

Maps and Morse Function Preliminaries

Lecture 2 - CMSE 890

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Where were we?

- Basics of topology - open, closed, connected,
- Maps $f : A \rightarrow B$

Section 1

Maps

Maps

Definition

A function $f : T \rightarrow U$ is continuous if for every open set $Q \subseteq U$, $f^{-1}(Q)$ is open. Continuous functions are also called maps.

Ex1. $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2$

Ex2. $g : \mathbb{R} \rightarrow \mathbb{R}$ $g(x) = \lfloor x \rfloor$

Embedding

Definition

A map $g : T \rightarrow U$ is an embedding of T into U if g is injective.

Injective (1-1): $f(x) = f(y)$ iff $x = y$

Ex1. $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^3$

Ex2. $g : \mathbb{S}^1 \rightarrow \mathbb{R}^2$

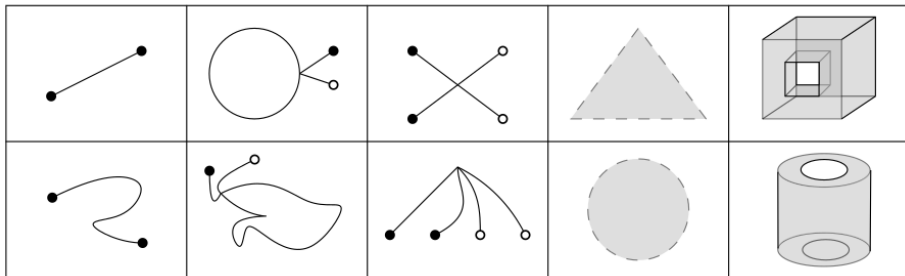
Homeomorphism

Definition

Let T and U be topological spaces.

A *homeomorphism* is a bijective map $h : T \rightarrow U$ whose inverse is also continuous.

Two topological spaces are *homeomorphic* if there exists a homeomorphism between them.



Homeomorphism: Cheap trick

Proposition

If T and U are compact metric spaces, every bijective map from T to U has a continuous inverse.

Homotopy

Definition

Let $g : X \rightarrow U$ and $h : X \rightarrow U$ be maps.

A homotopy is a map $H : X \times [0, 1] \rightarrow U$ such that $H(\cdot, 0) = g$ and $H(\cdot, 1) = h$.

Two maps are homotopic if there is a homotopy connecting them.

Annulus example - Homotopy

Annulus: $A = \{(\theta, r) \mid 1 \leq r \leq 2\}$. Circle: $\mathbb{S}^1 = \{(\theta, r) \mid r = 1\}$

$g : A \rightarrow A$ identity. $h : A \rightarrow A$, $h(\theta, r) = (\theta, 1)$.

Show $R(\theta, r, t) = (\theta, (1 - t)r + t)$ is a homotopy.

Homotopy equivalent

Definition

Two topological spaces T and U are homotopy equivalent if there exist maps $g : T \rightarrow U$ and $h : U \rightarrow T$ such that $h \circ g$ and $g \circ h$ are homotopic to the appropriate identity maps.

Annulus example - Homotopy equivalent

Show A is homotopy equivalent to $\mathbb{S}^1 \subset A$

Retract

Definition

Let T be a topological space, and let $U \subset T$ be a subspace.

A retraction r of T to U is a map $r : T \rightarrow U$ such that $r(x) = x$ for every $x \in U$.

Ex: Annulus to circle

Deformation retract

Definition

The space $U \subseteq T$ is a deformation retract of T if the identity map on T can be continuously deformed to a retraction with no motion of the points already in U .

Specifically, there is a homotopy $R : T \times [0, 1] \rightarrow T$ such that

- $R(\cdot, 0)$ is the identity map on T ,
- $R(\cdot, 1)$ is a retraction of T to U , and
- $R(x, t) = x$ for every $x \in U$ and every $t \in [0, 1]$.

Annulus example: Deformation retract

$$R(\theta, r, t) = (\theta, (1 - t)r + t).$$

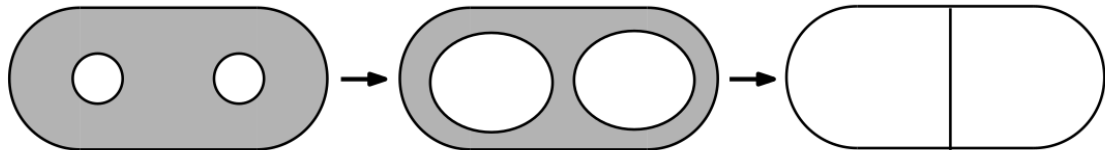
Check that R satisfies the three properties to be a deformation retract.

Why do I care?

Theorem

If U is a deformation retract of T , then T and U are homotopy equivalent.

Ex. 1



Ex.2
A to O

TRY IT: Alphabet

Divide the alphabet into *equivalence classes*: collections of letters that are all homotopy equivalent to every other letter in their collection.

