Chapter 6 Review

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Chapter 6: Set Theory

Section 6.1: Definitions and the Element Method of Proof

Element Argument: The Basic Method for Proving that One Set is a Subset of Another

- Let sets X and Y be given. To prove that $X \subseteq Y$,
    1. Suppose that x is a particular but arbitrarily chosen element of X
    2. Show that x is an element of Y

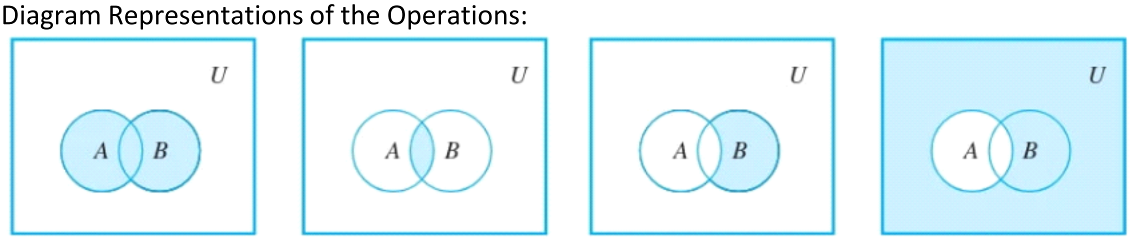
Set Equality

- Given sets A and B, A **equals** B, written **A = B**, IFF every element of A is in B and every element of B is in A.
- Symbolically: $A = B \iff A \subseteq B \text{ and } B \subseteq A$

  Distinct Sets: $A \neq B \iff A \nsubseteq B \text{ or } B \nsubseteq A$

Operations on Sets

- Let A and B be subsets of a universal set U.
    - The **union** of A and B, denoted $\boldsymbol{A} \cup \boldsymbol{B}$, is the set of all elements that are in at least one of A or B.
        - $A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$
    - The **intersection** of A and B, denoted $\boldsymbol{A} \cap \boldsymbol{B}$, is the set of all elements that are common to both A and B.
        - $A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$
    - The **difference** of B minus A (or **relative complement** of A in B), denoted $\boldsymbol{B} - \boldsymbol{A}$, is the set of all elements that are in B and not A.
        - $B - A = \{x \in U \mid x \in B \text{ and } x \notin A\}$
    - The **complement** of A, denoted $\boldsymbol{A}^{\boldsymbol{c}}$, is the set of all elements in U that are not in A.
        - $A^c = \{x \in U \mid x \notin A\}$
- Venn Diagram Representations of the Operations:



Shaded region represents $A \cup B$. Shaded region represents $A \cap B$. Shaded region represents $B - A$. Shaded region represents A^c.

Notation for Subsets of Real Numbers that are Intervals

- Given real numbers a and b with $a \leq b$:
    - $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$
    - $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$
    - $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$
    - $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$

- The symbols ∞ and -∞ are used to indicate intervals that are unbounded either on the right or on the left:
  - $(a, \infty) = \{x \in \mathbb{R} \mid x > a\}$
  - $(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$
  - $[a, \infty) = \{x \in \mathbb{R} \mid x \geq a\}$
  - $[-\infty, b) = \{x \in \mathbb{R} \mid x \leq b\}$

Unions and Intersection of an Indexed Collection of Sets

- Given sets A_0, A_1, A_2,… that are subsets of a universal set U and given a nonnegative integer n,
  - $\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for at least one } i = 0, 1, 2, \ldots, n\}$
  - $\bigcup_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for at least one nonnegative integer } i\}$
  - $\bigcap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for all } i = 0, 1, 2, \ldots, n\}$
  - $\bigcap_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for all nonnegative integers } i\}$
- Alternative notations
  - $\bigcup_{i=0}^{n} A_i = A_0 \cup A_1 \cup \cdots \cup A_n$
  - $\bigcap_{i=0}^{n} A_i = A_0 \cap A_1 \cap \cdots \cap A_n$

Partition of Sets

Disjoint

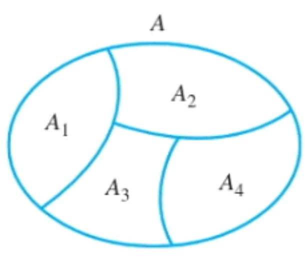
- Def: Two sets are called **disjoint** IFF they have no elements in common.
- Symbolically: A and B are disjoint ⇔ $A \cap B = \emptyset$

Mutually Disjoint

- Def: Sets A_1, A_2, A_3… are **mutually disjoint** (or **pairwise disjoint** or **nonoverlapping**) IFF no two sets A_i and A_j with distinct subscripts have any elements in common. More precisely, for all i, j = 1, 2, 3,… $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

Partition of a Set

- Def: A finite or infinite collection on nonempty sets {A_1, A_2, A_3…} is a **partition of a set A** IFF
  1. A is the union of all A_i
  2. The sets A_1, A_2, A_3… are mutually disjoint
- Venn Diagram Representation:



Power Sets

- Def: Given a set A, the **power set** of A, denoted $\mathcal{P}(A)$, is the set of all subsets of A.

