Chapter 7 Review

Saturday, April 24, 2021 7:19 PM

Chapter 7: Functions

Section 7.1: Functions Defined on General Sets

Function on Sets

- A **function *f* from a set *X* to a set *Y***, denoted $f:\ X\ \longrightarrow\ Y$, is a relation form *X*, the **domain**, to *Y*, the **co-domain**, that satisfies two properties: (1) every element in *X* is related to some element *Y*, and (2) no element in *X* is related to more than one element in *Y*.
- Thus, given any element x in *X*, there is a unique element in *Y* that is related to x by *f*. If we call this element y, the we say that "f sends x to y" or "f maps x to y" and write $x \xrightarrow{f} y$ or $f: x \longrightarrow y$. The unique element to which *f* sends x is denoted $f(x)$ and is called *"f* of x*"* or "the output of *f* for the input x" or "the value of *f* at x" or "**the image of x under *f***".
- The set of all values of *f* taken together is called the **range of *f*** or the image of X under *f*. Symbolically, $\text{range of } f\ =\ \text{image of } X \text{ under } f\ =\ \{y\ \in\ Y \mid \text{for some } x \text{ in } X\}$.
- Given an element y in *Y*, there may exist elements in *X* with y as their image. If $f(x)\ =\ y$, then x is called a **preimage of y** or an **inverse image of y**. The set of all inverse images of y in called the inverse image of y. Symbolically, $\text{inverse image of } y\ =\ \{x\ \in\ X \mid\ f(x) = y\}$.

A Test for Function Equality (Theorem 7.1.1): If $F: X \longrightarrow Y$ and $G: X \longrightarrow Y$ are functions, the $F\ =\ G$ IFF $F(x)\ =\ G(x)$ for all x ∈ X.

Logarithms and Logarithmic Functions

- Let b be a positive real number with $b \neq 1$. For each positive real number x, the **logarithm with base b of x**, written $\log_b x$, is the exponent to which b must be raised to obtain x. Symbolically,

  $$\log_b x = y \Leftrightarrow b^y = x.$$

- The **logarithmic function with base b** is the function from $\mathbb{R}^+$ to $\mathbb{R}$ that takes each positive real number x to $\log_b x$.

Boolean Function

- An (**n-place**) **Boolean function** *f* is a function whose domain is the set of all ordered n-tuples of 0's and 1's and whose co-domain is the set {0, 1}. More formally, the domain of a Boolean function can be described as the Cartesian product of n copies of the set $\{0, 1\}^n$. Thus $f: \{0, 1\}^n \longrightarrow \{0, 1\}$.

A Not Well Defined "Function"

- Def: We say that a "function" is **not well defined** if it fails to satisfy at least one of the requirements for being a function

Functions Acting on Sets

- If $f:\ X\ \longrightarrow\ Y$ is a function and $A \subseteq X$ and $C \subseteq Y$, then $f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \text{ in } A\}$ and $f^{-1}(C) = \{x \in X \mid f(x) \in C\}$. $f(A)$ is called the **image of A**, and $f^{-1}(C)$ is called the **inverse image of C**.

Section 7.2: One-to-One and Onto, Inverse Functions

One-to-One Functions

- Def: Let F be a function from a set X to a set Y. F is **one-to-one** (or **injective**) IFF for all elements x_1 and x_2 in X,

  If $F(x_1) = F(x_2)$, then $x_1 = x_2$

  or, equivalently,

  If $x_1 \neq x_2$, then $F(x_1) \neq F(x_2)$
- Symbolically, $F: X \rightarrow Y$ is one-to-one $\Leftrightarrow$ $\forall x_1, x_2 \in X$, if $F(x_1) = F(x_2)$ then $x_1 = x_2$

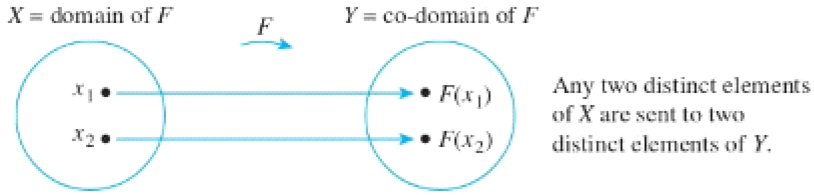
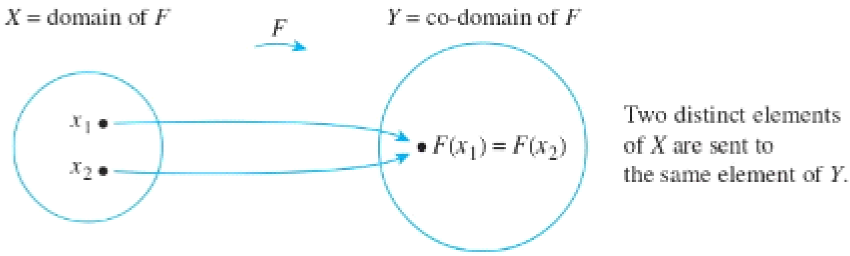


