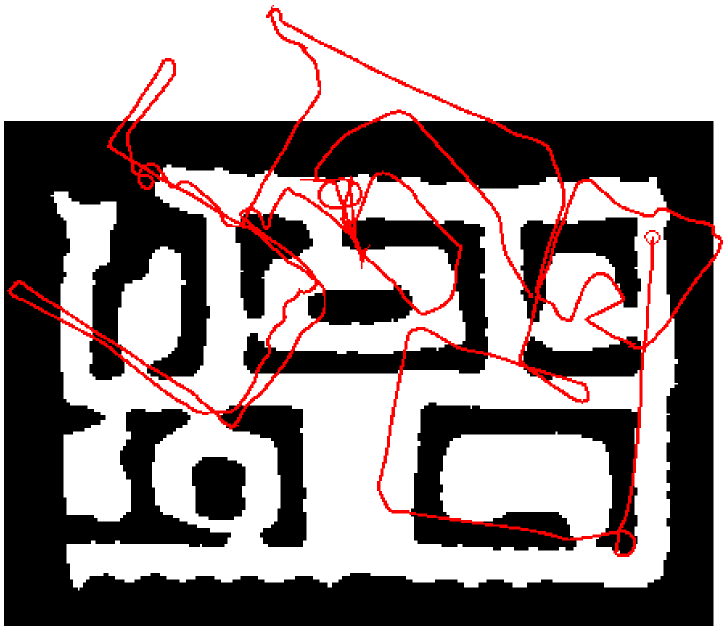
Outline

- Probability Fundamentals
- Odometry-based Models
- Velocity-based Models
- Map-consistent Models

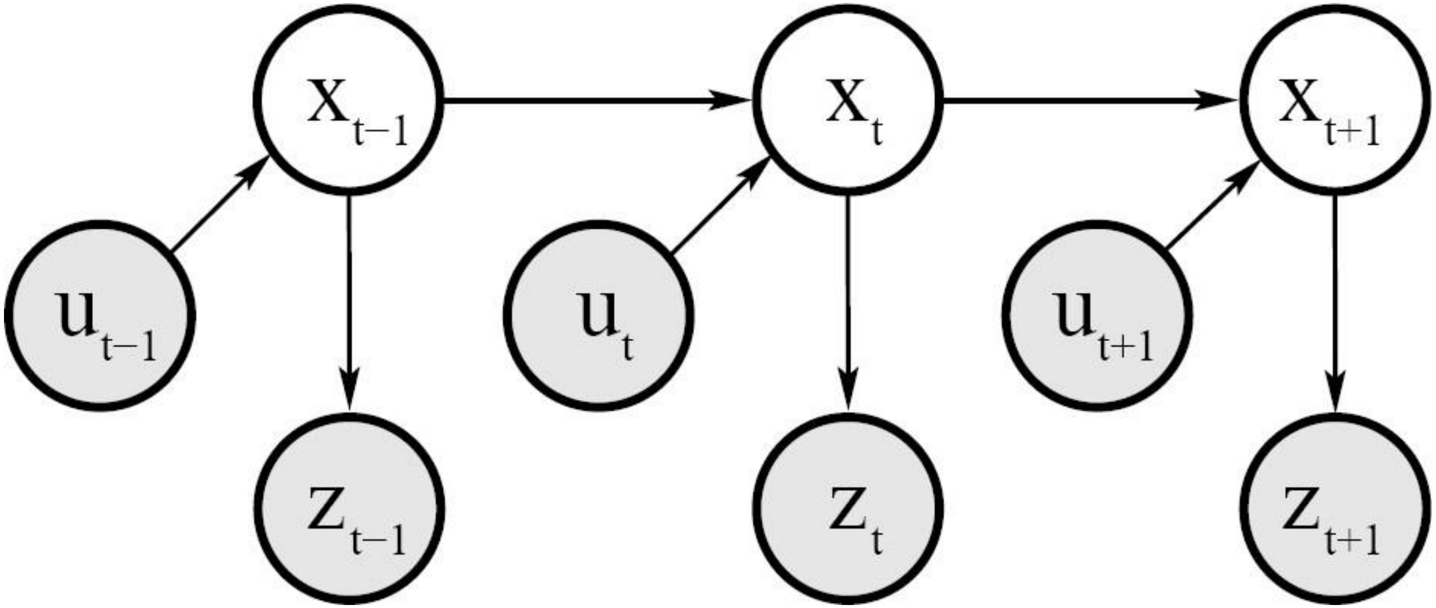
Robot Motion

- Robot motion is inherently uncertain.
- How can we model this uncertainty?





Dynamic Bayesian Network

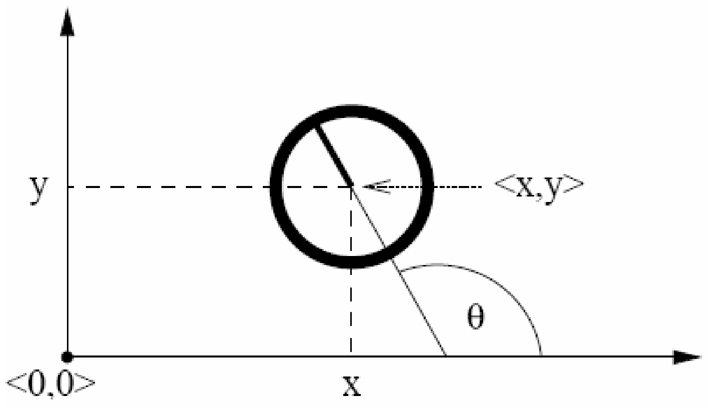


Probabilistic Motion Models

- To implement the Bayes Filter, we need the transition model *p(x* j *x', u)*.
- The term *p(x* j *x', u)* specifies a posterior probability, that action *u* carries the robot from *x'* to *x*.
- In this section we will specify, how *p(x* j *x', u)* can be modeled based on the motion equations.

Coordinate Systems

- In general the configuration of a robot can be described by six parameters.
- Three-dimensional Cartesian coordinates plus three Euler angles pitch, roll, and tilt.
- Throughout this section, we consider robots operating on a planar surface.
- The state space of such systems is three-dimensional (x,y,).

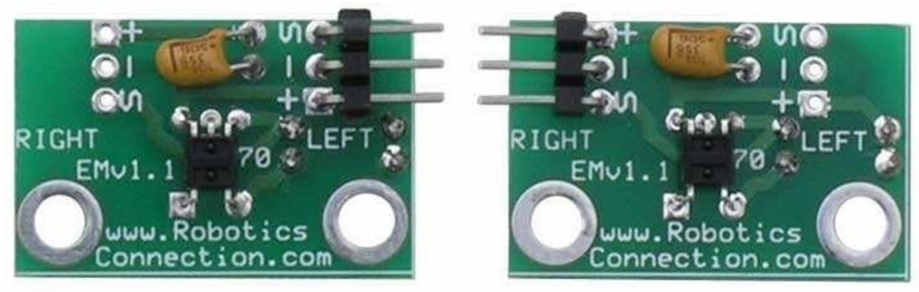


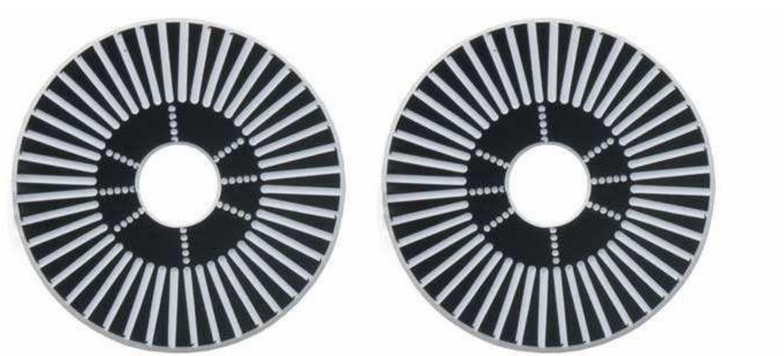
Typical Motion Models

- In practice, one often finds two types of motion models:
  - Odometry-based
  - Velocity-based (dead reckoning)
- Odometry-based models are used when systems are equipped with wheel encoders.
- Velocity-based models have to be applied when no wheel encoders are given.
- They calculate the new pose based on the velocities and the time elapsed.

