# Chapter 8 Review

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Chapter 8: Relations

Section 8.1: Relations on Sets

Binary Relation

- Def: A **binary relation** is a subset of a Cartesian product of two sets.

Congruent Modulo 2

- Def: When integers m and n are related by "m mod 2 = n mod 2" (that is both are even or both are odd), m and n are said to be **congruent modulo 2**.

The Inverse of a Relation ($R^{-1}$)

- Let R be a relation from A to B. Define the inverse relation $R^{-1}$ from B to A as follows:
  $R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R\}$
- For all $x \in A$ and $y \in B$, $(y, x) \in R^{-1} \Leftrightarrow (x, y) \in R$

Relation on a Set

- Def: A **relation on a set A** is a relation from A to A.
- When a relation R is defined on a set A, the arrow diagram of the relation can be modified to that it becomes a directed graph.

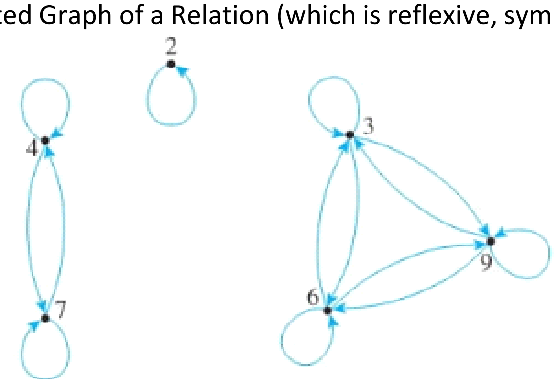
  Directed Graph of a Relation
  - Represent A only once, and draw an arrow from each point of A to each related point.
  - For all points x and y in A, there is an arrow from x to y $\Leftrightarrow$ x R y $\Leftrightarrow$ $(x, y) \in R$
  - If a point is related to itself, a loop id drawn that extends out from the point and goes back to it.

N-ary Relation

- Def: Given sets $A_1, A_2, \ldots, A_n$, an **n-ary relation** R on $A_1 \times A_2 \times \ldots \times A_n$ is a subset of $A_1 \times A_2 \times \ldots \times A_n$.
- The special cases of 2-ary, 3-ary, and 4-ary relations are called binary, ternary, and quaternary relations, respectively.

Section 8.2: Reflexivity, Symmetry, and Transitivity

Directed Graph of a Relation (which is reflexive, symmetric, and transitive)




Reflexivity

- Def: R is **reflexive** IFF for all $x \in A$, x R x.
- Reflexive means each element is related to itself.

- The directed graph of R if R is reflexive should have the property:
  - Each point of the graph has an arrow looping around from it back to itself.
- Def: R is **not reflexive** IFF ∃x ∈ A such that (x, x) ∉ R.

Symmetry
- Def: R is **symmetric** IFF for all x, y ∈ A, if x R y then y R x.
- Symmetric means if any one element is related to another element, then the second element is related to the first.
- The directed graph of R if R is symmetric should have the property:
  - In each case where there is an arrow going from one point to a second, there is an arrow going from the second point back to the first.
- Def: R is **not symmetric** IFF ∃x, y ∈ A such that (x, y) ∈ R but (y, x) ∉ R.

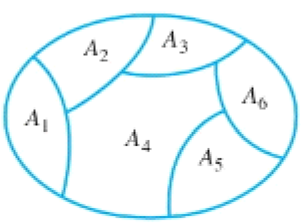
Transitivity
- Def: R is **transitive** IFF for all x, y, z ∈ A, if x R y and y R z then x R z.
- Transitive means if any one element is related to a second and that second element is related to a third, then the first element is related to the third.
- The directed graph of R if R is transitive should have the property:
  - In each case where there is an arrow going from one point to a second and from the second point to a third, there is an arrow going from the first point to the third. That is, there are no "incomplete directed triangles" in the graph.
- Def: R is **not transitive** IFF ∃x, y, z ∈ A such that (x, y) ∈ R and (y, z) ∈ R but (x, z) ∉ R.

