1.2 a) $Ey = \mu$

$E(y-\mu)^2 = \sigma^2$

$E(y-\mu)^3 = 0$

Overidentified $l > k$.

l equations, k unknowns.

$$B = \begin{Bmatrix} \mu \\ \sigma \end{Bmatrix}$$

$$\Rightarrow g_j(b) = g_j\begin{pmatrix} \mu \\ \sigma \end{pmatrix} = \begin{pmatrix} y-\mu \\ (y-\mu)^2 - \sigma^2 \\ (y-\mu)^3 \end{pmatrix}$$ so $Eg_j(b) = 0$

Estimate of $\Omega = \hat{\Omega}(b)$

$$= \frac{1}{N}\sum_{j=1}^{N} g_j(b)\, g_j(b)^T$$

$$b = \arg\min_b N\, g_N(b)^T \hat{\Omega}^{-1} g_N(b)$$

$$\Rightarrow \hat{\Omega}(b) = \frac{1}{N}\sum_j \begin{bmatrix} (y-\mu)(y-\mu)^T \\ ((y-\mu)^2 - \sigma^2)((y-\mu)^2-\sigma^2)^T \\ [(y-\mu)^3][(y-\mu)^3]^T \end{bmatrix}$$

$\Rightarrow$ ~~$b_{GMM} =$~~ No closed form solution?

1.2b $y = \alpha + x\beta + u$.

$\mathbb{E}(x^T u) = \mathbb{E} u = 0$

$B = [\alpha \; \beta]$ Exactly Identified

$\Rightarrow \quad g_j(b) = g_j(\beta) = \left[(y_j - \hat{\alpha} - x_j' b) x_j \quad y - \alpha - x_j' b \right]$

$$\hat{\Omega}(b) = \frac{1}{N} \sum_{j=1}^{N} g_j(b)\, g_j(b)^T$$

$$= \frac{1}{N} \sum_{j=1}^{N} \left[\begin{array}{l} ((y_j - \alpha - x_j' b) x_j)(y_j - \alpha - x_j' b) x_j)^T \\ (y - \alpha - x_j' b)(y_j - \alpha - x_j' b)^T \end{array} \right.$$

$$b = \arg\min_b \; N\, g_N(b)^T \hat{\Omega}^{-1} g_N(b).$$

$$\Rightarrow b_{GLS} = (x^T \hat{\Omega}^{-1} x)^{-1} x^T \hat{\Omega}^{-1} y$$

Assuming $Ey = Ex$ to get rid of α

1.2c) $y = \alpha + x\beta + u$

$E(x^T u) = E u = 0$ and $E(u^2) = \sigma^2$.

$B = \begin{bmatrix} \alpha \\ \beta \\ \sigma \end{bmatrix}$

Exactly Identified.

$$g_j(b) = \begin{bmatrix} (y_j - \alpha - x_j' b) x_j \\ (y_j - \alpha - x_j b) \\ (y_j - \alpha - x_j b)^2 - \sigma^2 \end{bmatrix}$$

$$b = \arg\min_b \; N g_N(b)^T \hat{\Omega}^{-1} g_N(b)$$

$$\hat{\Omega}(b) = \frac{1}{N}\sum_{j=1}^{N} \begin{bmatrix} ((y_j - \alpha - x_j' b) x_j)((y_j - \alpha - x_j' b) x_j)^T \\ (y_j - \alpha - x_j b)(y_j - \alpha - x_j b)^T \\ ((y_j - \alpha - x_j b)^2 - \sigma^2)((y_j - \alpha - x_j b)^2 - \sigma^2)^T \end{bmatrix}$$

$$bGLS = (X^T \hat{\sigma}^{2^{-1}} X)^{-1} X^T \hat{\sigma}^{2^{-1}} y.$$

d) $y = \alpha + x\beta + u$

$E(x^T u) = \quad Eu = 0 \quad$ and $\quad E(u^2) = e^{x\sigma}$

$$B = \begin{bmatrix} \alpha \\ \beta \\ \sigma \end{bmatrix}$$

$$g_j(b) = \begin{bmatrix} (y_j - \alpha - x_j'\beta)x_j \\ (y_j - \alpha - x_j'\beta) \\ (y_j - \alpha - x_j'\beta)^2 - e^{x\sigma} \end{bmatrix}$$

e) $y = \alpha + x\beta + u$

$E(Z^T u) = Eu = 0$ and $EZ^T x = \varrho$

if we assume $Ey = Ex$ to get rid of α,

$b_{Gmm} = \left[(x^T z)\hat{\Omega}^{-1}(z^T x)\right]^{-1}\left[x^T z\, \hat{\Omega}^{-1} Z^T\right] y$

where $E(Z^T x) = \varrho$

$\hat{\Omega}$ is the estimator of Ω as per FGLS.

* $\cancel{g_j(b) = (y_j - \alpha - x_j b) Z^T}$

$\Rightarrow \cancel{b = \arg\min_b (y_j - \alpha - x_j b) Z^T \hat{\Omega}^{-1} (y_j - \alpha -}$

f) $y = f(x\beta) + u$

$f \rightarrow$ known scalar function

$E(Z^T u) = Eu = 0$ and $E Z^T x f'(x\beta) = \varrho(\beta)$

$(\varrho(\beta) =$ Jacobian)

$$g_j(b) = \begin{bmatrix} (y_j - f(x_j b)) Z^T \\ (y_j - f(x_j b) \end{bmatrix}$$

$\hat{\Omega}$ is the estimator for Ω.

$b = \arg\min_b (y - f(xb)) Z^T \hat{\Omega}^{-1} (y - f(xb))^T Z$

$\Rightarrow b_{Gmm} = \left[x f'(xb)^T Z\, \hat{\Omega}^{-1} Z^T x f'(x\beta)\right]^{-1} \left[x^T f'(xb)^T Z\, \hat{\Omega}^{-1} Z^T\right] y$