

# Maps and Morse Function Preliminaries

## Lecture 2 - CMSE 890

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# Where were we?

- Basics of topology - open, closed, connected,
- Maps  $f : A \rightarrow B$

# Section 1

## Maps

# Maps

## Definition

A function  $f : T \rightarrow U$  is continuous if for every open set  $Q \subseteq U$ ,  $f^{-1}(Q)$  is open. Continuous functions are also called maps.

Ex1.  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = x^2$

Ex2.  $g : \mathbb{R} \rightarrow \mathbb{R}$   $g(x) = \lfloor x \rfloor$

# Embedding

## Definition

A map  $g : T \rightarrow U$  is an embedding of  $T$  into  $U$  if  $g$  is injective.

*Injective (1-1):*  $f(x) = f(y)$  iff  $x = y$

Ex1.  $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^3$

Ex2.  $g : \mathbb{S}^1 \rightarrow \mathbb{R}^2$

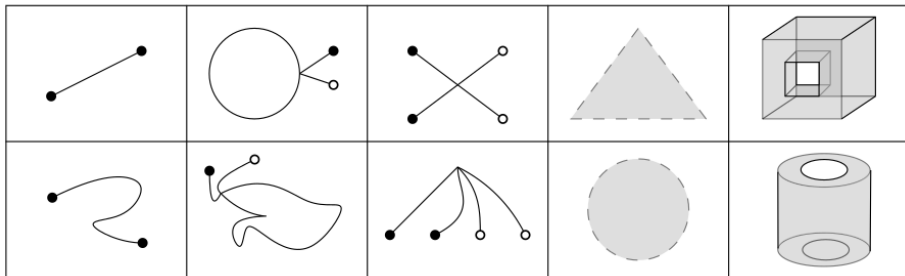
# Homeomorphism

## Definition

Let  $T$  and  $U$  be topological spaces.

A *homeomorphism* is a bijective map  $h : T \rightarrow U$  whose inverse is also continuous.

Two topological spaces are *homeomorphic* if there exists a homeomorphism between them.



# Homeomorphism: Cheap trick

## Proposition

*If  $T$  and  $U$  are compact metric spaces, every bijective map from  $T$  to  $U$  has a continuous inverse.*

# Homotopy

## Definition

Let  $g : X \rightarrow U$  and  $h : X \rightarrow U$  be maps.

A homotopy is a map  $H : X \times [0, 1] \rightarrow U$  such that  $H(\cdot, 0) = g$  and  $H(\cdot, 1) = h$ .

Two maps are homotopic if there is a homotopy connecting them.



## Annulus example - Homotopy

Annulus:  $A = \{(\theta, r) \mid 1 \leq r \leq 2\}$ .      Circle:  $\mathbb{S}^1 = \{(\theta, r) \mid r = 1\}$

$g : A \rightarrow A$  identity.       $h : A \rightarrow A$ ,  $h(\theta, r) = (\theta, 1)$ .

Show  $R(\theta, r, t) = (\theta, (1 - t)r + t)$  is a homotopy.

# Homotopy equivalent

## Definition

Two topological spaces  $T$  and  $U$  are homotopy equivalent if there exist maps  $g : T \rightarrow U$  and  $h : U \rightarrow T$  such that  $h \circ g$  and  $g \circ h$  are homotopic to the appropriate identity maps.

## Annulus example - Homotopy equivalent

Show  $A$  is homotopy equivalent to  $\mathbb{S}^1 \subset A$

# Retract

## Definition

Let  $T$  be a topological space, and let  $U \subset T$  be a subspace.

A retraction  $r$  of  $T$  to  $U$  is a map  $r : T \rightarrow U$  such that  $r(x) = x$  for every  $x \in U$ .

Ex: Annulus to circle

# Deformation retract

## Definition

The space  $U \subseteq T$  is a deformation retract of  $T$  if the identity map on  $T$  can be continuously deformed to a retraction with no motion of the points already in  $U$ .