Example: Wheel Encoders

These modules require
+5V and GND to power them, and provide a 0 to 5V output. They provide +5V output when they "see" white, and a 0V output when they "see" black.



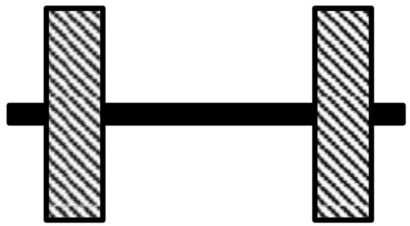


These disks are manufactured out of high quality laminated color plastic to offer a very crisp black to white transition. This enables a wheel encoder sensor to easily see the transitions.

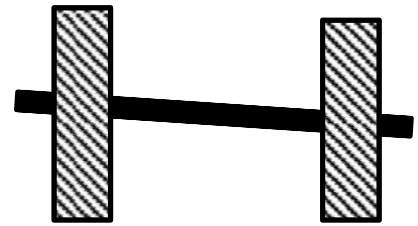
Dead Reckoning

- Derived from “deduced reckoning.”
- Mathematical procedure for determining the present location of a vehicle.
- Achieved by calculating the current pose of the vehicle based on its velocities and the time elapsed.

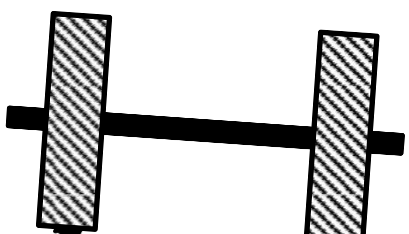
Reasons for Motion Errors



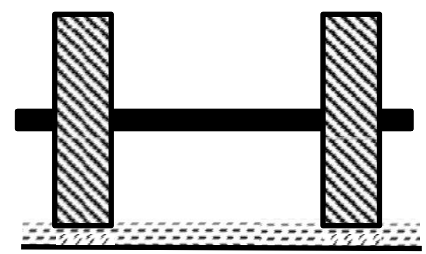
ideal case



different wheel diameters



bump



carpet

and many more …

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- Velocity-based Models
- Map-consistent Models

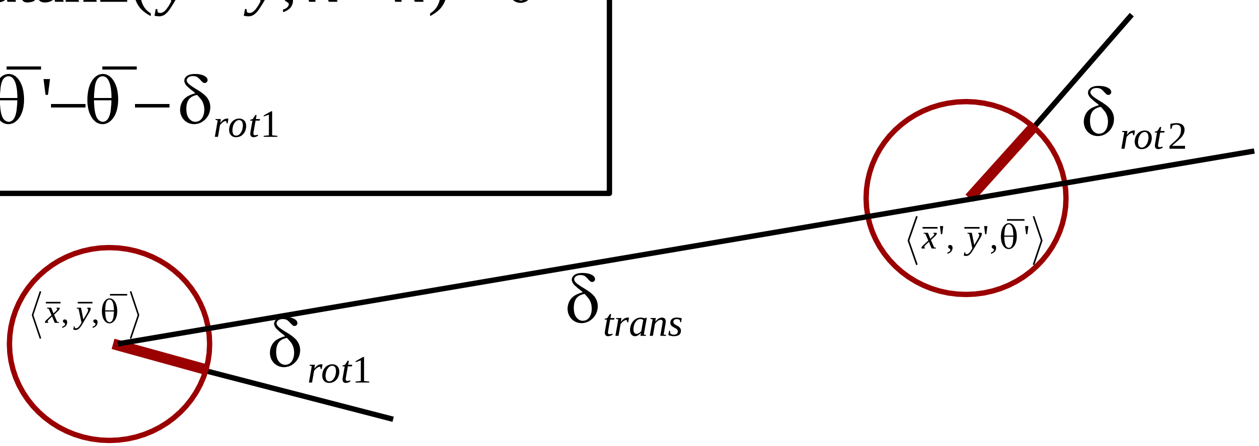
Odometry Model

- Robot moves from $\langle \bar{x}, \bar{y}, \bar{\theta} \rangle$ to $\langle \bar{x}', \bar{y}', \bar{\theta}' \rangle$.
- Odometry information $u = \langle \delta_{rot1}, \delta_{rot2}, \delta_{trans} \rangle$.

$$\delta_{trans} = \sqrt{(\bar{x}' - \bar{x})^2 + (\bar{y}' - \bar{y})^2}$$

$$\delta_{rot1} = \operatorname{atan2}(\bar{y}' - \bar{y}, \bar{x}' - \bar{x}) - \bar{\theta}$$

$$\delta_{rot2} = \bar{\theta}' - \bar{\theta} - \delta_{rot1}$$



The atan2 Function

- Extends the inverse tangent and correctly copes with the signs of x and y.

$$\mathrm{atan2}(y,x) = \begin{cases} \mathrm{atan}(y/x) & \text{if } x > 0 \\ \mathrm{sign}(y)\,(\pi - \mathrm{atan}(|y/x|)) & \text{if } x < 0 \\ 0 & \text{if } x = y = 0 \\ \mathrm{sign}(y)\,\pi/2 & \text{if } x = 0, y \neq 0 \end{cases}$$

Noise Model for Odometry

- The measured motion is given by the true motion corrupted with noise.

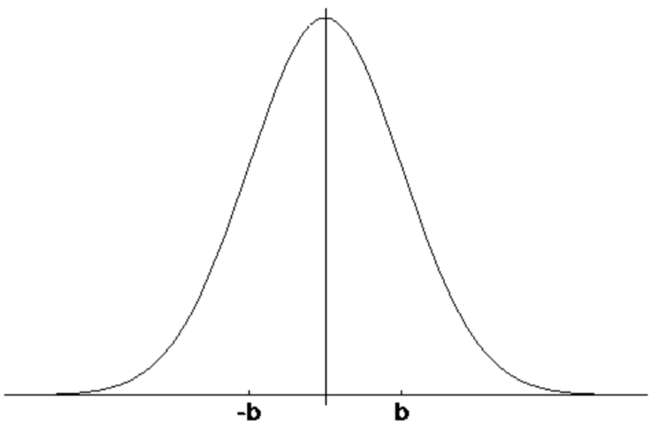
$$\hat{\delta}_{rot1} = \delta_{rot1} + \varepsilon_{\alpha_1 |\delta_{rot1}| + \alpha_2 |\delta_{trans}|}$$

$$\hat{\delta}_{trans} = \delta_{trans} + \varepsilon_{\alpha_3 |\delta_{trans}| + \alpha_4 |\delta_{rot1} + \delta_{rot2}|}$$

$$\hat{\delta}_{rot2} = \delta_{rot2} + \varepsilon_{\alpha_1 |\delta_{rot2}| + \alpha_2 |\delta_{trans}|}$$