Ordered n-Tuple

- The notation for an ordered n-tuple is a generalization of the notation for an ordered pair.
- Def: Let n be a positive integer and let $x_1, x_2, \dots, x_n$ be (not necessarily distinct) elements. The **ordered n-tuple**, $(x_1, x_2, \dots, x_n)$, consists of $x_1, x_2, \dots, x_n$ together with the ordering: first x_1, then x_2, and so forth up to x_n.
- An ordered 2-tuple is called an **ordered pair**, and an ordered 3-tuple is called an **ordered triple**.
- Two ordered n-tuples $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$ are equal IFF $x_1 = y_1, x_2 = y_2, \dots, x_n = y_n$.
  - Symbolically: $(x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_n) \Leftrightarrow x_1 = y_1, x_2 = y_2, \dots, x_n = y_n$
  - In particular, $(a, b) = (c, d) \Leftrightarrow a = c$ and $b = d$

Cartesian Product

- Given sets $A_1, A_2, \dots, A_n$, the **Cartesian product** of $A_1, A_2, \dots, A_n$ denoted $A_1 \times A_2 \times \cdots \times A_n$, is the set of all ordered n-tuples $(a_1, a_2, \dots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n$.
- Symbolically: $A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \dots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n\}$
- In particular, $A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1$ and $a_2 \in A_2\}$ in the Cartesian product of A_1 and A_2

Testing Whether $A \subseteq B$ (Algorithm 6.1.1)

Algorithm 6.1.1 Testing Whether $A \subseteq B$

[Input sets A and B are represented as one-dimensional arrays $a[1], a[2], \dots, a[m]$ and $b[1], b[2], \dots, b[n]$, respectively. Starting with $a[1]$ and for each successive $a[i]$ in A, a check is made to see whether $a[i]$ is in B. To do this, $a[i]$ is compared to successive elements of B. If $a[i]$ is not equal to any element of B, then answer is given the value "$A \nsubseteq B$." If $a[i]$ equals some element of B, the next successive element in A is checked to see whether it is in B. If every successive element of A is found to be in B, then answer never changes from its initial value "$A \subseteq B$."]

Input: *m [a positive integer], $a[1], a[2], \dots, a[m]$ [a one-dimensional array representing the set A], n [a positive integer], $b[1], b[2], \dots, b[n]$ [a one-dimensional array representing the set B]*

Algorithm Body:

$i := 1$, *answer* := "$A \subseteq B$"
while ($i \le m$ and *answer* = "$A \subseteq B$")
  $j := 1$, *found* := "no"
    while ($j \le n$ and *found* = "no")
      if $a[i] = b[j]$ **then** *found* := "yes"
      $j := j + 1$
    end while
    [If found has not been given the value "yes" when execution reaches this point, then $a[i] \notin B$.]
    if *found* = "no" **then** *answer* := "$A \nsubseteq B$"
    $i := i + 1$
end while

Output: *answer [a string]*

Section 6.2: Properties of Sets

Some Subset Relations (Theorem 6.2.1)

1. Inclusion of Intersection: For all sets A and B,
   a. $A \cap B \subseteq A$
   b. $A \cap B \subseteq B$
2. Inclusion in Union: For all sets A and B,
   a. $A \subseteq A \cup B$
   b. $B \subseteq A \cup B$
3. Transitive Property of Subsets: For all sets A, B, and C,
   a. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$

Procedural Versions of Set Definitions

- Let X and Y be subsets of a universal set U and suppose x and y are elements of U.
  1. $x \in X \cup Y \;\Leftrightarrow x \in X \; or \; x \in Y$
  2. $x \in X \cap Y \;\Leftrightarrow x \in X \; and \; x \in Y$
  3. $x \in X - Y \;\Leftrightarrow x \in X \; and \; x \notin Y$
  4. $x \in X^C \;\Leftrightarrow x \notin X$
  5. $(x, y) \in X \times Y \;\Leftrightarrow x \in X \; and \; y \in Y$

Set Identities (Theorem 6.2.2)

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws:* For all sets A and B,

   (a) $A \cup B = B \cup A$ and (b) $A \cap B = B \cap A$.

