Chapter 5 Review

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Chapter 5: Sequences, Mathematical Induction, and Recursion

Section 5.1: Sequences

Sequence

- Def: A **sequence** is a function whose domain is either all the integers between two given integers or all the integers greater than or equal to a given integer.
- Typically represent a sequence as a set of elements written in a row.
- Finite sequences are denoted as a_m, a_{m+1}, a_{m+2},..., a_n
  - Each individual element a_k (read "a sub k") is called a **term**.
  - The k in a_k is called a **subscript** or **index**, m (which may be any integer) is the subscript of the **initial term**, and n (which must be greater than or equal to m) is the subscript of the **final term**.
- Infinite sequences are denoted as a_m, a_{m+1}, a_{m+2},...
- An **explicit formula** or **general formula** for a sequence is a rule that shows how the values of a_k depend on k.

Summation Notation

- Def: If m and n are integers and n <= n, the symbol $\sum_{k=m}^{n} a_k$, read the **summation from k equals n to n of a-sub-k**, is the sum of all the terms a_m, a_{m+1}, a_{m+2},..., a_n (the **expanded form** of the sum).
- Denoted as $\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$
  - The **index** is the summation is k.
  - The **lower limit** of the summation is m and the **upper limit** is n.

Product Notation

- Def: If m and n are integers and m $\leq$ n, the symbol $\prod_{k=m}^{n} a_k$, read the **product from k equals m to n of a-sub-k**, is the product of all the terms a_m, a_{m+1}, a_{m+2},..., a_n.
- Denoted as $\prod_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$

Properties of Summations and Products

Theorem 5.1.1: If a_m, a_{m+1}, a_{m+2},... and b_m, b_{m+1}, b_{m+2},... are sequences of real numbers and c is any real number, then the following equations hold for any integer n $\geq$ m:

1. $$\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$$
2. $$c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$$
3. $$\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n} (a_k + b_k)$$

Factorial Notation

- For each positive integer n, the quantity **n factorial** denoted **n!**, is defined to be the product of all the integers from 1 to n:
  $$n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$$
- **Zero factorial**, denoted 0!, id defined to be 1: 0! = 1.

"n Choose r" Notation

- Let n and r be integers with $0 \le r \le n$. The symbol $\binom{n}{r}$ is read "**n choose r**" and represents the number of subsets of size r that can be chosen from a set with n elements.
- Formula for computing $\binom{n}{r}$
  - For all integers n and r with $0 \le r \le n$,
    $$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

Decimal to Binary Conversion Using Repeated Division by 2 (Algorithm 5.1.1)

Algorithm 5.1.1 Decimal to Binary Conversion Using Repeated Division by 2

[In Algorithm 5.1.1 the input is a nonnegative integer n. The aim of the algorithm is to produce a sequence of binary digits $r[0], r[1], r[2], \ldots, r[k]$ so that the binary representation of a is

$$(r[k]r[k-1]\cdots r[2]r[1]r[0])_2.$$

That is,

$$n = 2^k \cdot r[k] + 2^{k-1} \cdot r[k-1] + \cdots + 2^2 \cdot r[2] + 2^1 \cdot r[1] + 2^0 \cdot r[0].]$$

Input: n *[a nonnegative integer]*

Algorithm Body:

$q := n, i := 0$

[Repeatedly perform the integer division of q by 2 until q becomes 0. Store successive remainders in a one-dimensional array $r[0], r[1], r[2], \ldots, r[k]$. Even if the initial-value of q equals 0, the loop should execute one time (so that r[0] is computed). Thus the guard condition for the **while** *loop is $i = 0$ or $q \neq 0$.]*

while $(i = 0 \text{ or } q \neq 0)$

$r[i] := q \text{ mod } 2$

$q := q \text{ div } 2$

[$r[i]$ and q can be obtained by calling the division algorithm.]

$i := i + 1$

end while

[After execution of this step, the values of $r[0], r[1], \ldots, r[i-1]$ are all 0's and 1's, and $a = (r[i-1]r[i-2]\cdots r[2]r[1]r[0])_2$.]

Output: $r[0], r[1], r[2], \ldots, r[i-1]$ *[a sequence of integers]*

Section 5.2: Mathematical Induction 1

Principle of Mathematical Induction

- Let P(n) be a property that is defined for integers n, and let a be a fixed integer. Suppose the following two statements are true:
  1. P(a) is true
  2. For all integers k $\geq$ a, if P(k) is true then P(k+1) is true.

