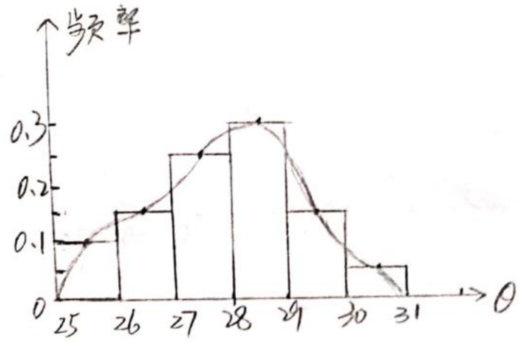
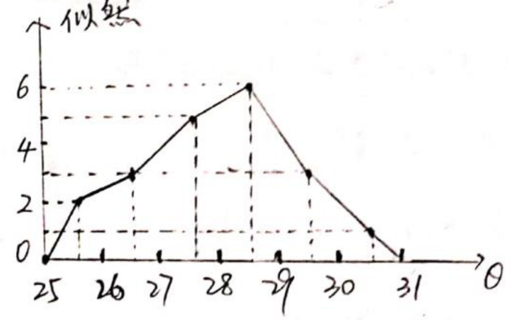
韦来生《贝叶斯统计》习题二

1. 如图所示.



2. 设 $\theta=25.5, 26.5, \cdots, 30.5$ 的可能性(似然)分别为 2, 3, 5, 6, 3, 1，$\theta=25$ 或 $\theta=31$ 的可能性为 0. 如下图

由 $\int_{\Theta}\pi(\theta)d\theta=1$ 得 $c=0.05$



3. (1) θ 的先验密度的 1/4 分位数为 27, 1/2 分位数为 28.

(2) 设 $\theta\sim N(\mu,\sigma^2)$, $\hat{\mu}=\theta(1/2)=28$,

由 $\frac{1}{4}=P(\theta\le 27)=P\left(\frac{\theta-28}{\sigma}\le\frac{27-28}{\sigma}\right)=P\left(Z\le-\frac{1}{\sigma}\right)$

故 $-\frac{1}{\sigma}=Z(1/4)=-0.674$ 因此 $\hat{\sigma}=1.483$

故 $\theta \sim N(28, 1.483^2)$

(3) (2)中所求 0.8、0.9 分位数分别为 29.248，29.900。主观决定的 0.8 分位数为 29，0.9 分位数为 29.667，与(2)中正态分布基本一致，因此不需要修改(2)中的正态密度。

4. (2) 设 $\theta \sim Cauchy(\mu, \lambda)$，则 $\mu = 28$.

$$f(\theta;\lambda) = \frac{\lambda}{\pi[\lambda^2+(\theta-28)^2]}, \quad F(\theta,\lambda) = \frac{1}{\pi}\arctan\left(\frac{\theta-28}{\lambda}\right)+\frac{1}{2}$$

由于 $\frac{1}{4} = P(\theta \le 27) = \frac{1}{\pi}\arctan\left(-\frac{1}{\lambda}\right)+\frac{1}{2} = \frac{1}{2}-\frac{1}{\pi}\arctan\left(\frac{1}{\lambda}\right)$

故 $\arctan\left(\frac{1}{\lambda}\right) = \frac{\pi}{4}$ 从而 $\frac{1}{\lambda} = \tan\frac{\pi}{4} = 1$，因此 $\lambda = 1$

故 $\theta \sim Cauchy(28, 1)$

(3) (2)中所求 0.8、0.9 分位数分别为 29.376，31.078。与主观决定的 0.8 分位数 29、0.9 分位数 29.667 相差较大。实际上主观决定的 0.95 分位数为 30，说明需要修改(2)中的结果。

