# Chapter 3 Review

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Chapter 3: The Logic of Quantified Statements

Section 3.1: Predicates and Quantified Statements 1

Predicate

- Def: A **predicate** is a sentence that contains a finite number of variables. Becomes a statement when specific values are substituted for the variables.
- Def: The **domain of a predicate** variable is the set of all values that may be substituted in place of the variable.

Truth Set

- Def: If P(x) is a predicate and x has domain D, the **truth set** of P(x) is the set of all elements of D that make P(x) true when they are substituted for x.
- The truth set of P(x) is denoted $\{x \in P(x)\}$.

Quantifier

- Def: **Quantifiers** are words that refer to quantities and tell for how many elements a given predicate is true.

  Universal Quantifier (∀)
  - Def: The **universal quantifier** is the symbol ∀ which denotes "for all".

  Existential Quantifier (∃)
  - Def: The **existential quantifier** is the symbol ∃ which denotes "there exists".

Universal Statement

- Def: Let Q(x) be a predicate and D the domain of x. A **universal statement** is a statement of the form "$\forall x \in D, Q(x)$" / "$\forall x(x \in D \rightarrow Q(x))$".
- It is defined to be <u>false</u> IFF Q(x) is false for at least one x in D.
- Def: A value for x for which Q(x) is false is called a **counterexample** to the universal statement.

  Method of Exhaustion
  - Def: The **method of exhaustion** consists of showing the truth of the predicate separately for each individual element of the domain.
  - This method can, in theory, be used whenever the domain of the predicate variable in finite.

Existential Statement

- Def: Let Q(x) be a predicate and D the domain of x. An **existential statement** is a statement of the form "$\exists x \in D, Q(x)$" / "$\exists x(x \in D \land Q(x))$".
- It is defined to be <u>true</u> IFF Q(x) for at least one x in D.
- It is defined to be <u>false</u> IFF Q(x) is false for all x in D.

Universal Conditional Statement

- Def: A universal conditional statement is a statement of the form "∀x, if P(x) then Q(x)" / "$\forall x \in D, P(x) \rightarrow Q(x)$".
- It is defined to be <u>false</u> IFF P(x) for at least on x in D is true but Q(x) for that x is false.

Formal vs. Informal Language

| Formal Language | Informal Language |
|---|---|
| Universal Statement Form:<br>$\forall x \in D, Q(x)$<br>$\forall x(x \in D \rightarrow Q(x))$ | Universal Statements Include:<br>"For all" "Each" "Every" "For any" |
| Existential Statement Form:<br>$\exists x \in D, Q(x)$<br>$\exists x(x \in D \land Q(x))$ | Existential Statements Include:<br>"Some" "At least one" |
| Universal Conditional Statement Form:<br>$\forall x \in D$, if $P(x)$ then $Q(x)$<br>$\forall x \in D, P(x) \rightarrow Q(x)$ | Universal Conditional Statements include:<br>"If, then" "Whenever" "Of any" "Of all" |

Equivalent Forms of Universal and Existential Statements

| Universal Conditional Statement | Universal Statement |
|---|---|
| $\forall x$, if $x \in D$ then $Q(x)$ | $\forall x \in D, Q(x)$ |

| Existential Statement | Existential Statement |
|---|---|
| $\exists x \in D, Q(x) \land P(x)$ | $\exists x \in D, Q(x) \mid P(x)$<br>$\exists x \in D, P(x) \mid Q(x)$ |

Implicit Quantification

- Let P(x) and Q(x) be predicates and suppose the common domain of x is D
  - The notation **P(x) ⇒ Q(x)** means that every element in the truth set of P(x) is in the truth set of Q(x), or, equivalently, $\forall x, P(x) \rightarrow Q(x)$
  - The notation **P(x) ⟺ Q(x)** means that P(x) and Q(x) have identical truth sets, or , equivalently, $\forall x, P(x) \leftrightarrow Q(x)$

Section 3.2: Predicates and Quantified Statements 2

Negation of Quantified Statements

| Statement | Negation | Theorem |
|---|---|---|
| Universal | $\sim(\forall x \in D, Q(x)) \equiv \exists x \in D, \sim Q(x)$<br>~("all are") ≡ "some are not" or "there is at least one that is not" | Negation of a Universal Statement (3.2.1) |
| Existential | $\sim(\exists x \in D, Q(x)) \equiv \forall x \in D, \sim Q(x)$<br>~("some are") ≡ "none are" or "all are not" | Negation of an Existential Statement (3.2.2) |
| Universal Conditional | $\sim(\forall x \in D$, if $P(x)$ then $Q(x)) \equiv \exists x \in D, P(x) \land \sim Q(x)$ | N/A |

