Chapter 9 Review

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Chapter 9: Counting and Probability

Section 9.1: Introduction

Random Process

- Def: To say that a process is **random** means that when it takes place, one outcome from some set of outcomes is sure to occur, but it is impossible to predict with certainty which outcome that will be.

Sample Space

- Def: A **sample space** is the set of all possible outcomes of a random process or experiment.
- Event
  - Def: An **event** is a subset of a sample space.

Notation: For any finite set A, **N(A)** denotes the **number of elements in A**.

Equally Likely Probability Formula

- If S is a finite sample space in which all outcomes are equally likely and E is an event in S, then the **probability of E**, denoted **P(E)**, is

$$P(E) = \frac{\text{the number of outcomes in } E}{\text{the total number of outcomes in } S} = \frac{N(E)}{N(S)}$$

The Number of Elements in a List (Theorem 9.1.1): If *m* and *n* are integers and $m \leq n$, then there are $n - m + 1$ integers from *m* to *n* inclusive.

Section 9.2: Possibility Trees and the Multiplication Rule

Possibility Trees

- A tree structure is a useful tool for keeping systematic track of all possibilities in situations in which events happen in order.

The Multiplication Rule (Theorem 9.2.1)

- If an operation consists of k steps and

  the first step can be performed in n_1 ways,

  the second step can be performed in n_2 ways [regardless of how the first step was performed],

  ⋮

  the kth step can be performed in n_k ways [regardless of how the first step was performed],

  then the entire operation can be performed in $n_1 \cdot n_2 \cdots n_k$ ways.

  When to apply the multiplication rule?
  - When you have n_1 options for one thing and n_2 options for another, but both are used to create one cohesive combination (like choosing a motherboard and case for a PC). Then there are $n_1 \cdot n_2$ ways.
  - When you have n options for x number of positions (like choosing a pin that allows repetitions). Then there are n^x ways.
  - When the n options/probability changes as you decrease x number of positions/spots (like taking objects out of a jar).

Beware: Don't use the multiplication rule when considering choosing people with restrictions for positions. Use the possibility tree.

Permutations

- Def: A **permutation** of a set of objects is an ordering of the objects in a row.
- There are $n!$ permutations of a set of n elements.

Theorem 9.2.2: For any integer n with $n \geq 1$, the number of permutations of a set with n elements is $n!$

Permutations of Selected Elements (r-permutation)

- Def: An **r-permutation** of a set of n elements is an ordered selection of r elements taken from the set of n elements. The number of r-permutations of a set of n elements is denoted **P(n, r)**.
- CALCULATOR: n [MATH][>>>][2] r [ENTER]

Theorem 9.2.3

- If n and r are integers and $1 \leq r \leq n$, then the number of r-permutations of a set of n elements is given by the formula

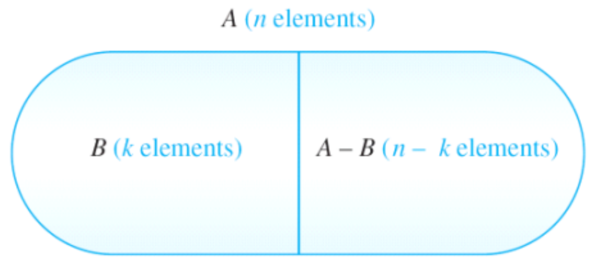
$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

Section 9.3: Counting Elements of Disjoint Sets: The Addition Rule

The Addition Rule (Theorem 9.3.1): Suppose a finite set A equals the union of k distinct mutually disjoint subsets $A_1, A_2, \ldots, A_k$. Then $N(A) = N(A_1) + N(A_2) + \cdots + N(A_k)$.

The Difference Rule (Theorem 9.3.2): If A is a finite set and B is a subset of A, then $N(A - B) = N(A) - N(B)$.

Illustration of the Difference Rule



The Difference Rule

Formula for the Probability for the Complement of an Event

- If S is a finite sample space and A is an event in S, then $P(A^c) = 1 - P(A)$.

The Inclusion/Exclusion Rule for Two or Three Sets (Theorem 9.3.3)

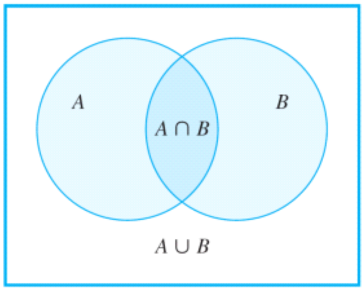
- If A, B, and C are any finite sets, then

$$N(A \cup B) = N(A) + N(B) - N(A \cap B)$$

And

$$N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C)$$

Illustration to Visualize the Inclusion/Exclusion Rule for Two Sets



Section 9.4: The Pigeonhole Principle

Pigeonhole Principle

- A function from one finite set to a smaller finite set cannot be one-to-one: There must be at least two elements in the domain that have the same image in the co-domain.

Generalized Pigeonhole Principle

- For any function f from a finite set X with n elements to a finite set Y with m elements and for any positive integer k, if $k < n/m$, then there is some $y \in Y$ such that y is the image of at least $k + 1$ distinct elements of X.

Generalized Pigeonhole Principle (Contrapositive Form)

- For any function f from a finite set X with n elements to a finite set Y with m elements and for any positive integer k, if for each $y \in Y$, $f^{-1}(y)$ has at most k elements, then X has at most km elements; in other words, $n \leq km$.

The Pigeonhole Principle (Theorem 9.4.1): For any function f from a finite set X with n elements to a finite set Y with m elements, if $n > m$, then f is not one-to-one.

