韦来生《贝叶斯统计》习题一

1. $\theta \sim Be(\alpha, \beta)$ $E\theta = \frac{\alpha}{\alpha+\beta} = \frac{1}{3}$, $D\theta = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{1}{45}$

解得 $\alpha = 3$, $\beta = 6$. 故 $\theta \sim Be(3, 6)$

2. 设 $\theta \sim \Gamma(r, \lambda)$. $E\theta = \frac{r}{\lambda} = 10$. $D\theta = \frac{r}{\lambda^2} = 5$

解得 $r = 20$, $\lambda = 2$, 故 $\theta \sim \Gamma(20, 2)$

3. 设 X 为产品不合格的数量, 则 $X \sim B(8, \theta)$

$$\pi(\theta=0.1|X=2) = \frac{\pi(\theta=0.1)f(X=2|\theta=0.1)}{\pi(\theta=0.1)f(X=2|\theta=0.1)+\pi(\theta=0.2)f(X=2|\theta=0.2)} = \frac{0.7\cdot\binom{8}{2}0.1^2\cdot 0.9^6}{0.7\binom{8}{2}0.1^2\cdot0.9^6+0.3\binom{8}{2}0.2^2 0.8^6}$$

$$= \frac{0.7\times0.1^2\times0.9^6}{0.7\times0.1^2\times0.9^6+0.3\times0.2^2\times0.8^6} = 0.5418$$

$$\pi(\theta=0.2|X=2) = \frac{0.3\times0.2^2\times0.8^6}{0.7\times0.1^2\times0.9^6+0.3\times0.2^2\times0.8^6} = 0.4582$$

4. $$\pi(\lambda=1.0|X=3) = \frac{\pi(1.0)f(X=3|\lambda=1.0)}{\pi(1.0)f(X=3|\lambda=1.0)+\pi(1.5)f(X=3|\lambda=1.5)} = \frac{0.4\times\frac{e^{-1.0}\cdot1.0^3}{3!}}{0.4\times\frac{e^{-1.0}\cdot1.0^3}{3!}+0.6\times\frac{e^{-1.5}\cdot1.5^3}{3!}}$$

$$= \frac{0.4\times e^{-1.0}\times1.0^3}{0.4\times e^{-1.0}\times1.0^3+0.6\times e^{-1.5}\times1.5^3} = 0.2457$$

$\pi(\lambda=1.5|X=3) = 0.7543$

5. (1) $\theta \sim U(0,1)$, $X|\theta \sim B(8,\theta)$

$$\pi(\theta|X=3)=\frac{\pi(\theta)\cdot f(X=3|\theta)}{\int_0^1 \pi(\theta)\cdot f(X=3|\theta)\,d\theta}=\frac{\binom{8}{3}\theta^3(1-\theta)^5}{\int_0^1\binom{8}{3}\theta^3(1-\theta)^5 d\theta}I_{[0,1]}$$

$$=\frac{\theta^3(1-\theta)^5}{C}\,I_{[0,1]}$$

故 $\theta|X \sim Be(4,6)$

(2) $$\pi(\theta|X=3)=\frac{\pi(\theta)\cdot f(X=3|\theta)}{\int_0^1\pi(\theta)f(X=3|\theta)\,d\theta}\propto \pi(\theta)f(X=3|\theta)$$

$$=2(1-\theta)\cdot\binom{8}{3}\theta^3(1-\theta)^5 \propto \theta^3(1-\theta)^6$$

故 $\theta|X\sim Be(4,7)$

6. (1) $$\pi(\theta|x_1)=\frac{\pi(\theta)\cdot P(x_1|\theta)}{\int_\Theta \pi(\theta)P(x_1|\theta)\,d\theta}\propto P(x_1|\theta)\pi(\theta)$$