5. $$m(x) = \int_{\Theta} \pi(\theta)\cdot f(x|\theta)\,d\theta = \int_0^{+\infty}\left(100\theta^{-2}e^{-\frac{100}{\theta}}\right)\left(\theta^{-1}e^{-\frac{x}{\theta}}\right)d\theta$$

$$= \int_0^{+\infty} 100\theta^{-3}e^{-\frac{100+x}{\theta}}\,d\theta = \frac{100}{(100+x)^2}\int_0^{+\infty}(100+x)^2\theta^{-3}e^{-\frac{100+x}{\theta}}\,d\theta = \frac{100}{(100+x)^2}\ (x>0)$$

$$P(X<200) = \int_0^{200} m(x)\,dx = \int_0^{200}\frac{100}{(100+x)^2}\,dx = -\frac{100}{x+100}\Big|_0^{200} = \frac{2}{3}$$

6. $m(x_i)=\int_0^{\infty}\left(\frac{\lambda^{\gamma}}{\Gamma(\gamma)}\theta_i^{\gamma-1}e^{-\lambda\theta_i}\right)\cdot\frac{e^{-\theta_i}\cdot\theta_i^{x_i}}{x_i!}d\theta_i$

$$=\int_0^{\infty}\frac{\lambda^{\gamma}}{\Gamma(\gamma)\cdot x_i!}\cdot\theta_i^{x_i+\gamma-1}e^{-\theta_i(\lambda+1)}d\theta_i$$

$$=\frac{\lambda^{\gamma}}{\Gamma(\gamma)\cdot x_i!}\cdot\frac{\Gamma(x_i+\gamma)}{(\lambda+1)^{x_i+\gamma}}\int_0^{\infty}\frac{(\lambda+1)^{x_i+\gamma}}{\Gamma(x_i+\gamma)}\theta_i^{x_i+\gamma-1}e^{-\theta_i(\lambda+1)}d\theta_i$$

$$=\frac{\lambda^{\gamma}\,\Gamma(x_i+\gamma)}{\Gamma(\gamma)\,x_i!\,(\lambda+1)^{x_i+\gamma}}$$

故 $m(\vec{x})=\prod_{i=1}^{n}m(x_i)=\prod_{i=1}^{n}\frac{\lambda^{\gamma}\Gamma(x_i+\gamma)}{\Gamma(\gamma)\,x_i!\,(\lambda+1)^{x_i+\gamma}}$，$x_i=0,1,2,\cdots$，$\forall i$.

7. 若 $n=3$，$x_1=3$，$x_2=0$，$x_3=5$，$\gamma=4$. 找出6题中 ML-II 先验.

由似然函数 $L=\prod_{i=1}^{n}\frac{\lambda^{\gamma}\Gamma(x_i+\gamma)}{\Gamma(\gamma)\,x_i!\,(\lambda+1)^{x_i+\gamma}}=C\cdot\frac{\lambda^{n\gamma}}{(\lambda+1)^{\sum x_i+n\gamma}}$

$\frac{1}{C}\log L=l=n\gamma\log\lambda-(\sum x_i+n\gamma)\log(\lambda+1)$

由 $0=\frac{\partial l}{\partial\lambda}=\frac{n\gamma}{\lambda}-\frac{\sum x_i+n\gamma}{\lambda+1}=0$ 得 $\lambda=\frac{n\gamma}{\sum x_i}=\frac{\gamma}{\bar{x}}$

代入数据得，$\hat{\lambda}=\frac{4}{\frac{8}{3}}=\frac{3}{2}$

故 ML-II 先验 $\hat{\theta}$ 为 $\Gamma(4,\frac{3}{2})$

8. 先验分布 $\theta\sim\Gamma(\gamma,\lambda)$，$X|\theta\sim P(\theta)$.

总体均值 $\mu(\theta)=\theta$，总体方差 $\sigma^2(\theta)=\theta$.