Figure 7.2.1(a) A One-to-One Function Separates Points

- Alternatively, $F: X \rightarrow Y$ is not one-to-one $\Leftrightarrow$ $\exists x_1, x_2 \in X,\ F(x_1) = F(x_2)$ and $x_1 \neq x_2$



gure 7.2.1(b) A Function That Is Not One-to-One Collapses Points Together

Hash Functions: **Hash functions** are functions defined form larger to smaller sets of integers, frequently using the mod function.

Onto Functions

- Def: Let F be a function from a set X to a set Y. F in **onto** (or **surjective**) IFF given any element y in Y, it is possible to find an element x in X with the property that $y = F(x)$.
- Symbolically, $F: X \rightarrow Y$ is onto $\Leftrightarrow$ $\forall y \in Y,\ \exists x \in X$ such that $F(x) = y$

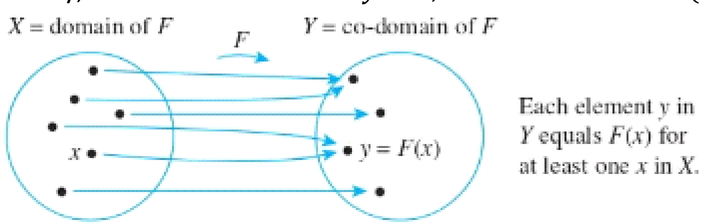


Figure 7.2.3(a) A Function That Is Onto

- Alternatively, $F: X \rightarrow Y$ is not onto $\Leftrightarrow$ $\exists y \in Y$ such that $\forall x \in X,\ F(x) \neq y$

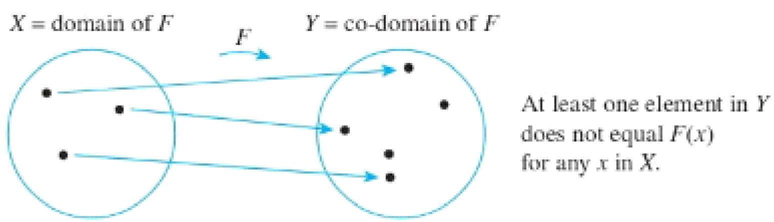


Figure 7.2.3(b) A Function That Is Not Onto

Exponential Function

- For positive numbers $b \neq 1$, the **exponential function with base b**, denoted exp_b, is the function from $\mathbb{R}$ to $\mathbb{R}^+$ defined as follows: For all real numbers x, $exp_b(x) = b^x \text{ where } b^0 = 1 \; and \; b^{-x} = {}^1/_{b^x}$

  Laws of Exponents
  - If b and c are any positive real numbers and u and v are any real numbers, the following laws of exponents hold true:
    - $b^u b^v = b^{u+v}$
    - $(b^u)^v = b^{uv}$
    - $\dfrac{b^u}{b^v} = b^{u-v}$
    - $bc^u = b^u c^u$

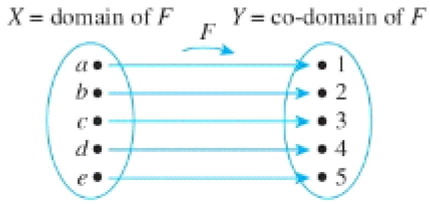
<u>Note</u>: $\log_b y = x \Leftrightarrow b^x = y$

Properties of Logarithms (<u>Theorem 7.2.1</u>): For any positive real numbers b, c, and x with $b \neq 1$ and $c \neq 1$,

- $\log_b(xy) = \log_b x + \log_b y$
- $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$
- $\log_b(x^a) = a \log_b x$
- $\log_c x = \dfrac{\log_b x}{\log_b c}$

One-to-One Correspondence

- Def: A **one-to-one correspondence** (or **bijection**) from a set X to a set Y is a function $F: X \to Y$ that is <u>both one-to-one and onto</u>.
- Diagrammatically,



An Arrow Diagram for a One-to-One Correspondence

Inverse Functions (F^{-1})

<u>Theorem 7.2.2</u>

- Suppose $F: X \to Y$ is a one-to-one correspondence; that is, suppose F is one-to-one and onto. Then there is a function $F^{-1}: X \to Y$ that is defined as follows:

  $$F^{-1}(y) = \text{that unique element } x \text{ in } X \text{ such that } F(x) \text{ equals } y$$

  In other words,

  $$F^{-1}(y) = x \Leftrightarrow y = F(x)$$

<u>Theorem 7.2.3</u>: If X and Y are sets and $F: X \to Y$ is one-to-one, then $F^{-1}: X \to Y$ is also one-to-one and onto.