(2) $$\pi(\theta|x_1,x_2)=\frac{\pi(\theta)\cdot P(x_1|\theta)\cdot P(x_2|x_1,\theta)}{\int_\Theta\pi(\theta)P(x_1|\theta)P(x_2|x_1,\theta)}=\frac{\pi(\theta)P(x_1|\theta)P(x_2|\theta)}{\int_\Theta\pi(\theta)P(x_1|\theta)P(x_2|\theta)}$$

$$\propto\pi(\theta|x_1)P(x_2|\theta)$$

(3) $$\pi(\theta|\vec{x})\propto P(\vec{x},\theta)=\underline{\pi(\theta)\cdot P(x_1,\cdots,x_{n-1}|\theta)}\cdot P(x_n|\theta)$$

$$\propto P(x_n|\theta)\,\pi(\theta|x_1,\cdots,x_{n-1})$$

7. $\pi(\theta)=\frac{192}{\theta^4}I_{[4,\infty)}$. $X\sim U(0,\theta)$

由第6题知, $\pi(\theta|x_1,x_2,x_3)\propto\pi(\theta)\cdot f(x_1|\theta)\cdot f(x_2|\theta)\cdot f(x_3|\theta)$

$$=\frac{192}{\theta^4}\cdot\frac{1}{\theta}\cdot\frac{1}{\theta}\cdot\frac{1}{\theta}\ I_{[8,\infty)}$$

$$\propto\frac{I_{[8,\infty)}}{\theta^7}$$

因此 $\theta|X\sim Pa(8,6)$

8. (1) $\pi(\theta)=\frac{1}{10}I_{[10,20]}$, $X\sim U(\theta-\frac{1}{2},\theta+\frac{1}{2})$

$\pi(\theta|X)\propto\pi(\theta)\cdot f(X|\theta)=\frac{1}{10}I_{\theta\in[10,20]}\cdot I_{\theta\in[11.5,12.5]}$

$\propto I_{[11.5,12.5]}$

因此 $\theta|X=12\sim U(11.5,12.5)$

(2) $\pi(\theta|X)\propto I_{[10,20]}\cdot I_{[11.5,12.5]}\cdot I_{[11.0,12.0]}\cdot I_{[11.2,12.2]}$

$\cdot I_{[10.6,11.6]}\cdot I_{[10.9,11.9]}\cdot I_{[11.4,12.4]}$

$=I_{[11.5,11.6]}$

从而 $\theta|\vec{X}\sim U(11.5,11.6)$

9. (1) $\pi(\theta|x)\propto\pi(\theta)\cdot p(x|\theta)=I_{[0,1]}\cdot\frac{2x}{\theta^2}I_{[x,1]}\propto\frac{1}{\theta^2}I_{[x,1]}$

从而 $\pi(\theta|x)=\frac{x}{1-x}\frac{1}{\theta^2}I_{[x,1]}$

(2) $\pi(\theta|x) \propto \pi(\theta) p(x|\theta) = I_{[0,1]} \cdot 3\theta^2 \cdot \frac{2x}{\theta^2} \cdot I_{[x,1]}$

$\propto I_{[x,1]}$.

从而 $\theta|x \sim U(x,1)$

10. 由于 $T=T(X)$ 是充分统计量. 由因子分解定理,

$f(x,\theta) = g(T(x),\theta)\, h(x)$

由于 $S=G(T)$ 为一一对应的变换, 故 $T=G^{-1}(S(x))$

因此 $f(x,\theta) = g(G^{-1}(S(x)),\theta)\, h(x) = g'(S(x),\theta) h(x)$

由因子分解定理. S 也是充分统计量.

11. (1) $X_i \sim P(\lambda)$, $f(x_i,\lambda) = \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$

故 $f(\vec{x},\lambda) = \frac{\lambda^{\sum_{i=1}^{n} x_i} \cdot e^{-n\lambda}}{\prod_{i=1}^{n}(x_i!)} = g(T(\vec{x}),\lambda)\, h(\vec{x})$,

其中 $g(T(\vec{x}),\lambda) = \lambda^{T(\vec{x})} \cdot e^{-n\lambda} = \lambda^{\sum_{i=1}^{n} X_i} e^{-n\lambda}$, $h(\vec{x}) = \frac{1}{\prod_{i=1}^{n}(x_i!)}$

由因子分解定理, $T(x) = \sum_{i=1}^{n} X_i$ 为充分统计量.