设 $\vec{\lambda}=(\gamma,\lambda)$

从而 $\mu_m(\vec{\lambda}) = E^{\theta|\vec{\lambda}}(\theta) = \frac{\gamma}{\lambda}$

$$\sigma_m^2(\vec{\lambda}) = E^{\theta|\vec{\lambda}}(\theta) + E^{\theta|\vec{\lambda}}\left(\theta - \frac{\gamma}{\lambda}\right)^2$$

$$= E^{\theta|\vec{\lambda}}\left(\theta^2 + \left(1 - \frac{2\gamma}{\lambda}\right)\theta + \frac{\gamma^2}{\lambda^2}\right)$$

$$= Var^{\theta|\vec{\lambda}}(\theta) + \left(E^{\theta|\vec{\lambda}}\theta\right)^2 + \left(1 - \frac{2\gamma}{\lambda}\right)E^{\theta|\vec{\lambda}}(\theta) + \frac{\gamma^2}{\lambda^2}$$

$$= \frac{\gamma}{\lambda^2} + \left(\frac{\gamma}{\lambda}\right)^2 + \left(1 - \frac{2\gamma}{\lambda}\right)\cdot\frac{\gamma}{\lambda} + \frac{\gamma^2}{\lambda^2}$$

$$= \frac{\gamma}{\lambda^2} + \frac{\gamma}{\lambda}$$

由 $\begin{cases} \frac{\gamma}{\lambda} = \overline{X} \\ \frac{\gamma}{\lambda^2} + \frac{\gamma}{\lambda} = S^2 \end{cases}$ 解得 $\begin{cases} \gamma = \frac{\overline{X}^2}{S^2 - \overline{X}} \\ \lambda = \frac{\overline{X}}{S^2 - \overline{X}} \end{cases}$

由于 $\gamma > 0, \lambda > 0$，$X_i \geq 0, \forall i$，从而 $0 < \overline{X} < S^2$.

9. $X \sim Exp(\theta)$，$\theta \sim \Gamma(\alpha, \lambda)$

边缘分布 $m(x) = \int_{\Theta} f(x|\theta)\pi(\theta)\,d\theta = \int_{\Theta} \theta e^{-\theta x} \frac{\lambda^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\lambda\theta}\,d\theta$

$$= \int_0^{+\infty} \frac{\lambda^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+1)}{(x+\lambda)^{\alpha+1}}\left[\frac{(x+\lambda)^{\alpha+1}}{\Gamma(\alpha+1)}\theta^\alpha e^{-(x+\lambda)\theta}\right]d\theta.$$

$$= \frac{\alpha\lambda^\alpha}{(x+\lambda)^{\alpha+1}} \quad (x > 0)$$

设 $\vec{\lambda} = (\alpha, \lambda)$

总体均值 $\mu(\theta)=\frac{1}{\theta}$，总体方差 $\sigma^2(\theta)=\frac{1}{\theta^2}$.

故 $\mu_m(\vec{\lambda})=E^{\theta|\vec{\lambda}}\left(\frac{1}{\theta}\right)=\int_0^{\infty}\frac{1}{\theta}\cdot\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\lambda\theta}d\theta$

$$=\int_0^{\infty}\frac{\Gamma(\alpha-1)\cdot\lambda}{\Gamma(\alpha)}\left(\frac{\lambda^{\alpha-1}}{\Gamma(\alpha-1)}\theta^{\alpha-2}e^{-\lambda\theta}\right)d\theta=\frac{\lambda}{\alpha-1}$$

$$\sigma_m^2(\lambda)=E^{\theta|\vec{\lambda}}\left(\frac{1}{\theta^2}\right)+E^{\theta|\vec{\lambda}}\left(\frac{1}{\theta}-\frac{\lambda}{\alpha-1}\right)^2$$

$$=2E^{\theta|\vec{\lambda}}\left(\frac{1}{\theta^2}\right)-\frac{2\lambda}{\alpha-1}E^{\theta|\vec{\lambda}}\left(\frac{1}{\theta}\right)+\frac{\lambda^2}{(\alpha-1)^2}$$

由于 $E^{\theta|\vec{\lambda}}\left(\frac{1}{\theta^2}\right)=\int_0^{\infty}\frac{1}{\theta^2}\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\lambda\theta}d\theta=\int_0^{\infty}\frac{\Gamma(\alpha-2)\lambda^2}{\Gamma(\alpha)}\left(\frac{\lambda^{\alpha-2}}{\Gamma(\alpha-2)}\theta^{\alpha-3}e^{-\lambda\theta}\right)d\theta=\frac{\lambda^2}{(\alpha-1)(\alpha-2)}$