Properties of Relations on Infinite Sets
- To show that a relation R on an infinite set A is <u>reflexive</u>, you suppose that x is any element of A and you show that x R x.
- To show that a relation R on an infinite set A is <u>symmetric</u>, you suppose that x and y are any elements of A such that x R y and you show that y R x.
- To show that a relation R on an infinite set A is <u>transitive</u>, you suppose that x, y, z are any elements of A such that x R y and y R z and you show that x R z.

Transitive Closure of a Relation
- Let A be a set and R a relation on A. The **transitive closure** of R is the relation $R^t$ on A that satisfies the following three properties:
  1. $R^t$ is transitive
  2. $R \subseteq R^t$
  3. If S is any other transitive relation that contains R, then $R^t \subseteq S$

Section 8.3: Equivalence Relations

Partition of a Set




$A_i \cap A_j = \emptyset$, whenever $i \neq j$
$A_i \cup A_2 \cup \cdots \cup A_6 = A$

**Figure 8.3.1** A Partition of a Set

- Def: A **partition** of a set A is a finite or infinite collection of nonempty, mutually disjoint subsets whose union is A.

The Relation Induced by a Partition

- Def: Given a partition of a set A, the relation **induced by the partition**, R, is defined on A as follows:

  For all x, y ∈ A, x R y ⟺ there is a subset $A_i$ of the partition such that both x and y are in $A_i$

- Theorem 8.3.1: Let A be a set with a partition and let R be the relation induced by the partition. Then R is reflexive, symmetric, and transitive.

Equivalence Relation

- Def: Let A be a set and R a relation on A. R is an **equivalence relation** IFF R is reflexive, symmetric, and transitive.

  Equivalence Classes of an Equivalence Relation

  - Def: Suppose A is a set and R is an equivalence relation on A. For each element a in A, the **equivalence class of a**, denoted **[a]** and called the **class of a** for short, is the set of all elements x in A such that x is related to a by R. In symbols: [a] = {x ∈ A | x R a}

Lemma 8.3.2: Suppose A is a set, R is an equivalence relation on A, and a and b are elements of A. If a R b, the [a] = [b].

Lemma 8.3.3: If A is a set, R is an equivalence relation on A, and a and b are elements of A, then either [a] ∩ [b] = ∅ or [a] = [b]

The Partition Induced by an Equivalence Relation (Theorem 8.3.4)

- If A is a set and R is an equivalence relation on A, then the distinct equivalence classes of R form a partition of A; that is, the union of the equivalence classes is all of A, and the intersection of any two distinct classes is empty.

Representative of an Equivalence Class

- Def: Suppose R is an equivalence relation on a set A and S is an equivalence class of R. A **representative** of the class S is an element a such that [a] = S.

**Congruence Relations**

- Def: Let m and n be integers and let d be a positive integer. We say that m is congruent to n modulo d and write $m \equiv n \pmod{d}$ IFF d | (m - n).
- Symbolically: $m \equiv n \pmod{d} \iff d \mid (m - n)$
- The number of equivalence classes for a congruence relation is d. Those with the same remainder from mod d are equal to each other.
- Congruence relations are equivalence relations (reflexive, symmetric, and transitive).
- d | (m - n) is said as "d divides (m-n)"

Section 8.5: Partial Order Relations

Antisymmetry

- Def: Let R be a relation on a set A. R is **antisymmetric** IFF for all a and b in A, if a R b and b R a then a = b.
- Def: Relation R is **not antisymmetric** IFF there are elements a and b in A such that a R b and b R a but a ≠ b

Partial Order Relations

- Def: Let R be a relation defined on a set A. R is a **partial order relation** IFF R is reflexive, antisymmetric, and transitive.
- Two fundamental partial order relations are the "less than or equal to" relation on a set of real numbers and the "subset" relation on a set of sets. Also x|y.

  Notation (≼): The symbol ≼ is often used to refer to a **general partial order relation** and the notation x ≼ y is read "x is less than or equal to y" or "y is greater than or equal to x".