(2) $f(\vec{x} \mid T(\vec{x})) = \frac{f(\vec{x}, T(\vec{x}))}{f(T(\vec{x}))} = \frac{f(\vec{x}) \cdot f(T(\vec{x})|\vec{x})}{f(T(\vec{x}))}$

$= \frac{\frac{\cancel{e^{-(n\lambda)}} \cdot \lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n}(x_i!)} \cdot f(T(\vec{x})|\vec{x})}{\frac{\cancel{e^{-(n\lambda)}} (n\lambda)^T}{T!}} = \frac{T!}{n^T \cdot \prod_{i=1}^{n}(x_i!)} \cdot f(T(\vec{x})|\vec{x})$

由于给定 $\vec{x}$ 的分布后，$T(\vec{x})$ 分布确定，即对 $X_1, X_2, \cdots, X_n$ 的参数，与 λ 无关，因此 $f(\vec{x}|T(\vec{x}))$ 与 λ 无关。因此 $T(x)=\sum_{i=1}^{n}X_i$ 为充分统计量

12. (1) $f(\vec{x}|T(\vec{x})) = \dfrac{f(\vec{x})\, f(T(x)|\vec{x})}{f(T(x))} = \dfrac{p^n (1-p)^{\sum_{i=1}^{n}X_i - n} f(T(x)|\vec{x})}{\binom{T-1}{n-1} p^n (1-p)^{T-n}}$

$= \dfrac{f(T(x)|\vec{x})}{\binom{T-1}{n-1}}$

上述使用了若 $X_1, \cdots, X_n$ i.i.d. $\sim Ge(p)$，则 $\sum_{i=1}^{n}X_i \sim Nb(n, p)$

由于给定 $\vec{x}$ 的分布后，$T(\vec{x})$ 分布确定，故上述 $f(\vec{x}|T(\vec{x}))$ 与 p 无关，故 $T(x)=\sum_{i=1}^{n}X_i$ 为充分统计量。

(2) $f(\vec{x}, p) = p^n (1-p)^{\sum_{i=1}^{n}X_i - n} = g(T(x), p)\, h(x)$

其中 $g(T(x), p) = p^n (1-p)^{T-n}$，$h(x) = 1$.

由因子分解定理，$T(x) = \sum_{i=1}^{n}X_i$ 是充分统计量.

13. $f(\vec{x}, \theta) = \dfrac{1}{\theta^n} I_{X_i \in (\theta-\frac{1}{2}, \theta+\frac{1}{2})} = \dfrac{1}{\theta^n} I_{\theta-\frac{1}{2} \le X_{(1)} \le X_{(n)} \le \theta+\frac{1}{2}}$

$= \dfrac{1}{\theta^n} I_{X_{(1)} \ge \theta-\frac{1}{2}} \cdot I_{X_{(n)} \le \theta+\frac{1}{2}}$

由因子分解定理，$(X_{(1)}, X_{(n)})$ 为充分统计量.

14. $$f(\vec{X},\vec{Y};a,b,\sigma^2)=\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^{m+n}\exp\left\{-\frac{\sum_{i=1}^{m}(X_i-a)^2+\sum_{i=1}^{n}(Y_i-b)^2}{2\sigma^2}\right\}$$

$$=\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^{m+n}\exp\left\{-\frac{\sum_{i=1}^{m}(X_i-\bar{X}+\bar{X}-a)^2+\sum_{i=1}^{n}(Y_i-\bar{Y}+\bar{Y}-b)^2}{2\sigma^2}\right\}$$