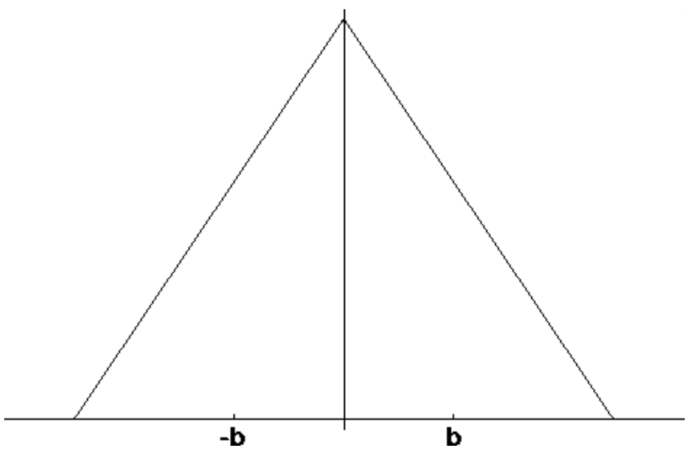
Typical Distributions for PMM

Normal distribution



$$\varepsilon_{o^2}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$$

Triangular distribution



$$\varepsilon_{o^2}(x) = \begin{cases} 0 \text{ if } |x| > \sqrt{6\sigma^2} \\ \dfrac{\sqrt{6\sigma^2} - |x|}{6\sigma^2} \end{cases}$$

Calculating the Probability

- For a normal distribution

1. Algorithm prob_normal_distribution(a,b):
2. return $\frac{1}{\sqrt{2\pi\, b^2}} \exp\left\{-\frac{1}{2}\frac{a^2}{b^2}\right\}$

- For a triangular distribution

1. Algorithm prob_triangular_distribution(a,b):
2. return $\max\left\{0, \frac{1}{\sqrt{6}\, b} - \frac{|a|}{6\, b^2}\right\}$

Calculating the Posterior

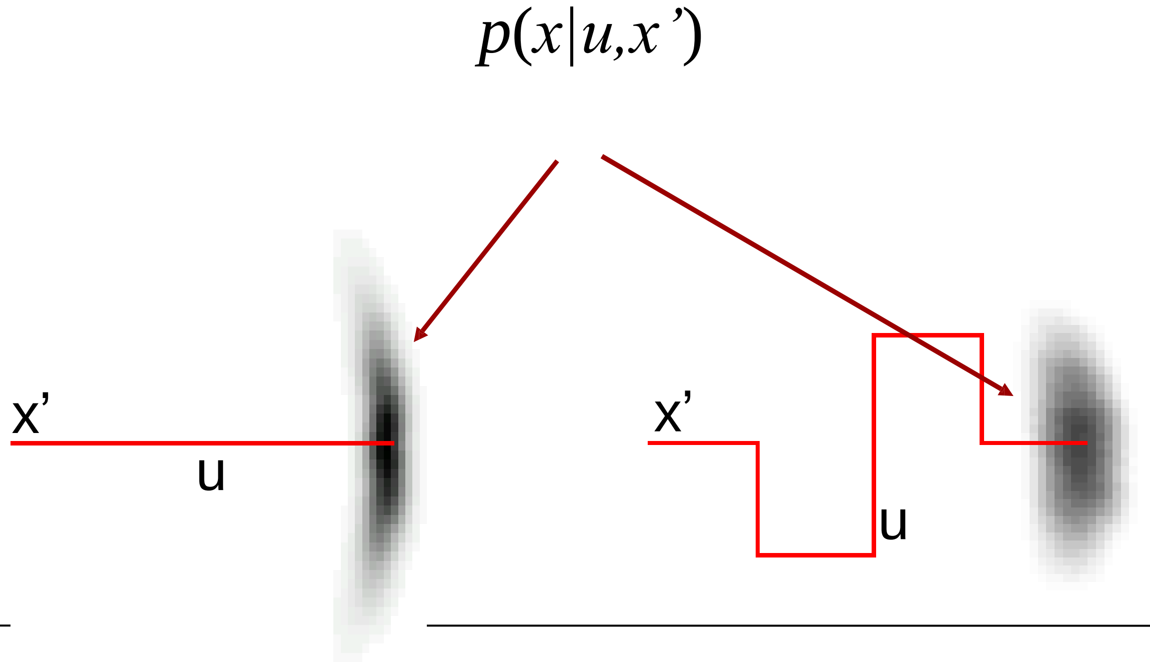
1. Algorithm motion_model_odometry(x,x',u)
2. $\delta_{trans} = \sqrt{(\bar{x}'-\bar{x})^2 + (\bar{y}'-\bar{y})^2}$
3. $\delta_{rot1} = \text{atan2}(\bar{y}'-\bar{y}, \bar{x}'-\bar{x}) - \bar{\theta}$
4. $\delta_{rot2} = \bar{\theta}' - \bar{\theta} - \delta_{rot1}$
5. $\hat{\delta}_{trans} = \sqrt{(x'-x)^2 + (y'-y)^2}$
6. $\hat{\delta}_{rot1} = \text{atan2}(y'-y, x'-x) - \bar{\theta}$
7. $\hat{\delta}_{rot2} = \theta' - \theta - \hat{\delta}_{rot1}$
8. $p_1 = \text{prob}(\delta_{\text{rot1}} - \hat{\delta}_{\text{rot1}}, \alpha_1 |\hat{\delta}_{\text{rot1}}| + \alpha_2 \hat{\delta}_{\text{trans}})$
9. $p_2 = \text{prob}(\delta_{\text{trans}} - \hat{\delta}_{\text{trans}}, \alpha_3 \hat{\delta}_{\text{trans}} + \alpha_4 (|\hat{\delta}_{\text{rot1}}| + |\hat{\delta}_{\text{rot2}}|))$
10. $p_3 = \text{prob}(\delta_{\text{rot2}} - \hat{\delta}_{\text{rot2}}, \alpha_1 |\hat{\delta}_{\text{rot2}}| + \alpha_2 \hat{\delta}_{\text{trans}})$
11. return $p_1 \cdot p_2 \cdot p_3$

odometry values (u)