  Then the statement, for all integers n $\geq$ a, P(n) is true.

Method of Proof by Mathematical Induction

- Consider a statement of the (universal) form, "For all integers n $\geq$ a, a property P(n) is true." To probe such a statement, perform the following two steps:
  1. Basis Step: Show that P(a) is true.
  2. Inductive Step: Show that for all integers k $\geq$ a, if P(k) is true then P(k+1) is true. To perform this step
     - Inductive Hypothesis: Suppose that P(k) is true, where k is any particular but arbitrarily chosen integer with k $\geq$ a.
     - Then show that P(k+1) is true.

<u>Proposition 5.2.1</u>: For all integers n $\geq$ 8, n₵ can be obtained using 3₵ and 5₵ coins.

Sum of the First n Integers (<u>Theorem 5.2.2</u>): For all integers n $\geq$ 1, $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

Closed Form

- Def: If a sum with a variable number of terms is shown to be equal to a formula that does <u>not</u> contain either an ellipsis or a summation symbol, we say that it is written in **closed form**.

Geometric Sequence

- Def: In a **geometric sequence**, each term is obtained from the preceding on by multiplying by a constant factor.

  Sum of a Geometric Sequence (<u>Theorem 5.2.3</u>): For any real number r except 1, and any integer n $\geq$ 0,

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$$

Section 5.3: Mathematical Induction 2

<u>Proposition 5.3.1</u>: For all integers n $\geq$ 0, $2^{2n} - 1$ is divisible by 3.

<u>Proposition 5.3.2</u>: For all integers n $\geq$ 3, $2n + 1 < 2^n$.

Section 5.4: Strong Mathematical Induction and the Well-Ordering Principle for the Integers

Principle of Strong Mathematical Induction

- Let P(n) be a property that is defined for integers n, and let a and b be fixed integers with a $\leq$ b. Suppose the following two statements are true:
  1. Basis Step: P(a), P(a+1),…, and P(b) are all true.
  2. Inductive Step: For any integer k $\geq$ b, if P(i) is true for all integers i from a through k, then P(k+1) is true.

  Then the statement for all integer n $\geq$ a, P(n) is true. (The supposition that P(i) is true for all integers i from a through k is called the inductive hypothesis. Another way to state the inductive hypothesis is to say that P(a), P(a+1),…,P(k) are all true.)

Existence and Uniqueness of Binary Integer Representations (Theorem 5.4.1): Given any positive integer n, n has a unique representation in the form $n = c_r \cdot 2^r + c_{r-1} \cdot 2^{r-1} + \cdots + c_2 \cdot 2^2 + c_1 \cdot 2 + c_0$, where r is a nonnegative integer, $c_r = 1$, and $c_j = 1$ or 0 for all j = 0, 1, 2,…, r-1.

Well-Ordering Principle for the Integers

- Let S be a set of integers containing one or more integers all of which are greater than some fixed integer. The S has a least element.

Section 5.6: Defining Sequences Recursively

Recurrence Relation

- Def: A **recurrence relation** for a sequence $a_0, a_1, a_2, \ldots$ is a formula that relates each term a_k to certain of its predecessors $a_{k-1}, a_{k-2}, \ldots, a_{k-i}$, where i is an integer with $k - i \geq 0$. The **initial conditions** for such a recurrence relation specify the values of $a_0, a_1, a_2, \ldots, a_{i-1}$, if i is a fixed integer, or $a_0, a_1, a_2, \ldots, a_m$, where m is an integer with $m \geq 0$, if i depends on k.

Catalan Numbers

- Recurrence Relation: For each integer $n \geq 1$, $C_n = \frac{1}{n+1}\binom{2n}{n}$
- Note: "n choose k" = $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Recursive Definitions of Sum and Product

- Given numbers $a_1, a_2, \ldots, a_n$, where n is a positive integer
  - The **summation from i = 1 to n of the a_i,** denoted [$\sum_{i=1}^{n} a_i$], is defined as follows:

$$\sum_{i=1}^{n} a_i = a_1 \ and \ \sum_{i=1}^{n} a_i = \left(\sum_{i=1}^{n-1} a_i\right) + a_n \quad if\ n > 1$$

  - The **product from i = 1 to n of the a_i**, is denoted [$\prod_{i=1}^{n} a_i$], is defined as follows:

$$\prod_{i=1}^{n} a_i = a_1 \ and \ \prod_{i=1}^{n} a_i = \left(\prod_{i=1}^{n-1} a_i\right) \cdot a_n \quad if\ n > 1$$