2. *Associative Laws:* For all sets A, B, and C,

   (a) $(A \cup B) \cup C = A \cup (B \cup C)$ and
   (b) $(A \cap B) \cap C = A \cap (B \cap C)$.

3. *Distributive Laws:* For all sets, A, B, and C,

   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and
   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

4. *Identity Laws:* For all sets A,

   (a) $A \cup \emptyset = A$ and (b) $A \cap U = A$.

5. *Complement Laws:*

   (a) $A \cup A^c = U$ and (b) $A \cap A^c = \emptyset$.

6. *Double Complement Law:* For all sets A,

   $(A^c)^c = A$.

7. *Idempotent Laws:* For all sets A,

   (a) $A \cup A = A$ and (b) $A \cap A = A$.

8. *Universal Bound Laws:* For all sets A,

   (a) $A \cup U = U$ and (b) $A \cap \emptyset = \emptyset$.

9. *De Morgan's Laws:* For all sets A and B,

   (a) $(A \cup B)^c = A^c \cap B^c$ and (b) $(A \cap B)^c = A^c \cup B^c$.

10. *Absorption Laws:* For all sets A and B,

    (a) $A \cup (A \cap B) = A$ and (b) $A \cap (A \cup B) = A$.

11. *Complements of U and $\emptyset$:*

    (a) $U^c = \emptyset$ and (b) $\emptyset^c = U$.

12. *Set Difference Law:* For all sets A and B,

9. *De Morgan's Laws:* For all sets A and B,

$$\text{(a) } (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad \text{(b) } (A \cap B)^c = A^c \cup B^c.$$

10. *Absorption Laws:* For all sets A and B,

$$\text{(a) } A \cup (A \cap B) = A \quad \text{and} \quad \text{(b) } A \cap (A \cup B) = A.$$

11. *Complements of U and $\emptyset$:*

$$\text{(a) } U^c = \emptyset \quad \text{and} \quad \text{(b) } \emptyset^c = U.$$

12. *Set Difference Law:* For all sets A and B,

$$A - B = A \cap B^c.$$

Basic Method for Proving that Sets are Equal

- Let sets X and Y be given. To prove that X = Y:
    1. Prove that $X \subseteq Y$
    2. Prove that $Y \subseteq X$

Intersection and Union with a Subset (Theorem 6.2.3)

- For any sets A and B, if $A \subseteq B$, then
    a. $A \cap B = A$
    b. $A \cup B = B$

A Set with No Elements is a Subset of Every Set (Theorem 6.2.4): If E is a set with no elements and A is any set, then $E \subseteq A$

Uniqueness of the Empty Set (Corollary 6.2.5): There is only one set with no elements.

Element Method for Proving a Set Equals the Empty Set

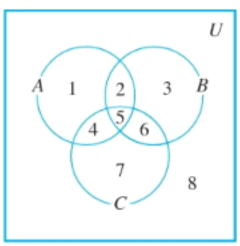
- To prove that a set X is equal to the empty set $\emptyset$, prove that X has no elements. To do this, suppose X has an element and derive a contradiction.

Proposition 6.2.6: For all sets A, B, and C, if $A \subseteq B$ and $B \subseteq C^c$, then $A \cap C = \emptyset$

Section 6.3: Disproofs, Algebraic Proofs, and Boolean Algebras

Disproving an Alleged Set Property

- Since an alleged set property would be a universal statement, to disprove it you provide a counterexample.
- Use a Venn diagram to show the sets and universal set as follows and number the sections to be the elements of the sets involved in the counterexample.



The Number of sets of a Set

Theorem 6.3.1: For all integers n ≥ 0, if a set X has n elements, then $\mathcal{P}(X)$ has 2^n elements.