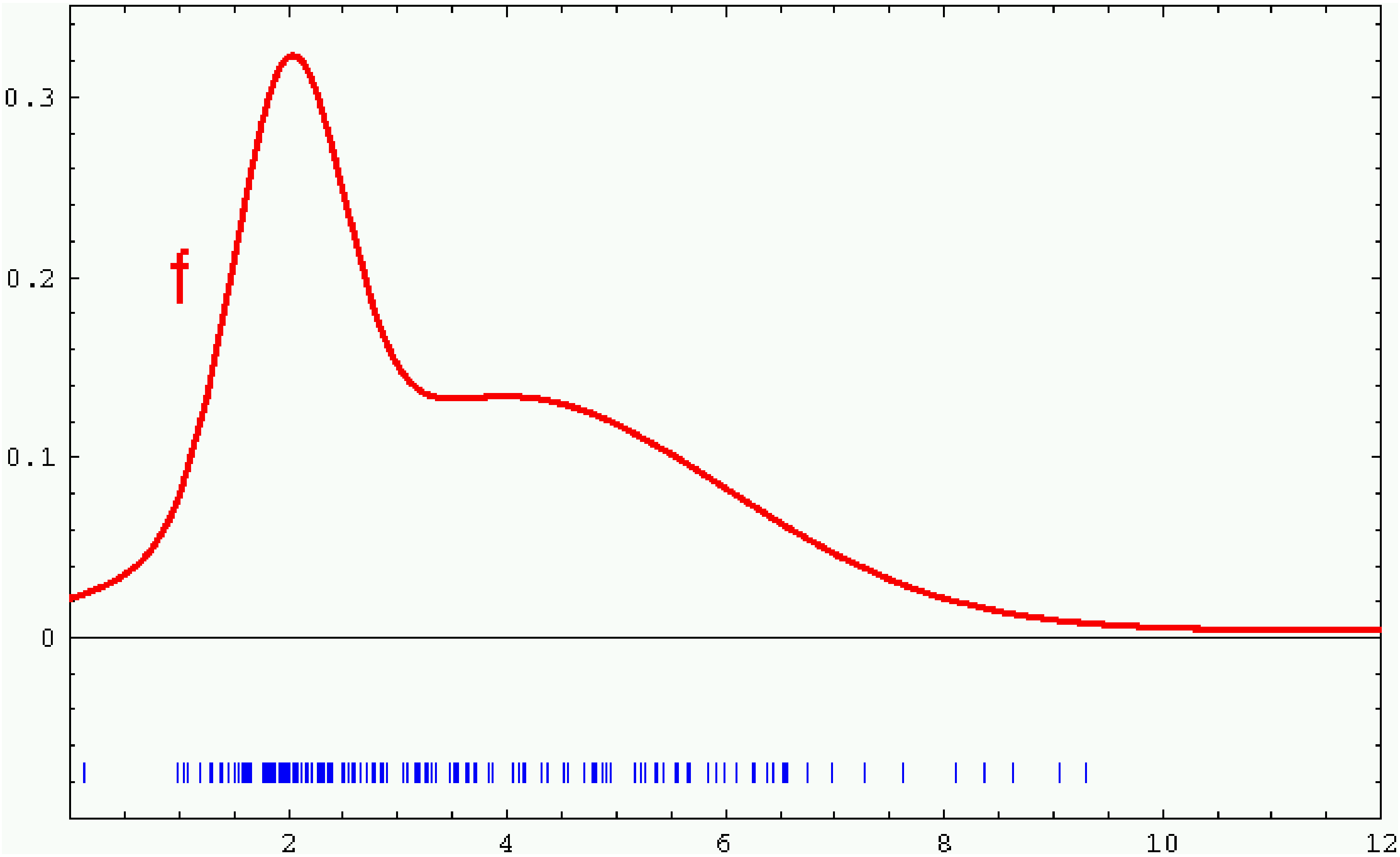
values of interest (x,x')

Application

- Repeated application of the sensor model for short movements.
- Typical banana-shaped distributions obtained for 2d-projection of 3d posterior.



Sample-based Density Representation



Sample from Distributions

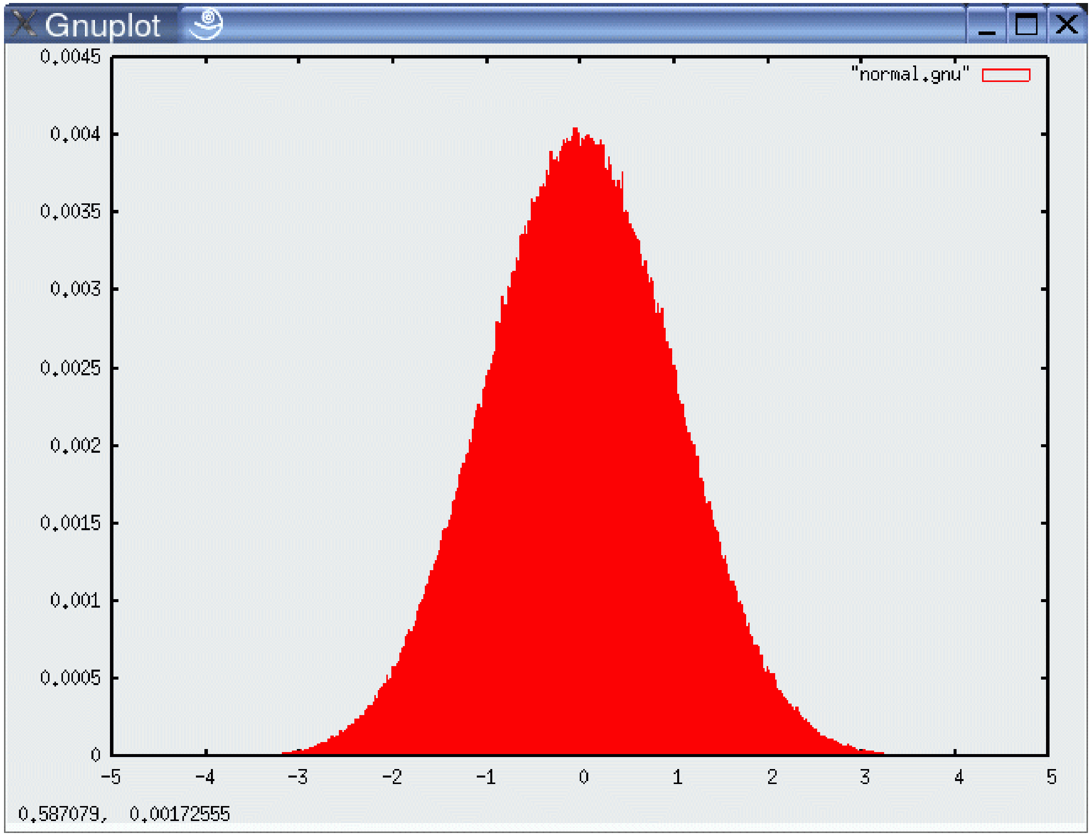
- Sampling from a normal distribution

1. Algorithm sample_normal_distribution(b):
2. return $\frac{1}{2}\sum_{i=1}^{12} rand(-b,b)$

- Sampling from a triangular distribution

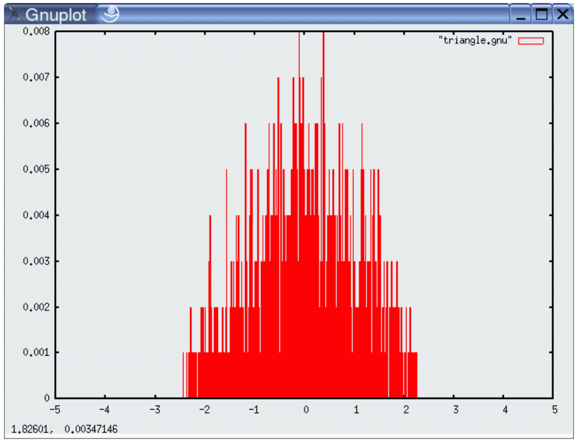
1. Algorithm sample_triangular_distribution(b):
2. return $\frac{\sqrt{6}}{2}\left[\mathbf{rand}(-b,b)+\mathbf{rand}(-b,b)\right]$