故 $\sigma_m^2(\lambda)=\frac{2\lambda^2}{(\alpha-1)(\alpha-2)}-\frac{2\lambda}{\alpha-1}\cdot\frac{\lambda}{\alpha-1}+\frac{\lambda^2}{(\alpha-1)^2}$

$$=\frac{2\lambda^2}{(\alpha-1)(\alpha-2)}-\frac{\lambda^2}{(\alpha-1)^2}=\frac{\lambda^2\alpha}{(\alpha-1)^2(\alpha-2)}$$

由 $\begin{cases}\frac{\lambda}{\alpha-1}=\overline{x}\\ \frac{\lambda^2\alpha}{(\alpha-1)^2(\alpha-2)}=S^2\end{cases}$ 解得 $\begin{cases}\hat{\lambda}=\frac{2S^2}{S^2-\overline{x}^2}\\ \hat{\alpha}=1+\frac{2S^2}{\overline{x}(S^2-\overline{x}^2)}\end{cases}$

代入数据解得 $\hat{\lambda}=\frac{2\times8}{8-2^2}=4$. $\hat{\alpha}=1+\frac{2\times8}{2\times(8-2^2)}=3$

故 θ 的先验分布为 $\Gamma(3,4)$

10、(1) X的密度函数为 $\frac{1}{2}I_{x\in[\theta-1,\theta+1]}=\frac{1}{2}I_{x-\theta\in[-1,1]}=f(x-\theta)$

故该分布族是位置参数族，无信息先验 $\pi(\theta)=1$

(2) X的密度函数为 $\frac{\beta}{\pi(x^2+\beta^2)}=\frac{1}{\pi\beta\left[\left(\frac{x}{\beta}\right)^2+1\right]}$

$=\frac{1}{\beta}\varphi\left(\frac{x}{\beta}\right)$ 故该分布族是刻度参数族，取无信息先验

$\pi(\beta)=\frac{1}{\beta}$，$\beta>0$

(3) X的密度函数为 $p=\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\sqrt{n\pi}\cdot\sigma}\left[1+\frac{1}{n}\left(\frac{x-\mu}{\sigma}\right)^2\right]^{-\frac{n+1}{2}}$

由于 $p=f(x-\mu)$，故它是位置参数族，μ的无信息先验 $\pi_1(\mu)=1$.

若令 $Y=X-\mu$，则 $p=\frac{1}{\sigma}\varphi\left(\frac{Y}{\sigma}\right)$，故它是刻度参数族，取$\sigma$的无信息先验 $\pi_2(\sigma)=\frac{1}{\sigma}$ $(\sigma>0)$

(4) X的密度函数为 $\frac{\alpha}{x_0}\left(\frac{x_0}{x}\right)^{\alpha+1}I_{x>x_0}=\frac{\alpha}{x_0}\cdot\left(\frac{x}{x_0}\right)^{-(\alpha+1)}\cdot I_{\frac{x}{x_0}>1}$

$=\frac{1}{x_0}\varphi\left(\frac{x}{x_0}\right)$，故它是刻度参数族，取无信息先验

$\pi(x_0)=\frac{1}{x_0}$，$0<x_0<x$.

11. (1) $l=\sum_{i=1}^{n}\ln f(x_i|\lambda)=\sum_{i=1}^{n}\ln\left(\frac{e^{-\lambda}\cdot\lambda^{x_i}}{x_i!}\right)$

$=\sum_{i=1}^{n}\left(-\lambda+x_i\ln\lambda-\ln x_i!\right)$

$=\left(\sum_{i=1}^{n}x_i\right)\ln\lambda-n\lambda+C$

Fisher信息量 $I(\lambda)=E\left\{-\frac{\partial^2 l}{\partial\lambda^2}\right\}=E\left[\frac{\sum X_i}{\lambda^2}\right]=\frac{n\lambda}{\lambda^2}=\frac{n}{\lambda}$

从而 $\sqrt{I(\lambda)}=\sqrt{\frac{n}{\lambda}}$.