$$=\frac{1}{(\sqrt{2\pi}\,\sigma)^{m+n}}\exp\left\{-\frac{\sum_{i=1}^{m}(X_i-\bar{X})^2+\sum_{i=1}^{n}(Y_i-\bar{Y})^2+2\left[\sum_{i=1}^{m}(X_i-\bar{X})(\bar{X}-a)+\sum_{i=1}^{n}(Y_i-\bar{Y})(\bar{Y}-b)\right]+m(\bar{X}-a)^2+n(\bar{Y}-b)^2}{2\sigma^2}\right\}$$

$$=\frac{1}{(\sqrt{2\pi}\sigma)^{m+n}}\exp\left\{-\frac{\sum_{i=1}^{m}(X_i-\bar{X})^2+\sum_{i=1}^{n}(Y_i-\bar{Y})^2+m(\bar{X}-a)^2+n(\bar{Y}-b)^2}{2\sigma^2}\right\}$$

$$=\frac{1}{(\sqrt{2\pi}\sigma)^{m+n}}\exp\left\{-\frac{(n+m-2)S^2+m(\bar{X}-a)^2+n(\bar{Y}-b)^2}{2\sigma^2}\right\}$$

由因子分解定理知，$(\bar{X},\bar{Y},S^2)$ 为 $\vec{\theta}=(a,b,\sigma^2)$ 的充分统计量

15. $$f(x,\sigma)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\}$$

$$\ln f(x,\sigma)=-\frac{1}{2}\ln(2\pi)-\ln\sigma-\frac{(x-a)^2}{2\sigma^2}$$

$$I(\sigma)=E_\sigma\left(\frac{\partial\ln f(X,\sigma)}{\partial\sigma}\right)^2=E_\sigma\left[-\frac{1}{\sigma}+\frac{(X-a)^2}{\sigma^3}\right]^2$$

$$=E_\sigma\left[\frac{1}{\sigma}\left(\left(\frac{X-a}{\sigma}\right)^2-1\right)\right]^2=\frac{1}{\sigma^2}E(Z^2-1)^2=\frac{EZ^4-2EZ^2+1}{\sigma^2}$$

$$=\frac{3-2\times1+1}{\sigma^2}=\frac{2}{\sigma^2}$$，其中 $Z\sim N(0,1)$

$$D_\sigma(S_a^2)=D_\sigma\left(\frac{\sigma^2}{n}\sum_{i=1}^{n}\left(\frac{X_i-a}{\sigma}\right)^2\right)=\frac{\sigma^4}{n^2}\cdot n\cdot D_\sigma(Z^2)=\frac{\sigma^4}{n}\left(EZ^4-(EZ^2)^2\right)$$

$$=\frac{\sigma^4}{n}(3-1^2)=\frac{2\sigma^4}{n}$$

$g(\sigma)=\sigma^2$, $$\frac{(g'(\sigma))^2}{nI(\sigma)}=\frac{(2\sigma)^2}{n\cdot\frac{2}{\sigma^2}}=\frac{2\sigma^4}{n}.$$

故 $D_\sigma(S_a^2) = \frac{(g'(\sigma))^2}{nI(\sigma)}$ 达到C-R下界

而 $E(S_a^2)=\sigma^2$，即 S_a^2 为 σ^2 的无偏估计

因此 $S_a^2=\frac{1}{n}\sum_{i=1}^{n}(X_i-a)^2$ 为 σ^2 的 UMVUE

16. $X \sim Exp(\frac{1}{\theta})$, $EX=\theta$, $DX=\theta^2$

显然，$\overline{X}$ 为 θ 的一个无偏估计，下证明其为 UMVUE.

先求C-R下界. $I(\theta)=E_\theta\left(\frac{\partial \ln f(x,\theta)}{\partial\theta}\right)^2=E_\theta\left(\frac{\partial(-\ln\theta-\frac{x}{\theta})}{\partial\theta}\right)^2$

$=E_\theta\left[\frac{x}{\theta^2}-\frac{1}{\theta}\right]^2=\frac{EX^2}{\theta^4}-\frac{2EX}{\theta^3}+\frac{1}{\theta^2}$

$=\frac{\theta^2+\theta^2}{\theta^4}-\frac{2\theta}{\theta^3}+\frac{1}{\theta^2}=\frac{1}{\theta^2}$

故 $\frac{1}{nI(\theta)}=\frac{\theta^2}{n}$

而 $Var(\overline{X})=\frac{Var(X)}{n}=\frac{\theta^2}{n}$，从而 $\overline{X}$ 为 θ 的 UMVUE. 与C-R下界相等，因此为有效估计.