Normally Distributed Samples

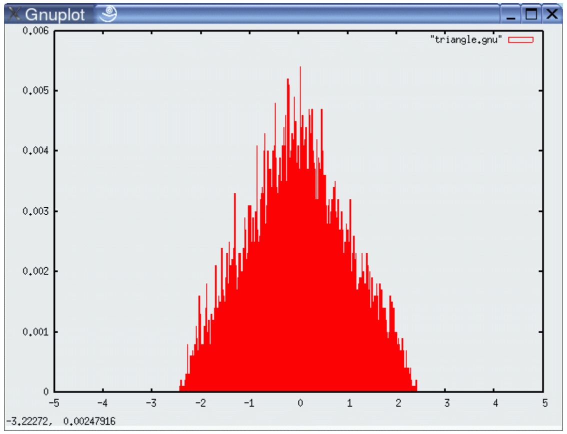


10^6 samples

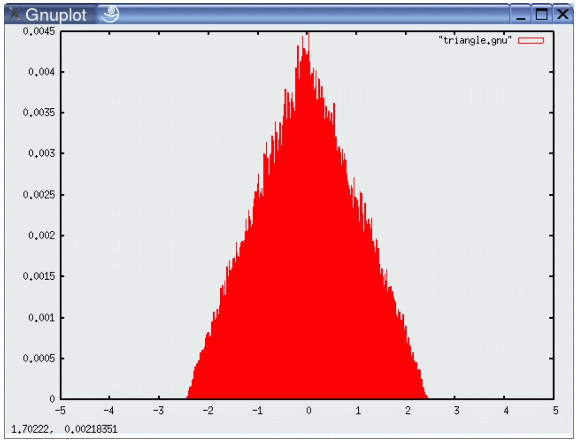
Triangular Distribution Samples



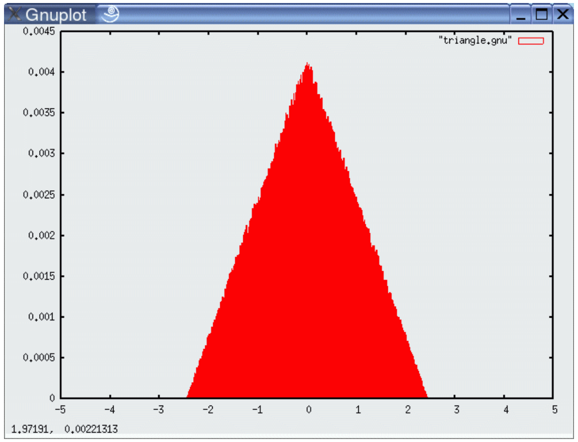
10^3 samples



10^4 samples



10^5 samples



10^6 samples

Rejection Sampling

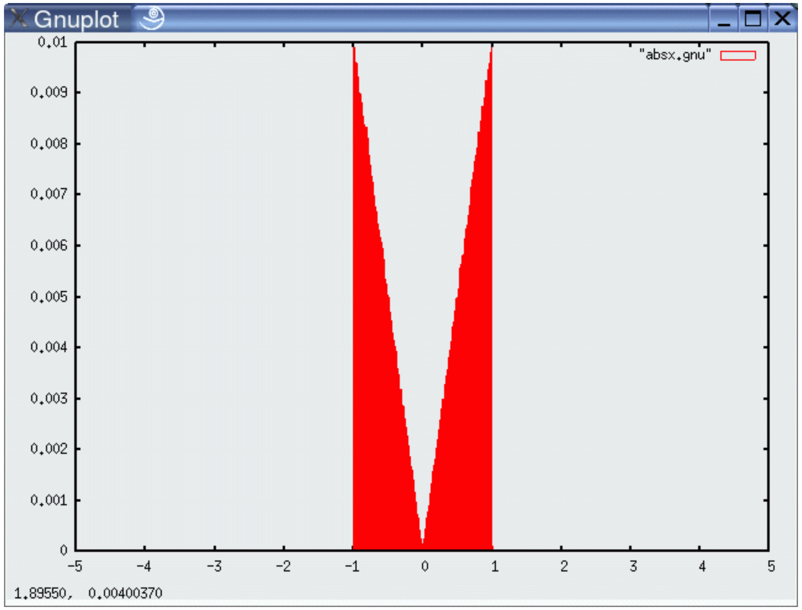
- Sampling from arbitrary distributions

1. Algorithm **sample_distribution**(f, b):
2. repeat
3. $\boxed{\begin{array}{lcl} x & = & \mathbf{rand}(-b, b) \\ y & = & \mathbf{rand}(0, \max\{f(x) \mid x \in (-b, b)\}) \end{array}}$
4.
5. until $(y \leq f(x))$
6. return x

Example

- Sampling from

$$f(x) = \begin{cases} \mathsf{abs}(x) & x \in [-1;1] \\ 0 & \text{otherwise} \end{cases}$$



Sampling Odometry Motion Model

1. **Algorithm sample_motion_model**(u, x):

   $u = \langle \delta_{rot1}, \delta_{rot2}, \delta_{trans} \rangle, x = \langle x, y, \theta \rangle$

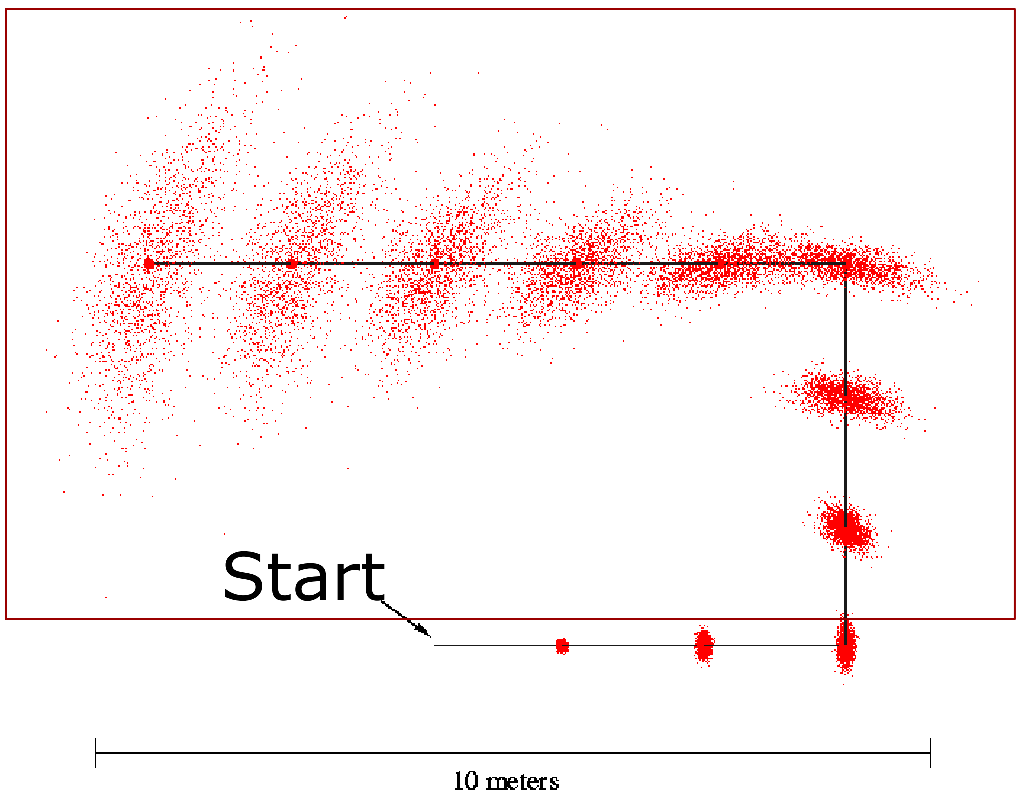
2. $\hat{\delta}_{rot1} = \delta_{rot1} + \text{sample}(\alpha_1 |\delta_{rot1}| + \alpha_2 \delta_{trans})$
3. $\hat{\delta}_{trans} = \delta_{trans} + \text{sample}(\alpha_3 \delta_{trans} + \alpha_4 (|\delta_{rot1}| + |\delta_{rot2}|))$
4. $\hat{\delta}_{rot2} = \delta_{rot2} + \text{sample}(\alpha_1 |\delta_{rot2}| + \alpha_2 \delta_{trans})$