One-to-One and Onto for Finite Sets (Theorem 9.4.2): Let X and Y be finite sets with the same number of elements and suppose f is a function from X to Y. Then f is one-to-one IFF f is onto.

Section 9.5: Counting Subsets of a Set: Combinations

Combinations

- Def: Let n and r be nonnegative integers with $r \leq n$. An **r-combination** of a set of n elements is a subset of r of the n elements. As indicated in Section 5.1, the symbol
  $$\binom{\mathrm{n}}{\mathrm{r}}$$
  which is read "n choose r", denotes the number of subsets of size r (r-combinations) that can be chosen from a set of n elements.
- Other symbols used to indicate "n choose r": $\mathrm{C}(n, r)$, $C_{n,r}$, nCr, ^{n}Cr
- CALCULATOR: n [MATH][>>>][3] r [ENTER]
- [Section 5.1] Formula for computing $\binom{n}{r}$
  - For all integers n and r with $0 \leq r \leq n$,
    $$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

Complete Enumeration: listing all possibilities of a combination

Relation of r-Permutation ($P(n, r)$) and r-Combinations($\binom{n}{r}$): $\mathrm{P}(n, r) = \binom{n}{r} \cdot r!$

Theorem 9.5.1

- The number of subsets of size r (or r-combinations) that can be chosen from a set of n elements, $\binom{n}{r}$, is given by the formula

  $$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} \; or \; \binom{n}{r} = \frac{P(n,r)}{r!}$$

  where n and r are nonnegative integers with $0 \le r \le n$.

Phrases Meaning

- The phrase "at least n" means "n or more"
- The phrase "at most n" means "n or fewer"

Permutations with Sets of Indistinguishable Objects (Theorem 9.5.2)

- Suppose a collection consists of n objects of which

  n_1 are of type 1 and are indistinguishable from each other

  n_2 are of type 2 and are indistinguishable from each other

  ⋮

  n_k are of type k and are indistinguishable from each other
- And suppose that $n_1 + n_2 + \cdots + n_k = n$. Then the number of distinguishable permutations of the n objects is

  $$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$$

  $$= \frac{n!}{n_1!\,n_2!\,n_3!\cdots n_k!}$$

The Number of Partitions of a Set into r Subsets

- **Stirling numbers of the second kind** ($S_{n,r}$)
- $S_{n,r}$ = number of ways a set of size n can be partitioned into r subsets

Section 9.6: R-Combinations with Repetition Allowed

r-Combinations with Repetition Allowed

- Def: An **r-combination with repetition allowed**, or **multiset of size r**, chosen from a set of X of n elements is an unordered selection of elements taken from X with repetition allowed. If $X = \{x_1, x_2, \cdots, x_n\}$, we write an r-combination with repetition allowed, or multiset of size r, as $[x_{i_1}, x_{i_2}, \cdots, x_{i_r}]$ where each x_{i_j} is in X and some of the x_{i_j} may equal each other.

Theorem 9.6.1: The number of r-combinations with repetition allowed (multisets of size r) that can be selected from a set of n elements is $\binom{r+n-1}{r}$. This equals the numbers of ways r objects can be selected from n categories of objects with repetition allowed.

Which Formula to Use? To find number of combinations possible

	Order Matters	Order Does Not Matter
Repetition Is Allowed	n^k	$\binom{k+n-1}{k}$
Repetition Is Not Allowed	$P(n, k)$	$\binom{n}{k}$

Section 9.7: Pascal's Formula and the Binomial Theorem

Important Evaluations

- $\binom{n}{0} = 1$
- $\binom{n}{n} = 1$
- $\binom{n}{1} = n$
- $\binom{n}{n-1} = n$
- $\binom{n}{r} = \binom{n}{n-r}$ $for\ all\ nonnegative\ integers\ n\ and\ r\ with\ r \leq n.$

Pascal's Formula (Theorem 9.7.1)

- Let n and r be positive integers and suppose $r \leq n$. Then

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

Table 9.7.1 Pascal's Triangle for $\left(\text{Values of } \binom{n}{r}\right)$

n \ r	0	1	2	3	4	5	...	$r-1$	r	...
0	1							·	·	...
1	1	1						·	·	...
2	1	2	1					·	·	...
3	1	3	3	1				·	·	...
4	1	4	6	+ 4	1			·	·	...
5	1	5	10	= 10	5	1		·	·	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮		⋮	⋮	⋮
n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	...	$\binom{n}{r-1}$ +	$\binom{n}{r}$	...
$n+1$	$\binom{n+1}{0}$	$\binom{n+1}{1}$	$\binom{n+1}{2}$	$\binom{n+1}{3}$	$\binom{n+1}{4}$	$\binom{n+1}{5}$	...	=	$\binom{n+1}{r}$	...
·		·	·	·	·	·		·	·	...
·		·	·	·	·	·		·	·	...
·		·	·	·	·	·		·	·	...

Binomial: In algebra, the sum of two terms is called a **binomial**.

Binomial Theorem (Theorem 9.7.2)

- Given any real numbers a and b and any nonnegative integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$= a^n + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + b^n$$

Nonnegative Integer Powers

- Def: For any real number a and any nonnegative integer n, the **nonnegative integer powers of a** are defined as follows:

$$a^n = \begin{cases} 1 & if\ n = 0 \\ a \cdot a^{n-1} & if\ n > 0 \end{cases}$$

Binomial Coefficient: If n and r are nonnegative integers and $r \leq n$, then $\binom{n}{r}$ is called a **binomial coefficient**.