17. $E(\frac{\overline{X}}{a})=\frac{EX}{a}=\frac{a/\lambda}{a}=\frac{1}{\lambda}=g(\lambda)$.

已知 $EX=a/\lambda$, $DX=a/\lambda^2$

$\ln f(x,\lambda)=\ln\left(\frac{\lambda^a}{\Gamma(a)}x^{a-1}e^{-\lambda x}\right)=a\ln\lambda-\ln\Gamma(a)+(a-1)\ln x-\lambda x$

$I(\lambda)=E_\lambda\left[\frac{\partial\ln f(x,\lambda)}{\partial\lambda}\right]^2=E_\lambda\left[\frac{a}{\lambda}-x\right]^2=EX^2-\frac{2a}{\lambda}EX+\frac{a^2}{\lambda^2}$

$$=\left(\frac{\alpha}{\lambda^2}+\frac{\alpha^2}{\lambda^2}\right)-\frac{2\alpha}{\lambda}\cdot\frac{\alpha}{\lambda}+\frac{\alpha^2}{\lambda^2}=\frac{\alpha}{\lambda^2}$$

故 $\frac{(g'(\lambda))^2}{nI(\lambda)}=\frac{\frac{1}{\lambda^4}}{\frac{n\alpha}{\lambda^2}}=\frac{1}{n\alpha\lambda^2}$

而 $D(\overline{X}/\alpha)=\frac{1}{\alpha^2}D(\overline{X})=\frac{1}{n\alpha^2}D(X)=\frac{1}{n\alpha^2}\cdot\frac{\alpha}{\lambda^2}=\frac{1}{n\alpha\lambda^2}$

由 $\frac{(g'(\lambda))^2}{nI(\lambda)}=D(\overline{X}/\alpha)$ 达到 C-R 下界，从而为有效估计

18. 由 $E\hat{\theta}_1=E\hat{\theta}_2=\theta$. 设 $D\hat{\theta}_1=2a$, $D\hat{\theta}_2=a$.

由无偏性, $\theta=E(c_1\hat{\theta}_1+c_2\hat{\theta}_2)=c_1E(\hat{\theta}_1)+c_2E(\hat{\theta}_2)=(c_1+c_2)\theta$

故 $c_1+c_2=1$.

$D(c_1\hat{\theta}_1+c_2\hat{\theta}_2)=c_1^2D(\hat{\theta}_1)+c_2^2D(\hat{\theta}_2)=c_1^2\cdot 2a+(1-c_1)^2\cdot a$

$=a(3c_1^2-2c_1+1)\geq a\cdot\frac{4\cdot 3-4}{4\cdot 3}=\frac{2}{3}a$.

当且仅当 $c_1=\frac{1}{3}$, $c_2=\frac{2}{3}$ 时取等号.

故 $c_1=\frac{1}{3}$, $c_2=\frac{2}{3}$

19. $f(\vec{x},\mu)=(2\pi\sigma^2)^{-\frac{n}{2}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\}$

$\Theta=\{\mu:\mu\in(-\infty,+\infty)\}$. $\Theta_0:\{\mu=\mu_0\}$

在 Θ 上，$\hat{\mu}_{MLE}=\overline{X}$ 在 Θ_0 上，$\mu=0$

故 $\sup\limits_{\mu\in\Theta} f(\vec{x},\mu)=(2\pi\sigma^2)^{-\frac{n}{2}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\overline{X})^2\right\}$