Section 6.4: Boolean Algebras, Russell's Paradox, and the Halting Problem

Logical Equivalences and Set Properties

Logical Equivalences	**Set Properties**
For all statement variables p, q, and r:	For all sets A, B, and C:
a. $p \vee q \equiv q \vee p$ b. $p \wedge q \equiv q \wedge p$	a. $A \cup B = B \cup A$ b. $A \cap B = B \cap A$
a. $p \wedge (q \wedge r) \equiv p \wedge (q \wedge r)$ b. $p \vee (q \vee r) \equiv p \vee (q \vee r)$	a. $A \cup (B \cup C) \equiv A \cup (B \cup C)$ b. $A \cap (B \cap C) \equiv A \cap (B \cap C)$
a. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ b. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	a. $A \cap (B \cup C) \equiv (A \cap B) \cup (A \cap C)$ b. $A \cup (B \cap C) \equiv (A \cup B) \cap (A \cup C)$
a. $p \vee \mathbf{c} \equiv p$ b. $p \wedge \mathbf{t} \equiv p$	a. $A \cup \emptyset = A$ b. $A \cap U = A$
a. $p \vee \sim p \equiv \mathbf{t}$ b. $p \wedge \sim p \equiv \mathbf{c}$	a. $A \cup A^c = U$ b. $A \cap A^c = \emptyset$
$\sim(\sim p) \equiv p$	$(A^c)^c = A$
a. $p \vee p \equiv p$ b. $p \wedge p \equiv p$	a. $A \cup A = A$ b. $A \cap A = A$
a. $p \vee \mathbf{t} \equiv \mathbf{t}$ b. $p \wedge \mathbf{c} \equiv \mathbf{c}$	a. $A \cup U = U$ b. $A \cap \emptyset = \emptyset$
a. $\sim(p \vee q) \equiv \sim p \wedge \sim q$ b. $\sim(p \wedge q) \equiv \sim p \vee \sim q$	a. $(A \cup B)^c = A^c \cap B^c$ b. $(A \cap B)^c = A^c \cup B^c$
a. $p \vee (p \wedge q) \equiv p$ b. $p \wedge (p \vee q) \equiv p$	a. $A \cup (A \cap B) \equiv A$ b. $A \cap (A \cup B) \equiv A$
a. $\sim \mathbf{t} \equiv \mathbf{c}$ b. $\sim \mathbf{c} \equiv \mathbf{t}$	a. $U^c = \emptyset$ b. $\emptyset^c = U$

Boolean Algebra

- Def: A **Boolean algebra** is a set B together with two operations, generally denoted '+' and '·', such that for all a and b in B both 'a + b' and 'a · b' are in B and the following properties hold:
  - Commutative Laws: For all a and b in B,
    - a + b = b + a
    - a · b = b · a
  - Associative Laws: For all a, b, and c in B,
    - (a + b) + c = a + (b + c)
    - (a · b) · c = a · (b · c)
  - Distributive Laws: For all a, b, and c in B,
    - a + (b · c) = (a + b) · (a + c)
    - a · (b + c) = (a · b) + (a · c)
  - Identity Laws: There exist distinct elements 0 and 1 in B such that for all a in B,
    - a + 0 = 0
    - a · 1 = 1 a
  - Complement Laws: For each a in B, there exists an element in B, denoted $\bar{a}$ and called the **complement** or **negation** of a, such that
    - a + $\bar{a}$ = 1
    - a · $\bar{a}$ = 0

Properties of a Boolean Algebra (<u>Theorem 6.4.1</u>)

- Let B be any Boolean algebra
    1. Uniqueness of the Complement Law: For all a and x in B, if a + x = 1 and a · x = 0, then x = $\bar{a}$.
    2. Uniqueness of 0 and 1: If there exists x in B such that a + x = a for all a in B, then x = 0, and if there exists y in B such that a · y = a for all a in B, then y = 1.
    3. Double Complement Law: For all a ∈ B, $\overline{(\bar{a})} = a$
    4. Idempotent Law: For all a ∈ B,
        a. a + a = a
        b. a · a = a
    5. Universal Bound Law: For all a ∈ B,
        a. a + 1 = 1
        b. a · 0 = 0
    6. De Morgan's Laws: For all a and b ∈ B,
        a. $\overline{a+b} = \bar{a} \cdot \bar{b}$
        b. $\overline{a \cdot b} = \bar{a} + \bar{b}$
    7. Absorption Laws: For all a and b ∈ B,
        a. (a + b) · a = a
        b. (a · b) + a = a
    8. Complements of 0 and 1
        a. $\bar{0} = 1$
        b. $\bar{1} = 0$

<u>Theorem 6.4.2</u>: There is no computer algorithm that will accept any algorithm X and data set D as input and then will output "halts" of "loops forever" to indicate whether or not X terminates in a finite number of steps when X is run with data set D.