Specifically, there is a homotopy  $R : T \times [0, 1] \rightarrow T$  such that

- $R(\cdot, 0)$  is the identity map on  $T$ ,
- $R(\cdot, 1)$  is a retraction of  $T$  to  $U$ , and
- $R(x, t) = x$  for every  $x \in U$  and every  $t \in [0, 1]$ .

## Annulus example: Deformation retract

$$R(\theta, r, t) = (\theta, (1 - t)r + t).$$

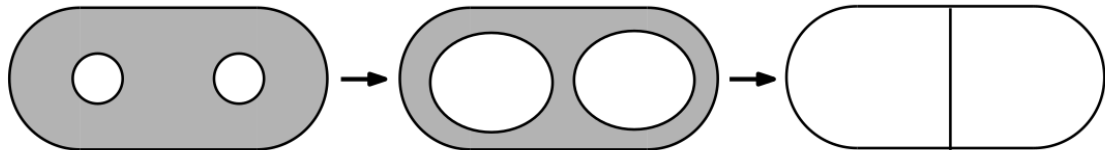
Check that  $R$  satisfies the three properties to be a deformation retract.

# Why do I care?

## Theorem

*If  $U$  is a deformation retract of  $T$ , then  $T$  and  $U$  are homotopy equivalent.*

Ex. 1



Ex.2  
A to O

# TRY IT: Alphabet

Divide the alphabet into *equivalence classes*: collections of letters that are all homotopy equivalent to every other letter in their collection.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z



## Section 2

### Manifolds

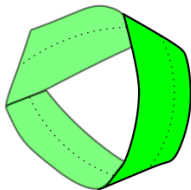
# Manifold definition

## Definition

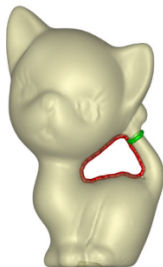
A topological space  $M$  is an  $m$ -manifold if every point  $x \in M$  has a point homeomorphic to the  $m$ -ball  $\mathbb{B}_o^d$  or the  $m$ -hemisphere  $\mathbb{H}^d$ .

$$\mathbb{B}_o^d = \{y \in \mathbb{R}^d \mid \|y\| < 1\}$$

$$\mathbb{H}^d = \{y \in \mathbb{R}^d \mid d(y, 0) < 1 \text{ and } y_d \geq 0\}.$$



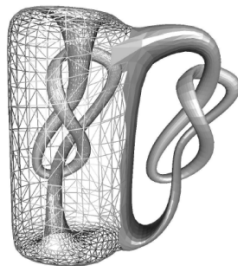
(a)



(b)



(c)



(d)

## More manifold vocab

- Manifold with/without boundary
- Surface: 2-manifold subspace of  $\mathbb{R}^d$
- Non-orientable: can start at point  $p$ , stand on one side of the manifold and walk back to  $p$  but be on the other side
- Loop: 1-manifold without boundary
- Genus: of a surface is  $g$  if  $2g$  is the maximum number of loops that can be removed without disconnecting the surface.
- Smooth embedded manifold: No wrinkles, no zero directional derivative

# Gradients

## Definition

Given a smooth function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ , the gradient vector field  $\nabla f : \mathbb{R}^d \rightarrow \mathbb{R}^d$  at  $x$  is:

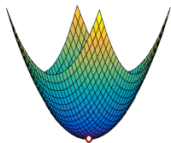
$$\nabla f(x) = \left[ \frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_d}(x) \right]$$

*Note: This definition can be extended to more general settings  $f : M \rightarrow \mathbb{R}$ .*

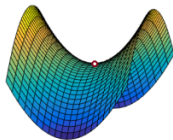
Ex.  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $f(x_1, x_2) = x^2 + y^2$  at  $(0,0)$  and  $(1,0)$