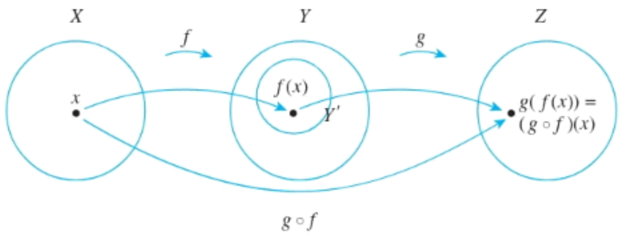
Section 7.3: Composition of Functions

Composition Two Functions

- Let $f: X \to Y'$ and $g: Y \to Z$ be functions with the property that the range of *f* is a subset of the domain of g. Define a new function $g \circ f: X \to Z$ as follows:

  $(g \circ f)(x) = g(f(x))$ for all $x \in X$

  where $g \circ f$ is read "*g* circle *f*" and $g(f(x))$ is read "*g* of *f* of *x*". The function $g \circ f$ is called the **composition of *f* and *g***.
- Shown schematically,



Composition with an Identity Function (Theorem 7.3.1): If *f* is a function from a set X to a set Y, and I_X is the identity function on X, and I_Y is the identity function on Y, then

- $f \circ I_X = f$
- $I_Y \circ f = f$

Composition of a Function with Its Inverse (Theorem 7.3.2): If $f: X \to Y$ is a one-to-one and onto function with inverse function $f^{-1}: Y \to X$ then

- $f^{-1} \circ f = I_X$
- $f \circ f^{-1} = I_Y$

Theorem 7.3.3: If $f: X \to Y$ and $g: Y \to Z$ are both one-to-one fucntions, then $g \circ f$ is one-to-one.

Composition of Functions is One-To-One: $g \circ f$ is one-to-one $\Leftrightarrow \forall x_1, x_2 \in X,$ if $(g \circ f)(x_1) = (g \circ f)(x_2)$ then $x_1 = x_2$

Theorem 7.3.4: If $f: X \to Y$ and $g: Y \to Z$ are both onto function, then $g \circ f$ is onto.

Composition of Functions is Onto: $g \circ f: x \to Z$ is onto $\Leftrightarrow$ given any element z of $Z,$ it is possible to find an element x of X such that $(g \circ f)(x) = z.$

Section 7.4: Cardinality with Applications to Computability

Finite Set: A **finite set** is one that has no elements at all or that can be put into one-to-one correspondence with a set of the form {1, 2,…, n} for some positive integer n.

Infinite Set: An **infinite set** is a nonempty set that cannot be put into one-to-one correspondence with a set of the form {1, 2,…, n} for some positive integer n.

Cardinality

- Def: Let A and B be any sets. **A has the same cardinality as B** IFF there is a one-to-one correspondence from A to B.
- In other words, A has the same cardinality of B IFF there is a function *f* from A to B that is one-to-one and onto.

Properties of Cardinality (Theorem 7.4.1): For all sets A, B, and C:

- **Reflexive property of cardinality**: A has the same cardinality as A.
- **Symmetric property of cardinality**: If A has the same cardinality as B, then B has the same cardinality as A.
- **Transitive property of cardinality**: If A has the same cardinality as B and B has the same cardinality as C, then A has the same cardinality as C.

Same Cardinality: A and B **have the same cardinality** IFF A has the same cardinality as B or B has the same cardinality as A.

Countable Sets

- A set is called **countably infinite** IFF it has the same cardinality as the set of positive integers $\mathbb{Z}^+$.
- A set is called **countable** IFF it is finite or countably infinite.
- A set that is not countable is called **uncountable**.

Note: Every real number, which is a measure of location on a number line, can be represented by a decimal expansion of the form $a_0.a_1a_2a_3$ ... where a_0 is an integer (positive, negative, or zero) and for each $i \geq 1, a_i$ is an integer from 0 to 9.

Cantor (Theorem 7.4.2): The set of all real numbers between 0 and 1 is uncountable.

Theorem 7.4.3: Any subset of any countable set is countable.

Corollary 7.4.4: Any set with an uncountable subset is uncountable.