故其Jeffrey先验为 $\pi(\lambda)=\sqrt{\frac{1}{\lambda}}$.

(2) $l=\sum_{i=1}^{n}\ln f(x_i|\theta)=\sum_{i=1}^{n}\ln\left[\binom{x_i-1}{r-1}\theta^{r}(1-\theta)^{x_i-r}\right]$

$=\sum_{i=1}^{n}\left[r\ln\theta+(x_i-r)\ln(1-\theta)+\ln\binom{x_i-1}{r-1}\right]$

$=nr\ln\theta+\left(\sum_{i=1}^{n}x_i-nr\right)\ln(1-\theta)+C$

Fisher信息量 $I(\theta)=E\left\{-\frac{\partial^2 l}{\partial\theta^2}\right\}=E\left(\frac{nr}{\theta^2}+\frac{\sum_{i=1}^{n}x_i-nr}{(1-\theta)^2}\right)$

$=\frac{nr}{\theta^2}+\frac{n\cdot\frac{r}{\theta}-nr}{(1-\theta)^2}=\frac{nr}{\theta^2(1-\theta)}$

故其Jeffrey先验为 $\pi(\theta)=\frac{1}{\theta\sqrt{1-\theta}}$.

(3) $l=\sum_{i=1}^{n}\ln f(x_i|\lambda)=\sum_{i=1}^{n}\ln\left[\frac{1}{\lambda}e^{-\frac{x_i}{\lambda}}\right]=\sum_{i=1}^{n}\left(-\frac{x_i}{\lambda}-\ln\lambda\right)$

$=-\frac{\sum_{i=1}^{n}x_i}{\lambda}-n\ln\lambda$.

$I(\lambda)=E\left\{-\frac{\partial^2 l}{\partial\theta^2}\right\}=E\left(\frac{2\sum_{i=1}^{n}x_i}{\lambda^3}-\frac{n}{\lambda^2}\right)=\frac{2n\lambda}{\lambda^3}-\frac{n}{\lambda^2}=\frac{n}{\lambda^2}$

故Jeffery先验 $\pi(\lambda)=\frac{1}{\lambda}$

(4) $l=\sum_{i=1}^{n}\ln f(x_i|\lambda)=\sum_{i=1}^{n}\ln\left(\frac{\lambda^{\alpha}}{\Gamma(\alpha)}x_i^{\alpha-1}e^{-\lambda x_i}\right)$

$=\sum_{i=1}^{n}\left(\alpha\ln\lambda-\lambda x_i+\ln\frac{x_i^{\alpha-1}}{\Gamma(\alpha)}\right)$

$=n\alpha\ln\lambda-\sum_{i=1}^{n}x_i\lambda+C$

$I(\lambda)=E\left\{-\frac{\partial^2 l}{\partial\lambda^2}\right\}=E\left(\frac{n\alpha}{\lambda^2}\right)=\frac{n\alpha}{\lambda^2}$

故 Jeffery先验为 $\pi(\lambda)=\frac{1}{\lambda}$.

(5). $l=\ln\left(\frac{n!}{x_1!\,x_2!\cdots x_k!}p_1^{x_1}p_2^{x_2}\cdots p_k^{x_k}\right)$

$=x_1\ln p_1+\cdots+x_k\ln p_k+C.$

当 $i\neq j$ 时, $I_{ij}(p)=E\left\{-\frac{\partial^2 l}{\partial p_i\partial p_j}\right\}=0$

而 $I_{ii}(p)=E\left\{-\frac{\partial^2 l}{\partial p_i^2}\right\}=E\left\{\frac{x_i}{p_i^2}\right\}=\frac{np_i}{p_i^2}=\frac{n}{p_i}$

故 $I(p)=\begin{pmatrix}\frac{n}{p_1} & & & \\ & \frac{n}{p_2} & & \\ & & \ddots & \\ & & & \frac{n}{p_n}\end{pmatrix}$

