Maps and Morse Function Preliminaries

Lecture 2 - CMSE 890

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Where were we?

- Basics of topology - open, closed, connected,
- Maps $f : A \to B$

Section 1

Maps

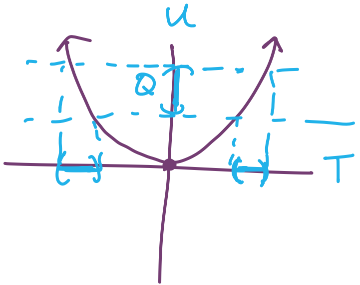
Maps

$f^{-1}(Q) = \{t \in T \mid f(t) \in Q\}$

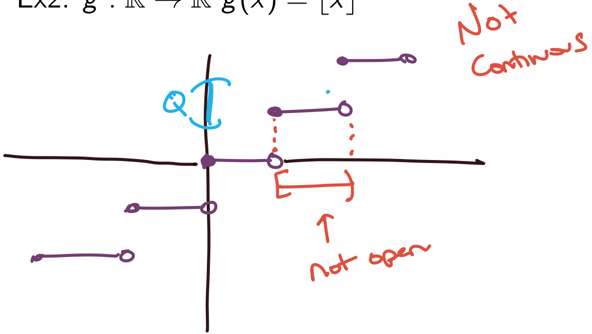
Definition

A function $f : T \to U$ is continuous if for every open set $Q \subseteq U$, $\boxed{f^{-1}(Q)}$ is open. Continuous functions are also called maps.

Ex1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$



Ex2. $g : \mathbb{R} \to \mathbb{R}$ $g(x) = \lfloor x \rfloor$



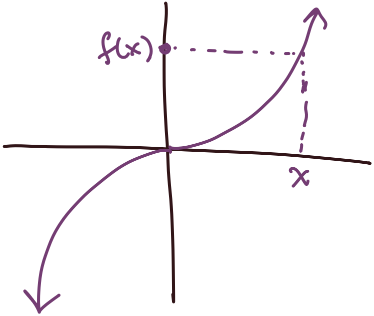
Embedding

Definition

A map $g : T \to U$ is an embedding of T into U if g is injective.
Injective (1-1): $f(x) = f(y)$ *iff* $x = y$

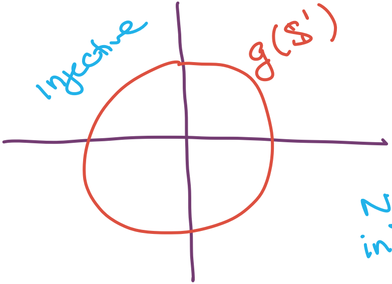
circle

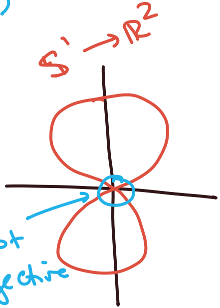
Ex1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$



Ex2. $g : \mathbb{S}^1 \to \mathbb{R}^2$

$[0, 2\pi]$ $(\cos(\theta), \sin\theta)$

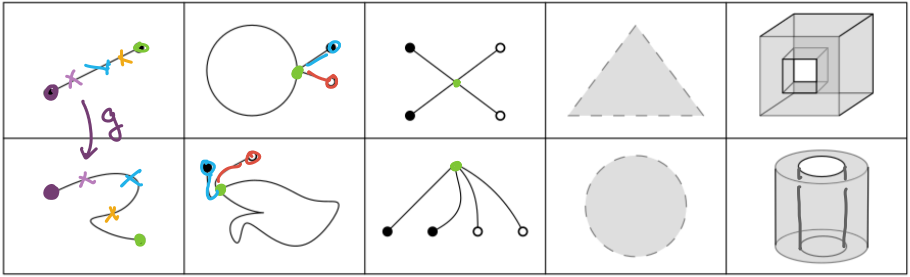




Homeomorphism

Definition

Let T and U be topological spaces.
A *homeomorphism* is a bijective map $h : T \to U$ whose inverse is also continuous.
Two topological spaces are *homeomorphic* if there exists a homeomorphism between them.



Homeomorphism: Cheap trick

Proposition

nice enough!

If T and U are (compact metric spaces,) every bijective map from T to U has a continuous inverse.