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

Section 2

Manifolds

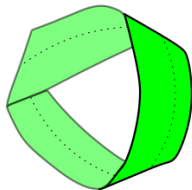
Manifold definition

Definition

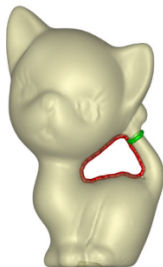
A topological space M is an m -manifold if every point $x \in M$ has a point homeomorphic to the m -ball \mathbb{B}_o^d or the m -hemisphere \mathbb{H}^d .

$$\mathbb{B}_o^d = \{y \in \mathbb{R}^d \mid \|y\| < 1\}$$

$$\mathbb{H}^d = \{y \in \mathbb{R}^d \mid d(y, 0) < 1 \text{ and } y_d \geq 0\}.$$



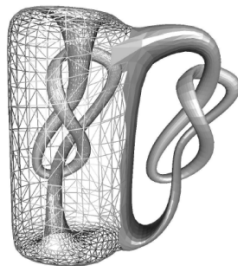
(a)



(b)



(c)



(d)

More manifold vocab

- Manifold with/without boundary
- Surface: 2-manifold subspace of \mathbb{R}^d
- Non-orientable: can start at point p , stand on one side of the manifold and walk back to p but be on the other side
- Loop: 1-manifold without boundary
- Genus: of a surface is g if $2g$ is the maximum number of loops that can be removed without disconnecting the surface.
- Smooth embedded manifold: No wrinkles, no zero directional derivative

Gradients

Definition

Given a smooth function $f : \mathbb{R}^d \rightarrow \mathbb{R}$, the gradient vector field $\nabla f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ at x is:

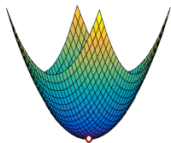
$$\nabla f(x) = \left[\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_d}(x) \right]$$

Note: This definition can be extended to more general settings $f : M \rightarrow \mathbb{R}$.

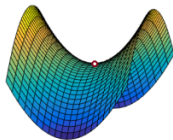
Ex. $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $f(x_1, x_2) = x^2 + y^2$ at $(0,0)$ and $(1,0)$

Critical points

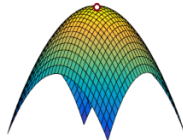
- Points in \mathbb{R}^d where $\nabla f(p) = [0, \dots, 0]$



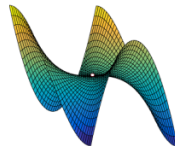
minimum (index-0)



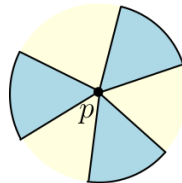
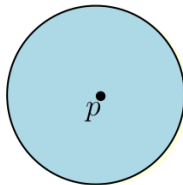
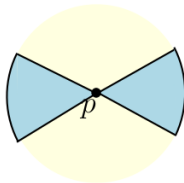
saddle (index-1)



maximum (index-2)



monkey-saddle



“Nice” critical points

Definition

For a smooth m -manifold M , the Hessian matrix of $f : M \rightarrow \mathbb{R}$ is the matrix of second order partial derivatives

$$\text{Hessian}(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_m}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_m}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_m \partial x_1}(x) & \frac{\partial^2 f}{\partial x_m \partial x_2}(x) & \cdots & \frac{\partial^2 f}{\partial x_m \partial x_m}(x) \end{bmatrix},$$

A critical point of f is non-degenerate if the Hessian is non-singular (has non-zero determinant); otherwise it is degenerate.

TRY IT: Saddle

Ex. $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $f(x, y) = x^2 - y^2$. Is the critical point at the origin degenerate?

Interactive plot: <https://www.desmos.com/3d/cw0km8przc>

TRY IT: Monkey saddle

Ex. $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $f(x, y) = x^3 - 3xy^2$. Is the critical point at the origin degenerate?

Interactive plot: <https://www.desmos.com/3d/cw0km8przc>

Morse lemma

Theorem

Given a smooth function $f : M \rightarrow \mathbb{R}$ defined on a smooth m -manifold M , let p be a non-degenerate critical point of f .

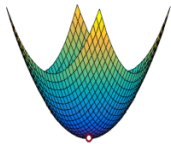
There is a local coordinate system in a neighborhood $U(p)$ so that

- $U(p) = (0, \dots, 0)$
- *Locally any x is of the form*

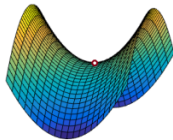
$$f(x) = f(p) - x_1^2 - \dots - x_s^2 + x_{s+1}^2 + \dots + x_m^2.$$

In this case, the integer s is called the index of the critical point p .

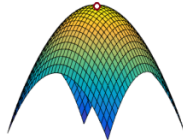
Back to critical points



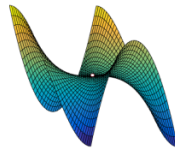
minimum (index-0)



saddle (index-1)



maximum (index-2)



monkey-saddle

Morse Functions

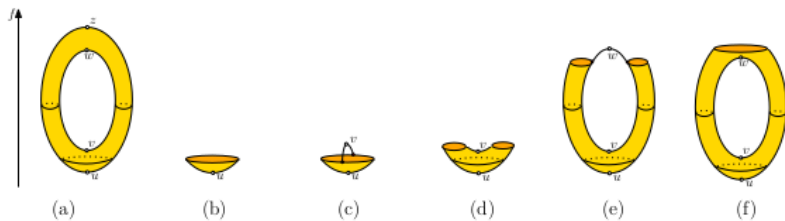
Definition

A smooth function $f : M \rightarrow \mathbb{R}$ defined on a smooth manifold M is a Morse function if

- none of f 's critical points are degenerate
- the critical points have distinct function values.

Why do I care? Every function is almost Morse.

Morse Functions and sublevelsets



Homework for next time

Choose one of the following to present.

- ① DW 1.6.9
- ② DW 1.6.10
- ③ DW 1.6.12