故 $\pi(p)=\sqrt{\prod_{i=1}^{n}\frac{n}{p_i}}=\frac{\sqrt{n^n}}{\sqrt{\prod_{i=1}^{n}p_i}}\propto\frac{1}{\sqrt{p_1p_2\cdots p_n}}$

12. $\vec{X} \sim f(x_1|\theta_1)\cdot f(x_2|\theta_2)\cdots f(x_n|\theta_n)$

$$l = \ln f = \sum_{i=1}^{k} \ln f(x_i|\theta_i)$$

当 $i \neq j$ 时 $I_{ij}(\theta) = E\left\{-\frac{\partial^2 l}{\partial\theta_i \partial\theta_j}\right\} = 0$.

$$I_{ii}(\theta) = E\left\{-\frac{\partial^2 l}{\partial\theta_i^2}\right\} = E\left\{-\frac{\partial^2 l_i}{\partial\theta_i^2}\right\} = \pi_i(\theta_i)^2,$$

其中 $l_i = \ln f(x_i|\theta_i)$

$$\text{从而 } I(\theta) = \begin{pmatrix} \pi_1(\theta_1)^2 & & \\ & \ddots & \\ & & \pi_k(\theta_k)^2 \end{pmatrix}$$

故 Jeffrey 先验为 $\pi(\theta) = \sqrt{\det I(\theta)} = \sqrt{\prod_{i=1}^{k} \pi_i(\theta_i)^2} = \prod_{i=1}^{k} \pi_i(\theta_i)$

13. $$\frac{\partial^2 l}{\partial\theta_i\partial\theta_j} = \frac{\partial^2 \log f(x)}{\partial\theta_i\partial\theta_j} = \frac{\partial\left(\frac{\frac{\partial f(x)}{\partial\theta_i}}{f(x)}\right)}{\partial\theta_j}$$

$$= \frac{\frac{\partial^2 f(x)}{\partial\theta_i\partial\theta_j}\cdot f(x) - \frac{\partial f(x)}{\partial\theta_i}\cdot\frac{\partial f(x)}{\partial\theta_j}}{f^2(x)}$$

$$= \frac{\partial^2 f(x)}{\partial\theta_i\partial\theta_j}\cdot\frac{1}{f(x)} - \frac{\frac{\partial f(x)}{\partial\theta_i}}{f(x)}\cdot\frac{\frac{\partial f(x)}{\partial\theta_j}}{f(x)}$$

$$= \frac{\partial^2 f(x)}{\partial\theta_i\partial\theta_j}\cdot\frac{1}{f(x)} - \frac{\partial l}{\partial\theta_i}\cdot\frac{\partial l}{\partial\theta_j}.$$

故 $E\left\{\frac{\partial l}{\partial\theta_i}\cdot\frac{\partial l}{\partial\theta_j}\right\} = E\left\{-\frac{\partial^2 l}{\partial\theta_i\partial\theta_j}\right\} + E\left\{\frac{\partial^2 f(x)}{\partial\theta_i\partial\theta_j}\cdot\frac{1}{f(x)}\right\}$,

而 $E\left\{\frac{\partial^2 f(x)}{\partial\theta_i\partial\theta_j}\cdot\frac{1}{f(x)}\right\} = \int \frac{\partial^2 f(x)}{\partial\theta_i\partial\theta_j}\,dx = \frac{\partial^2}{\partial\theta_i\partial\theta_j}\left\{\int f(x)\,dx\right\} = \frac{\partial^2}{\partial\theta_i\partial\theta_j}(1) = 0$

因此 $E\left\{\frac{\partial l}{\partial\theta_i}\cdot\frac{\partial l}{\partial\theta_j}\right\} = E\left\{-\frac{\partial^2 l}{\partial\theta_i\partial\theta_j}\right\}$.