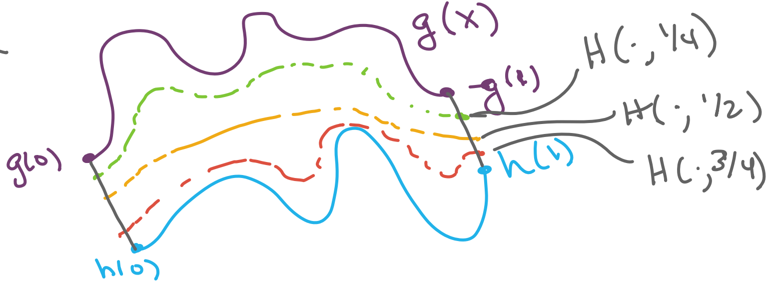
Homotopy

Definition

Let $g : X \to U$ and $h : X \to U$ be maps.
A homotopy is a map $H : X \times [0,1] \to U$ such that $H(\cdot, 0) = g$ and $H(\cdot, 1) = h$.
Two maps are homotopic if there is a homotopy connecting them.

$X = [0,1] \longrightarrow \mathbb{R}^2$

$H : X \times [0,1] \longrightarrow \mathbb{R}$



Annulus: $A = \{(\theta, r) \mid 1 \leq r \leq 2\}$. Circle: $\mathbb{S}^1 = \{(\theta, r) \mid r = 1\}$

$g : A \to A$ identity. $h : A \to A$, $h(\theta, r) = (\theta, 1)$.

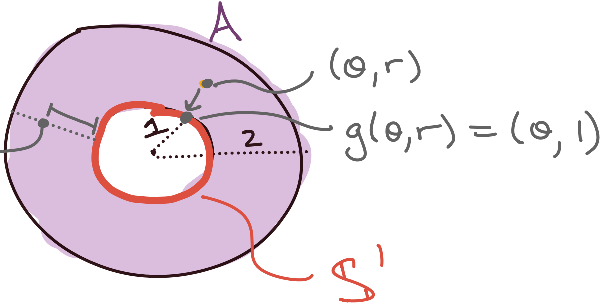
Show $R(\underbrace{\theta, r}_{A}, \underbrace{t}_{[0,1]}) = (\theta, (1 - t)r + t)$ is a homotopy.

$g \simeq h$

$R : A \times [0,1] \longrightarrow A$

$R(\theta, r, 0) = (\theta, r)$ ✓

$R(\theta, r, 1) = (\theta, 1)$



Homotopy equivalent

example: Annulus hom. equiv to S^1

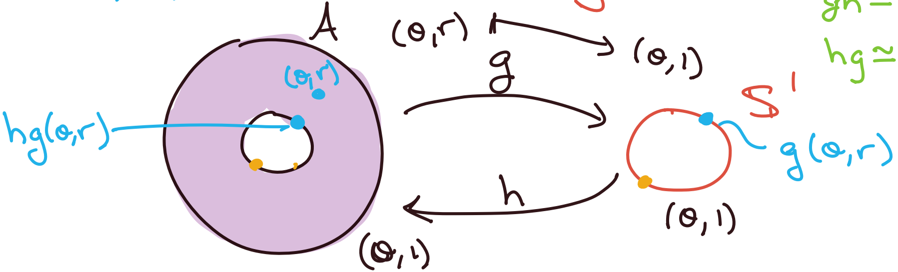
Definition

Two topological spaces T and U are homotopy equivalent if there exist maps $g : T \to U$ and $h : U \to T$ such that $\underbrace{h \circ g}_{A \to A}$ and $\underbrace{g \circ h}_{S^1 \to S^1}$ are homotopic to the appropriate identity maps.

$S^1 \to S^1$ equal to identity map on S^1

$gh \simeq \mathbb{1}_{S^1}$

$hg \simeq \mathbb{1}_A$



Warning: not homeomorphic, but they are hom. equiv.

Annulus example - Homotopy equivalent

Show A is homotopy equivalent to $\mathbb{S}^1 \subset A$

Retract

Definition

Let T be a topological space, and let $U \subset T$ be a subspace.
A retraction r of T to U is a map $r : T \to U$ such that $r(x) = x$ for every $x \in U$.