<u>Theorem 8.5.1</u>

**Theorem 8.5.1**

Let $A$ be a set with a partial order relation $R$, and let $S$ be a set of strings over $A$. Define a relation $\preceq$ on $S$ as follows:

For any two strings in $S$, $a_1a_2\cdots a_m$ and $b_1b_2\cdots b_n$, where $m$ and $n$ are positive integers,

1. If $m \le n$ and $a_i = b_i$ for all $i = 1, 2, \ldots, m$, then
$$a_1a_2\cdots a_m \preceq b_1b_2\cdots b_n.$$
2. If for some integer $k$ with $k \le m$, $k \le n$, and $k \ge 1$, $a_i = b_i$ for all $i = 1, 2, \ldots, k-1$, and $a_k \ne b_k$, but $a_k \; R \; b_k$ then
$$a_1a_2\cdots a_m \preceq b_1b_2\cdots b_n.$$
3. If $\varepsilon$ is the null string and $s$ is any string in $S$, then $\epsilon \preceq s$.

If no strings are related other than by these three conditions, then $\preceq$ is a partial order relation.

Lexicographic Order for S

- The partial order relation of Theorem 8.5.1 is called the lexicographic order of S that corresponds to the partial order R on A.

Hasse Diagram

- A simpler graph to represent a partial order relation defined on a finite set.
- To obtain a Hasse diagram: Start with a directed graph of the relation, placing vertices on the page so that all arrows point upwards. Then eliminate…
  1. The loops at all the vertices
  2. All arrows whose existence is implied by the transitive property
  3. The direction indicators on the arrows

Comparable

- Def: Suppose ≼ is a partial order relation on a set A. Elements a and b of A are said to be **comparable** IFF either a ≼ b or b ≼ a. Otherwise, a and b are called **noncomparable**.

Total Order Relation

- Def: If R is a partial order relation on a set A, and for any two elements a and b in A either a R b or b R a, the R is a **total order relation** on A.

Partially and Totally Ordered Sets

Partially Ordered Set

- Def: A set A is called a **partially ordered set** (or **poset**) with respect to a relation $\preccurlyeq$ IFF $\preccurlyeq$ is a partial order relation on A.

Totally Ordered Set

- Def: A set A is called a **totally ordered set** with respect to a relation $\preccurlyeq$ IFF A is partially ordered with respect to $\preccurlyeq$ and $\preccurlyeq$ is a total order.

Chains

- Def: Let A be a set that is partially ordered with respect to a relation $\preccurlyeq$. A subset B of A is called a **chain** IFF the elements in each pair of elements in B is comparable. In other words, $a \preccurlyeq b$ or $b \preccurlyeq a$ for all a and b in A. The **length of a chain** is one less than the number of elements in the chain.

Elements of a Set Partially Ordered with Respect to a Relation $\preccurlyeq$

- Let a set A be partially ordered with respect to a relation $\preccurlyeq$
  - An element a in A is called a **maximal element of A** IFF for all b in A, either $b \preccurlyeq a$ or b and a are not comparable.
  - An element a in A is called a **greatest element of A** IFF for all b in A, $b \preccurlyeq a$.
  - An element a in A is called a **minimal element of A** IFF for all b in A, either $a \preccurlyeq b$ or b and a are not comparable.
  - An element a in A is called a **least element of A** IFF for all b in A, $a \preccurlyeq b$.

Compatible

- Def: Given partial order relations $\preccurlyeq$ and $\preccurlyeq'$ on a set A, $\preccurlyeq'$ is **compatible** with $\preccurlyeq$ IFF for all a and b in A, if $a \preccurlyeq b$ then $a \preccurlyeq' b$.

Topological Sorting

- Def: Given partial order relations $\preccurlyeq$ and $\preccurlyeq'$ on a set A, $\preccurlyeq'$ is **topological sorting** with $\preccurlyeq$ IFF $\preccurlyeq'$ is total order that is compatible with $\preccurlyeq$.
- Constructing a topological sorting
  - Let $\preccurlyeq$ be a partial order relation on a nonempty finite set A. To contruct a topological sorting,
    1. Pick any minimal element x in A. [Such an element exists since A is nonempty]
    2. Set A' := A - {x}
    3. Repeat steps a-c while $A' \neq \emptyset$
       a. Pick any minimal element y in A'
       b. Define $x \preccurlyeq' y$
       c. Set A' := A' - {y} and x := y