5. $x' = x + \hat{\delta}_{trans} \cos(\theta + \hat{\delta}_{rot1})$
6. $y' = y + \hat{\delta}_{trans} \sin(\theta + \hat{\delta}_{rot1})$
7. $\theta' = \theta + \hat{\delta}_{rot1} + \hat{\delta}_{rot2}$

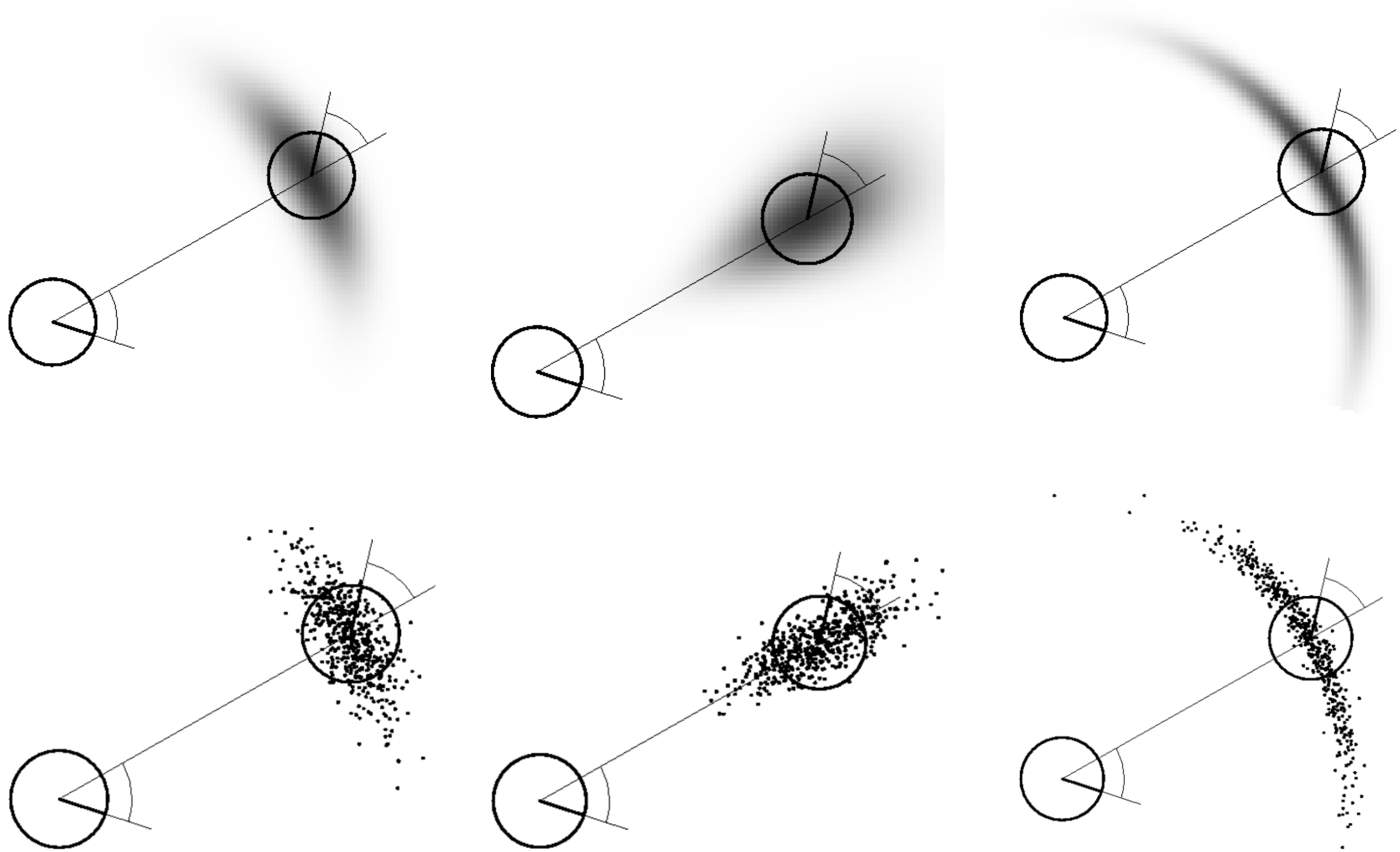
8. Return $\langle x', y', \theta' \rangle$

sample_normal_distribution

Sampling from Motion Model



Examples (Odometry-Based)



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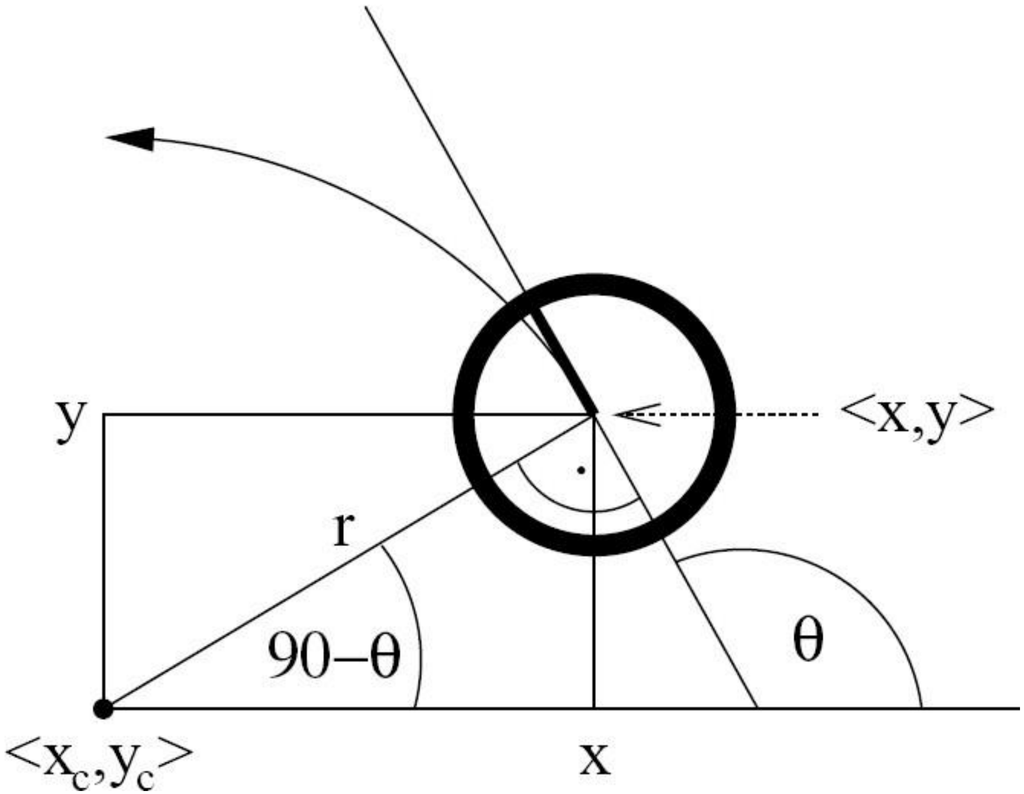
Velocity-Based Model



Equation for the Velocity Model

Center of circle:

$$\begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -\lambda \sin\theta \\ \lambda \cos\theta \end{pmatrix} = \begin{pmatrix} \frac{x+x'}{2} + \mu(y-y') \\ \frac{y+y'}{2} + \mu(x'-x) \end{pmatrix}$$

with

$$\mu = \frac{1}{2} \frac{(x-x')\cos\theta + (y-y')\sin\theta}{(y-y')\cos\theta - (x-x')\sin\theta}$$