Ex: Annulus to circle

Deformation retract

Definition

The space $U \subseteq T$ is a deformation retract of T if the identity map on T can be continuously deformed to a retraction with no motion of the points already in U.
Specifically, there is a homotopy $R : T \times [0,1] \to T$ such that

- $R(\cdot, 0)$ is the identity map on T,
- $R(\cdot, 1)$ is a retraction of T to U, and
- $R(x,t) = x$ for every $x \in U$ and every $t \in [0,1]$.

Annulus example: Deformation retract

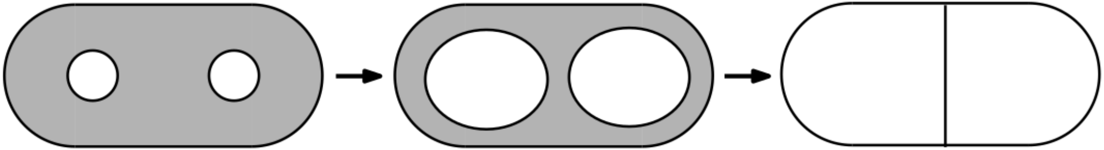
$R(\theta, r, t) = (\theta, (1-t)r + t)$.
Check that R satisfies the three properties to be a deformation retract.

Why do I care?

Theorem

If U is a deformation retract of T, then T and U are homotopy equivalent.

Ex. 1



Ex.2

A to O

TRY IT: Alphabet

Divide the alphabet into *equivalence classes*: collections of letters that are all homotopy equivalent to every other letter in their collection.
A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

Section 2

Manifolds

Manifold definition

dimension of the manifold

Definition

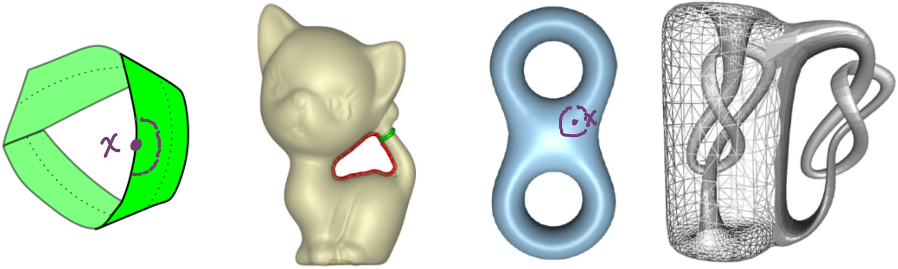
neighborhood

A topological space M is an m-manifold if every point $x \in M$ has a ~~point~~ homeomorphic to the m-ball $\mathbb{B}^d_o$ (or the m-hemisphere $\mathbb{H}^d$).

Manifold with boundary

$$\mathbb{B}^d_o = \{y \in \mathbb{R}^d \mid \|y\| < 1$$
$$\mathbb{H}^d = \{y \in \mathbb{R}^d \mid d(y,0) < 1 \text{ and } y_d \geq 0\}.$$

2-Torus



(a) (b) (c) (d)

More manifold vocab

- Manifold with/without boundary
- Surface: 2-manifold subspace of $\mathbb{R}^d$
- Non-orientable: can start at point p, stand on one side of the manifold and walk back to p but be on the other side
- Loop: 1-manifold without boundary
- Genus: of a surface is g if $2g$ is the maximum number of loops that can be removed without disconnecting the surface.
- Smooth embedded manifold: No wrinkles, no zero directional derivative

Gradients

Definition

Given a smooth function $f : \mathbb{R}^d \to \mathbb{R}$, the gradient vector field $\nabla f : \mathbb{R}^d \to \mathbb{R}^d$ at x is:

$$\nabla f(x) = \left[\frac{\partial f}{\partial x_1}(x), \cdots, \frac{\partial f}{\partial x_d}(x)\right]$$