$\sup\limits_{\mu\in\Theta_0} f(\vec{x},\mu)=(2\pi\sigma^2)^{-\frac{n}{2}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}X_i^2\right\}$

$$\lambda(\vec{x})=\frac{\sup\limits_{\mu\in\Theta}f(\vec{x},\mu)}{\sup\limits_{\mu\in\Theta_0}f(\vec{x},\mu)}=\exp\left\{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}(X_i-\overline{X})^2-\sum_{i=1}^{n}X_i^2\right)\right\}$$

$$=\exp\left\{-\frac{1}{2\sigma^2}\cdot(-n\overline{X}^2)\right\}=\exp\left\{\frac{n\overline{X}^2}{2\sigma^2}\right\}$$

$$=\exp\left\{\frac{1}{2}\left(\frac{\sqrt{n}\,\overline{X}}{\sigma}\right)^2\right\}$$

有 $2\log\lambda(\vec{x})=\left(\frac{\sqrt{n}\,\overline{X}}{\sigma}\right)^2$

拒绝域为 $D=\left\{X:\left(\frac{\sqrt{n}\,\overline{X}}{\sigma}\right)^2>c^2\right\}=\{X:|Z|>c\}$

设检验水平为 α，则 $c=Z_{\alpha/2}=1.96$

故当 $\left|\frac{\sqrt{n}\,\overline{X}}{\sigma}\right|>1.96$ 时拒绝 H_0，否则接受 H_0。

20. $f(\vec{x},\lambda)=\lambda^n e^{-\lambda\sum_{i=1}^n X_i}$

在 Θ 上，由 $\dfrac{\partial \ln f(\vec{x},\lambda)}{\partial\lambda}=\dfrac{\partial\left(n\ln\lambda-\lambda\cdot\sum_{i=1}^n X_i\right)}{\partial\lambda}=\dfrac{n}{\lambda}-\sum_{i=1}^n X_i=0$ 得

$$\hat{\lambda}_{MLE}=\frac{1}{\overline{X}}$$

在 Θ_0 上，$\lambda=\lambda_0$

$$\lambda(\vec{x})=\frac{\sup_{\mu\in\Theta} f(\vec{x},\mu)}{\sup_{\mu\in\Theta_0} f(\vec{x},\mu)}=\frac{\frac{1}{\overline{X}^n}e^{-\frac{\sum_{i=1}^n X_i}{\overline{X}}}}{\lambda_0^n e^{-\lambda_0\sum_{i=1}^n X_i}}=\frac{e^{-n}}{(\lambda_0\overline{X})^n e^{-\lambda_0\sum_{i=1}^n X_i}}$$

$$\log\lambda(\vec{x})=-n-n\log(\lambda_0\overline{X})+n\lambda_0\overline{X}=-n+n\left[\lambda_0\overline{X}-\log(\lambda_0\overline{X})\right]$$

拒绝域 $D=\{\vec{x}:\lambda(\vec{x})>\lambda\}=\{\vec{x}:\lambda_0\overline{X}-\log(\lambda_0\overline{X})>C_0\}$

$=\{\vec{x}:\overline{X}<l_0 \text{ 或 } \overline{X}>l_1\}$

由于 $2n\lambda_0\overline{X}\sim\chi^2_{2n}$ under H_0

故 $D=\{\vec{x}:2n\lambda_0\overline{X}<\chi^2_{2n}(1-\alpha/2)$ 或 $2n\lambda_0\overline{X}>\chi^2_{2n}(\alpha/2)\}$

$=\left\{\vec{x}:\overline{X}<\dfrac{\chi^2_{2n}(1-\alpha/2)}{2n\lambda_0}\right.$ 或 $\left.\overline{X}>\dfrac{\chi^2_{2n}(\alpha/2)}{2n\lambda_0}\right\}$

即当 $\overline{X}<\dfrac{\chi^2_{2n}(1-\alpha/2)}{2n\lambda_0}$ 或 $\overline{X}>\dfrac{\chi^2_{2n}(\alpha/2)}{2n\lambda_0}$ 时拒绝 H_0，否则接受 H_0.