14. 设 $\pi(p) \sim Be(\alpha, \beta)$

则 $\pi(p|x) \propto \pi(p)\cdot f(x|p) \propto p^{\alpha-1}\cdot(1-p)^{\beta-1}\cdot p^{k}\cdot(1-p)^{x-k}$

$$\propto p^{(\alpha+k-1)}(1-p)^{(x+\beta-1-k)}$$

故 $p|x \sim Be(\alpha+k,\ x+\beta-k)$

从而 p 的共轭先验分布族为 Beta 分布族

15. (1) 由 $\begin{cases} E\theta = \frac{r}{\lambda} = 0.0002 \\ D\theta = \frac{r}{\lambda^2} = 0.0001^2 \end{cases}$ 得 $\begin{cases} r = 4 \\ \lambda = 20000 \end{cases}$

(2) 设 $\pi(\theta) \sim \Gamma(r, \lambda)$

则 $\pi(\theta|x) \propto \pi(\theta)\cdot f(x|\theta) \propto \theta^{r-1} e^{-\lambda\theta}\cdot \theta\cdot e^{-\theta x}$

$$\propto \theta^{r} e^{-(\lambda+x)\theta}$$

故 $\theta|x \sim Be(r+1, \lambda+x)$.

从而 伽玛分布族是 θ 的共轭先验分布族

16. $h(\theta) = A e^{k_1 a(\theta) + k_2 c(\theta)}$. 设 $\pi(\theta) = h(\theta)$.

$\pi(\theta|x) \propto \pi(\theta)\cdot f(x|\theta) \propto e^{k_1 a(\theta) + k_2 c(\theta)}\cdot e^{a(\theta)b(x) + c(\theta) + d(x)}$

$$\propto e^{[k_1 + b(x)]a(\theta) + [k_2+1]c(\theta)}$$

令 $k_1' = k_1 + b(x)$, $k_2' = k_2 + 1$, 从而可知 $h(\theta)$ 是 θ 的共轭先验分布.

17. 设 $\lambda \sim \Gamma^{-1}(a,b)$

则 $\pi(\lambda|x) \propto \pi(\lambda)\cdot p(x|\lambda) \propto \left(\lambda^{-(a+1)} e^{-\frac{b}{\lambda}}\right)\left(\lambda^{-1} e^{-\frac{x}{\lambda}}\right)$

$$\propto \lambda^{-(a+2)} e^{-\frac{b+x}{\lambda}}$$

从而 $\lambda|x \sim \Gamma^{-1}(a+1,\ b+x)$

因此 λ 的共轭先验分布是逆伽玛分布.

18. $\pi(\theta|\vec{x}) \propto \pi(\theta)\cdot p(\vec{x}|\theta) \propto \frac{1}{\theta^{\alpha+1}}\cdot I_{\theta>\theta_0}\cdot\left(\frac{1}{\theta}\right)^n\cdot I_{\theta>X_{(n)}}$

$\propto \frac{1}{\theta^{\alpha+n+1}} I_{\theta>\theta_1}$ ，其中 $\theta_1=\max\{\theta_0, X_{(n)}\}$.

从而 $\theta|\vec{x} \sim Pa(\theta_1, \alpha+n)$

因此 均匀分布是 $U(0,\theta)$ 端点 θ 的共轭先验分布.

19. 设 $\lambda \sim \Gamma^{-1}(a,b)$

$\pi(\lambda|\vec{x}) \propto \pi(\lambda)\cdot f(\vec{x}|\lambda) \propto \lambda^{-(a+1)} e^{-\frac{b}{\lambda}}\cdot\left(\frac{1}{\lambda}\right)^{rn}\cdot e^{-\frac{\sum_{i=1}^{n}X_i}{\lambda}}$

$$\propto \lambda^{-(a+nr+1)} e^{-\frac{b+\sum_{i=1}^{n}X_i}{\lambda}}$$

故 $\lambda|\vec{x} \sim \Gamma^{-1}(a+nr,\ b+\sum_{i=1}^{n}X_i)$

从而 λ 的共轭先验分布是逆伽玛分布.

20. $X_i|\theta_i \sim N(\theta_i, 900)$

第一层: $\theta_i|\vec{\lambda} \sim N(\mu,\tau^2)$ ，$\vec{\lambda}=(\mu,\tau^2)$

第二层: $\mu \sim N(100, 20^2)$, $\pi(\tau^2)=1$. 假设 $\mu \perp \tau^2$.