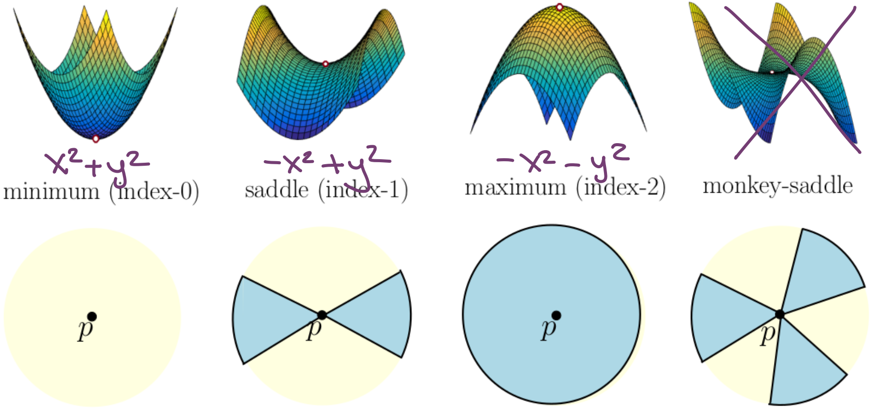
Note: This defnition can be extended to more general settings $f : M \to \mathbb{R}$.

Ex. $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = x^2 + y^2$ at $(0, 0)$ and $(1, 0)$

Critical points

- Points in $\mathbb{R}^d$ where $\nabla f(p) = [0, \cdots, 0]$

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$



“Nice” critical points

Definition

For a smooth m-manifold M, the Hessian matrix of $f : M \to \mathbb{R}$ is the matrix of second order partial derivatives

$$Hessian(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_m}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_m}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_m \partial x_1}(x) & \frac{\partial^2 f}{\partial x_m \partial x_2}2(x) & \cdots & \frac{\partial^2 f}{\partial x_m \partial x_m}(x) \end{bmatrix},$$

A critical point of f is non-degenerate if the Hessian is non-singular (has non-zero determinant); otherwise it is degenerate.

TRY IT: Saddle

Ex. $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 - y^2$. Is the critical point at the origin degenerate?

Interactive plot: `https://www.desmos.com/3d/cw0km8przc`

TRY IT: Monkey saddle

Ex. $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^3 - 3xy^2$. Is the critical point at the origin degenerate?

Interactive plot: `https://www.desmos.com/3d/cw0km8przc`

Morse lemma

Theorem

Given a smooth function $f : M \to \mathbb{R}$ defined on a smooth m-manifod M, let p be a non-degenerate critical point of f.
There is a local coordinate system in a neighborhood $U(p)$ so that

- $U(p) = (0, \cdots, 0)$
- *Locally any x is of the form*

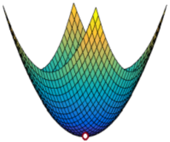
$$f(x) = f(p) \underbrace{- x_1^2 - \cdots - x_s^2 + x_{s+1}^2 + \cdots + x_m^2}.$$

In this case, the integer s is called the index of the critical point p.

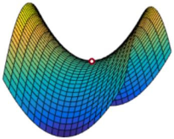
$S=0$

$f(x) = f(p) + x_1^2 + x_2^2$

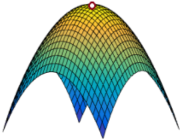
Back to critical points



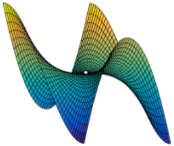
minimum (index-0)



saddle (index-1)



maximum (index-2)



monkey-saddle

Morse Functions

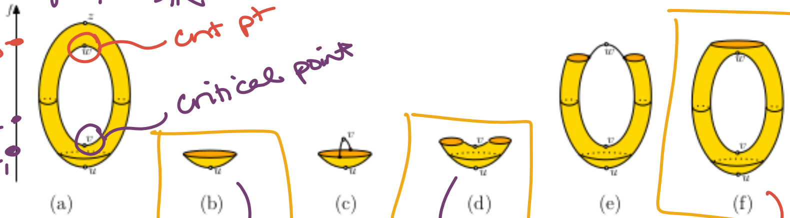
Definition

A smooth function $f : M \to \mathbb{R}$ defined on a smooth manifold M is a Morse function if

- none of f's critical points are degenerate
- the critical points have distinct function values.

Why do I care? Every function is almost Morse.

$f: T \to \mathbb{R}$



a_3

a_2

a_1

crit pt

critical point

$f^{-1}(-\infty, a_1]$

$f^{-1}(-\infty, a_2]$

$f^{-1}(-\infty, a_3]$

Fixed $a \in \mathbb{R}$

Sublevelset @ a

$f^{-1}(-\infty, a]$

Homework for next time

Choose one of the following to present.

1. DW 1.6.9
2. DW 1.6.10
3. DW 1.6.12