Posterior Probability for Velocity Model

Algorithm motion_model_velocity(x_t, u_t, x_{t-1}**):**

$$\mu = \frac{1}{2}\,\frac{(x - x')\cos\theta + (y - y')\sin\theta}{(y - y')\cos\theta - (x - x')\sin\theta}$$

$$x^* = \frac{x + x'}{2} + \mu(y - y')$$

$$y^* = \frac{y + y'}{2} + \mu(x' - x)$$

$$r^* = \sqrt{(x - x^*)^2 + (y - y^*)^2}$$

$$\Delta\theta = \operatorname{atan2}(y' - y^*, x' - x^*) - \operatorname{atan2}(y - y^*, x - x^*)$$

$$\hat{v} = \frac{\Delta\theta}{\Delta t}\, r^*$$

$$\hat{\omega} = \frac{\Delta\theta}{\Delta t}$$

$$\hat{\gamma} = \frac{\theta' - \theta}{\Delta t} - \hat{\omega}$$

return $\mathbf{prob}(v - \hat{v}, \alpha_1|v| + \alpha_2|\omega|) \cdot \mathbf{prob}(\omega - \hat{\omega}, \alpha_3|v| + \alpha_4|\omega|) \cdot \mathbf{prob}(\hat{\gamma}, \alpha_5|v| + \alpha_6|\omega|)$

Sampling from Velocity Model

Algorithm sample_motion_model_velocity(u_t, x_{t-1})**:**

$\hat{v} = v + \mathbf{sample}(\alpha_1|v| + \alpha_2|\omega|)$

$\hat{\omega} = \omega + \mathbf{sample}(\alpha_3|v| + \alpha_4|\omega|)$

$\hat{\gamma} = \mathbf{sample}(\alpha_5|v| + \alpha_6|\omega|)$

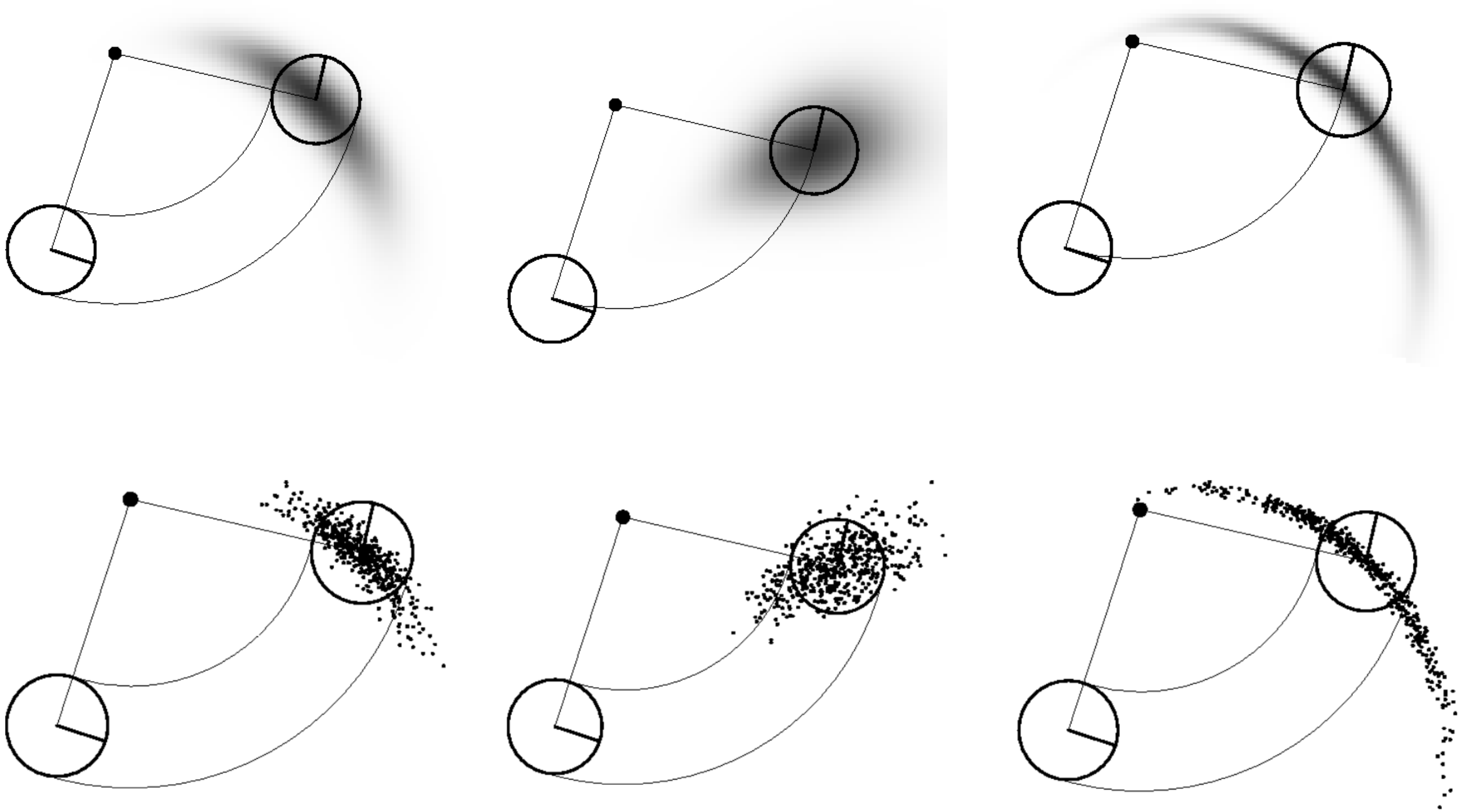
$x' = x - \frac{\hat{v}}{\hat{\omega}} \sin\theta + \frac{\hat{v}}{\hat{\omega}}\ sin(\theta + \hat{\omega}\Delta t)$

$y' = y + \frac{\hat{v}}{\hat{\omega}} \cos\theta - \frac{\hat{v}}{\hat{\omega}}\ cos(\theta + \hat{\omega}\Delta t)$

$\theta' = \theta + \hat{\omega}\Delta t + \hat{\gamma}\Delta t$

return $x_t = (x', y', \theta')^T$

Examples (velocity based)