# Critical points

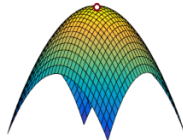
- Points in  $\mathbb{R}^d$  where  $\nabla f(p) = [0, \dots, 0]$



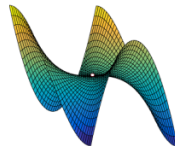
minimum (index-0)



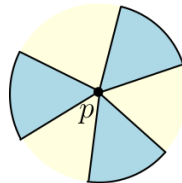
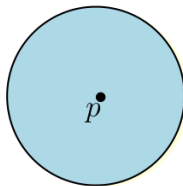
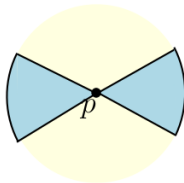
saddle (index-1)



maximum (index-2)



monkey-saddle



# “Nice” critical points

## Definition

For a smooth  $m$ -manifold  $M$ , the Hessian matrix of  $f : M \rightarrow \mathbb{R}$  is the matrix of second order partial derivatives

$$\text{Hessian}(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_m}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_m}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_m \partial x_1}(x) & \frac{\partial^2 f}{\partial x_m \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_m \partial x_m}(x) \end{bmatrix},$$

A critical point of  $f$  is non-degenerate if the Hessian is non-singular (has non-zero determinant); otherwise it is degenerate.

## TRY IT: Saddle

Ex.  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $f(x, y) = x^2 - y^2$ . Is the critical point at the origin degenerate?

Interactive plot: <https://www.desmos.com/3d/cw0km8przc>

## TRY IT: Monkey saddle

Ex.  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $f(x, y) = x^3 - 3xy^2$ . Is the critical point at the origin degenerate?

Interactive plot: <https://www.desmos.com/3d/cw0km8przc>



# Morse lemma

## Theorem

*Given a smooth function  $f : M \rightarrow \mathbb{R}$  defined on a smooth  $m$ -manifold  $M$ , let  $p$  be a non-degenerate critical point of  $f$ .*

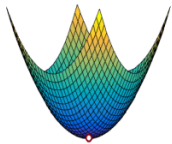
*There is a local coordinate system in a neighborhood  $U(p)$  so that*

- $U(p) = (0, \dots, 0)$
- *Locally any  $x$  is of the form*

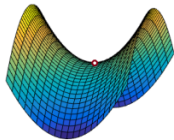
$$f(x) = f(p) - x_1^2 - \dots - x_s^2 + x_{s+1}^2 + \dots + x_m^2.$$

*In this case, the integer  $s$  is called the index of the critical point  $p$ .*

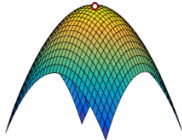
## Back to critical points



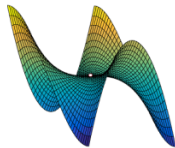
minimum (index-0)



saddle (index-1)



maximum (index-2)



monkey-saddle

# Morse Functions

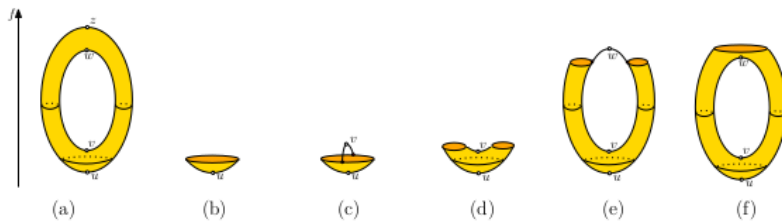
## Definition

A smooth function  $f : M \rightarrow \mathbb{R}$  defined on a smooth manifold  $M$  is a Morse function if

- none of  $f$ 's critical points are degenerate
- the critical points have distinct function values.

Why do I care? Every function is almost Morse.

# Morse Functions and sublevelsets



# Homework for next time

*Choose one of the following to present.*

- ① DW 1.6.9
- ② DW 1.6.10
- ③ DW 1.6.12