Variants of Universal Conditional Statements

- Consider the universal conditional statement form: $\forall x \in D$, if $P(x)$ then $Q(x)$
  - Its **contrapositive** is the statement: $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$
  - Its **converse** is the statement: $\forall x \in D$, if $Q(x)$ then $P(x)$
  - Its **inverse** is the statement: $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$

Necessary and Sufficient Conditions, Only If

- Def: "∀x, r(x) is a **sufficient condition** *for* s(x)" ≡ "∀x, if r(x) then s(x)"
- Def: "∀x, r(x) is a **necessary condition** *for* s(x)" ≡ "∀x, if ~r(x) then ~s(x)" ≡ "∀x, if s(x) then r(x)"
- Def: "∀x, r(x) **only if** *for* s(x)" ≡ "∀x, ~s(x) then ~r(x)" ≡ "∀x, if r(x) then s(x)"

Section 3.3: Statements with Multiple Quantifiers

Interpreting Statements with Two Different Quantifiers

- Universal Quantifier + Existential Quantifier
  - If you want to establish the truth of a statement of the form ∀x ∈ D, ∃y ∈ E such that P(x,y), your challenge is to allow someone else to pick whatever element x in D they wish and then you must find an element y in E that "works" for that particular x.
- Existential Quantifier + Universal Quantifier
  - If you want to establish the truth of a statement of the form ∃x ∈ D such that ∀y ∈ E, P(x,y), your job is to find one particular x in D that willl "work" no matter what y in E anyone might choose.

Reciprocal: The **reciprocal** of a real number *a* is a real number *b* such that *ab* = 1.

Negation of Multiply-Quantified Statements

- ~(∀x ∈ D, ∃y ∈ E such that P(x,y)) ≡ ∃x ∈ D such that ∀y ∈ E, ~P(x,y)
- ~(∃x ∈ D such that ∀y ∈ E, P(x,y)) ≡ ∀x ∈ D, ∃y ∈ E such that ~P(x,y)

Order of Quantifiers

- In a statement containing both A and E, changing the order of the quantifiers usually changes the meaning of the statement.

Section 3.4: Arguments and Quantified Statements

Universal Instantiation

- Rule of **universal instantiation**: If some property is true of *everything* in a set, then it is true of *any particular* thing in the set.
- The fundamental tool of deductive reasoning

Universal Modus Ponens

| Formal | Informal |
|---|---|
| ∀x, if P(x) then Q(x).<br>P(a) for a particular a.<br>∴ Q(a). | If x makes P(x) true, then x makes Q(x) true.<br>a makes P(x) true.<br>∴ a makes Q(x) true. |

- A valid argument form that combines universal instantiation and modus ponens.

Universal Modus Tollens

| Formal | Informal |
|---|---|
| ∀x, if P(x) then Q(x).<br>~Q(a) for a particular a.<br>∴ ~P(a). | If x makes P(x) true, then x makes Q(x) true.<br>a does not make Q(x) true.<br>∴ a does not make P(x) true. |

- A valid argument form that combines universal instantiation and modus tollens.

Using Diagrams to Test for Validity

- To test the validity of an argument diagrammatically, represent the truth of both premises with diagrams. Then analyze the diagrams to see whether they necessarily represent the truth of the conclusion as well.

Converse Error (Quantified Form)

| Formal | Informal |
|---|---|
| ∀x, if P(x) then Q(x).<br>Q(a) for a particular a.<br>∴ P(a). | If x makes P(x) true, then x makes Q(x) true.<br>a makes Q(x) true.<br>∴ a makes P(x) true. |

Inverse Error (Quantified Form)

| Formal | Informal |
|---|---|
| ∀x, if P(x) then Q(x).<br>~P(a) for a particular a.<br>∴ ~Q(a). | If x makes P(x) true, then x makes Q(x) true.<br>a does not make P(x) true.<br>∴ a does not make Q(x) true. |

Universal Transitivity

| Formal | Informal |
|---|---|
| ∀x, P(x) → Q(x).<br>∀x, Q(x) → R(x).<br>∴ ∀x, P(x) → R(x). | Any x that makes P(x) true makes Q(x) true.<br>Any x that makes Q(x) true makes R(x) true.<br>∴ Any x that makes P(x) true makes R(x) true. |

- A valid argument form that combines universal instantiation and transitivity.