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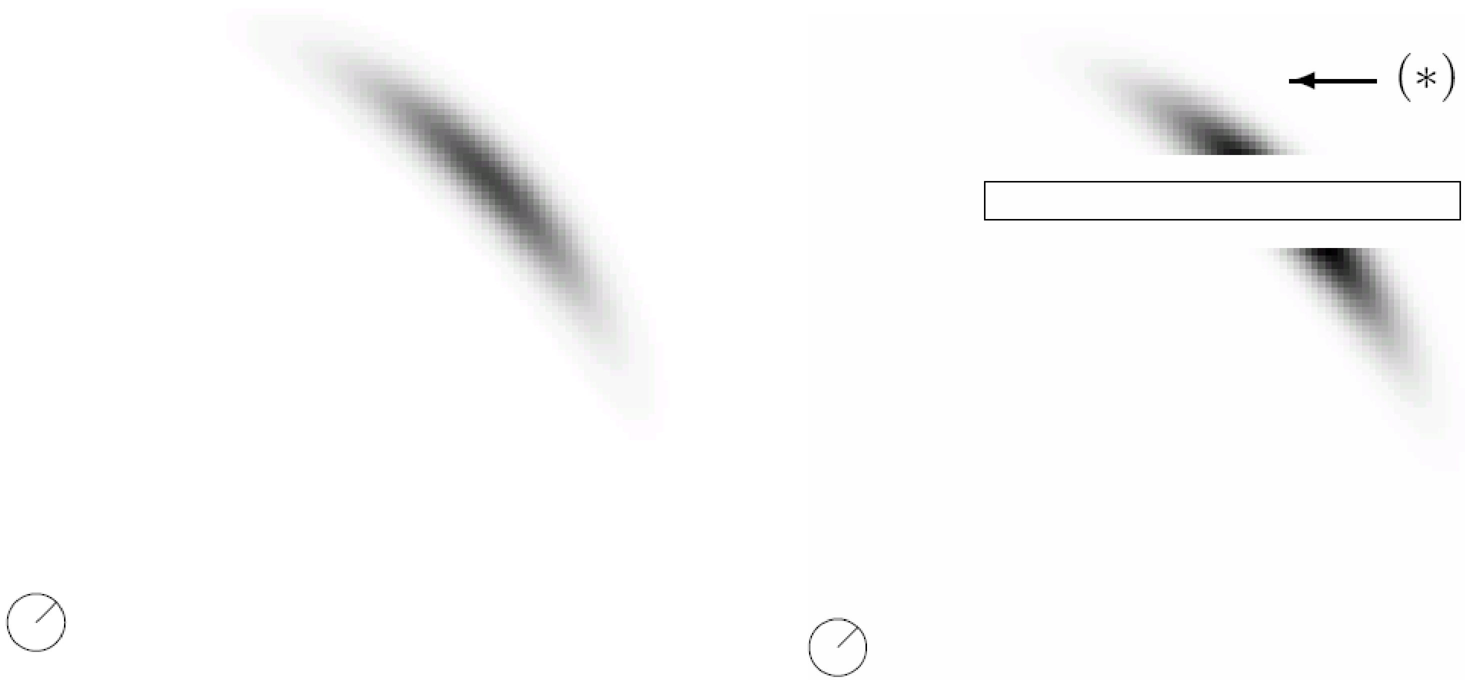
Map-Consistent Motion Model



$$p(x \mid u, x') \neq p(x \mid u, x', m)$$

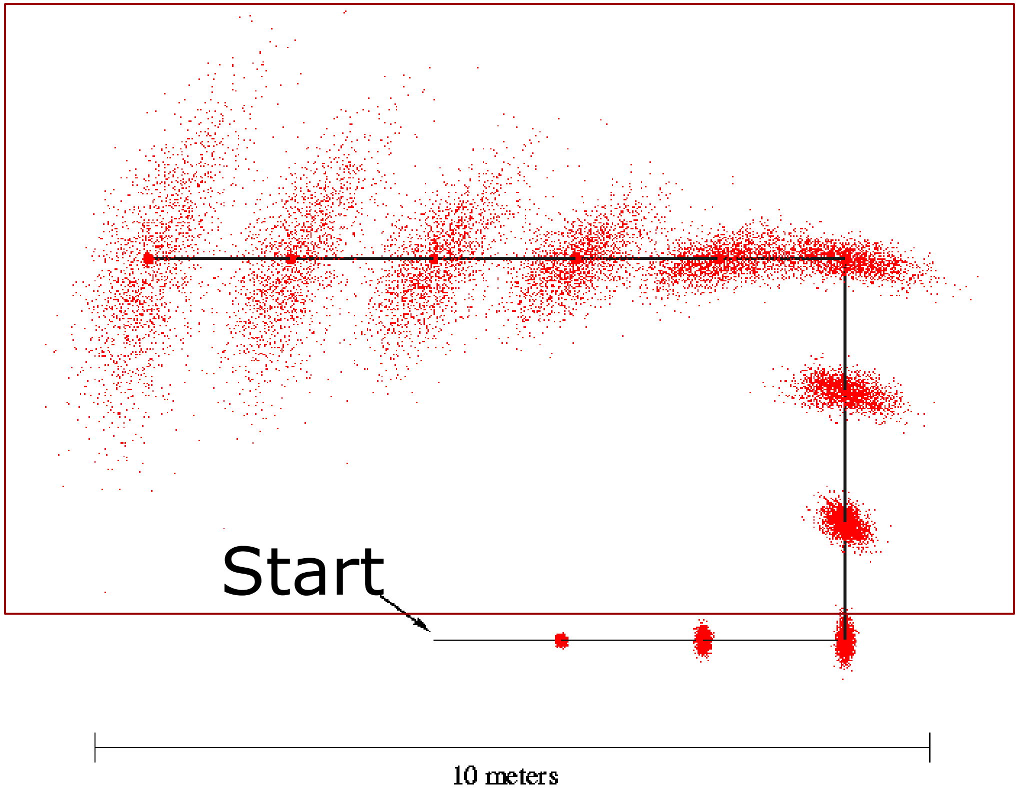
Approximation: $p(x \mid u, x', m) = \eta \; p(x \mid m) \; p(x \mid u, x')$

Summary

- We discussed motion models for odometry-based and velocity-based systems
- We discussed ways to calculate the posterior probability $p(x|\, x', u)$.
- We also described how to sample from $p(x|\, x', u)$.
- Typically the calculations are done in fixed time intervals $\triangle t$.
- In practice, the parameters of the models have to be learned.
- We also discussed an extended motion model that takes the map into account.

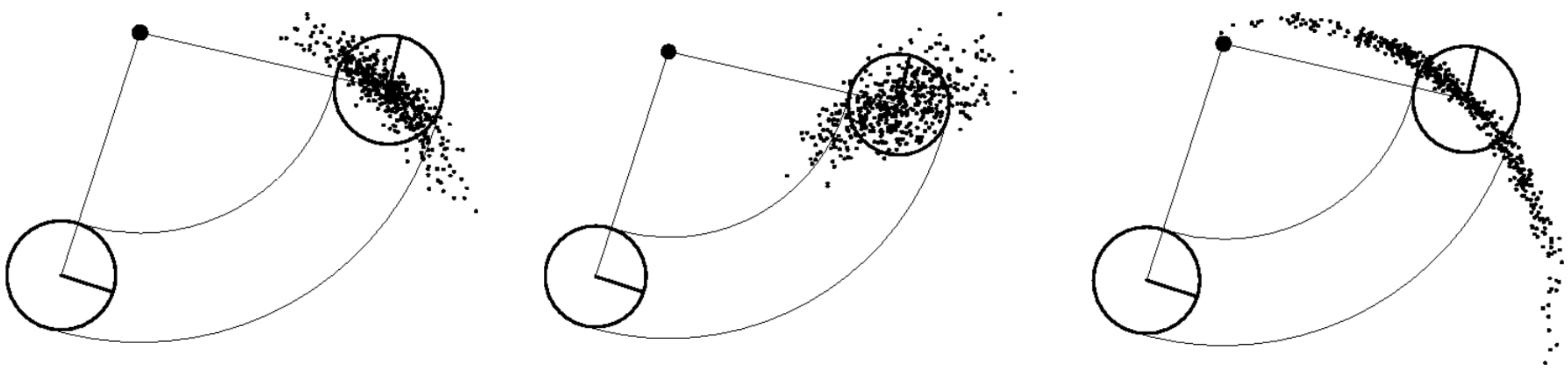
Homework 4

Problem 1: Please generate samples of the odometry-based motion model (N=500).



Homework 4

Problem 2: Please generate samples of the velocity-based motion model for following cases (N=500).



Homework 4

Problem 3: Please generate the map-consistent probability model in the following situation.