

# The Chow-Lam Form

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Slides: <https://lizziepratt.com/notes>  
The Chow-Lam Form (w/ Bernd Sturmfels)  
The Segre Determinant arXiv 2505.09204

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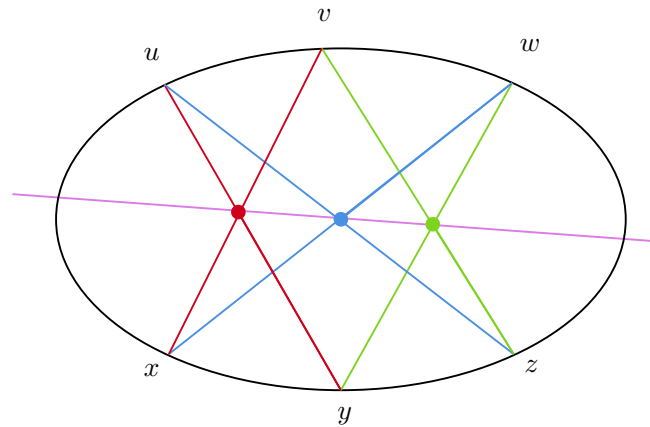
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Example (Pascal 1640)



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Given  $\begin{bmatrix} u_0 & v_0 & w_0 & x_0 & y_0 & z_0 \\ u_1 & v_1 & w_1 & x_1 & y_1 & z_1 \\ u_2 & v_2 & w_2 & x_2 & y_2 & z_2 \end{bmatrix}$ , when

$$\det \begin{bmatrix} u_0^2 & v_0^2 & w_0^2 & x_0^2 & y_0^2 & z_0^2 \\ u_0u_1 & v_0v_1 & w_0w_1 & x_0x_1 & y_0y_1 & z_0z_1 \\ u_0u_2 & v_0v_2 & w_0w_2 & x_0x_2 & y_0y_2 & z_0z_2 \\ u_1^2 & v_1^2 & w_1^2 & x_1^2 & y_1^2 & z_1^2 \\ u_1u_2 & v_1v_2 & w_1w_2 & x_1x_2 & y_1y_2 & z_1z_2 \\ u_2^2 & v_2^2 & w_2^2 & x_2^2 & y_2^2 & z_2^2 \end{bmatrix} = 0.$$

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### Theorem (First fundamental theorem of invariant theory)

Let  $x_{ij}$  be entries of a  $d \times n$  matrix  $X$ , and let  $SL_d$  act by multiplication on  $X$ . The invariant ring  $\mathbb{C}[x_{ij}]^{SL_d}$  is generated by the  $d \times d$  minors of  $X$ .

When do  $N := \binom{r+d}{d}$  points in  $\mathbb{P}^r$  lie on a degree  $d$  hypersurface?

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In the 20 minors  $q_{ijk}$ , this equals

$$q_{123} q_{145} q_{246} q_{356} - q_{124} q_{135} q_{236} q_{456} = 0.$$

When do  $N := \binom{r+d}{d}$  points in  $\mathbb{P}^r$  lie on a degree  $d$  hypersurface?

### Example

Bruxelles problem (1825): 10 points on a quadratic surface in  $\mathbb{P}^3$



When do  $N := \binom{r+d}{d}$  points in  $\mathbb{P}^r$  lie on a degree  $d$  hypersurface?

## Example

Bruxelles problem (1825): 10 points on a quadratic surface in  $\mathbb{P}^3$

- ▶ Synthetic construction due to [Traves 2024]
- ▶ Condition in the minors of

$$\begin{bmatrix} a_0 & b_0 & c_0 & d_0 & e_0 & f_0 & h_0 & g_0 & i_0 & j_0 \\ a_1 & b_1 & c_1 & d_1 & e_1 & f_1 & h_1 & g_1 & i_1 & j_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 & h_2 & g_2 & i_2 & j_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 & h_3 & g_3 & i_3 & j_3 \end{bmatrix}$$

given by [Turnbull-Young 1927] ; polynomial with 240 terms

- ▶ Straightened to a 148-term polynomial by [White 1990]

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Today: when do  $k\ell$  points lie on a bilinear hypersurface in  $\mathbb{P}^{k-1} \times \mathbb{P}^{\ell-1}$ ?

Fix vector spaces  $V, W$  of dimensions  $k, \ell$  and write  $n = k\ell$ . The *Segre embedding* is

$$\iota : \mathbb{P}(V) \times \mathbb{P}(W) \hookrightarrow \mathbb{P}(V \otimes W).$$

Let  $A_1 \times B_1, \dots, A_n \times B_n$  denote  $n$  points in  $\mathbb{P}(V) \times \mathbb{P}(W)$ . The *Segre determinant* is the polynomial

$$\text{Seg}_{k,\ell} = \det \begin{bmatrix} \vdots & & \vdots \\ \iota(A_1 \times B_1) & \dots & \iota(A_n \times B_n) \\ \vdots & & \vdots \end{bmatrix}.$$

### Example

Four points  $\begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix} \times \begin{bmatrix} b_{1,1} \\ b_{2,1} \end{bmatrix}, \dots, \begin{bmatrix} a_{1,4} \\ a_{2,4} \end{bmatrix} \times \begin{bmatrix} b_{1,4} \\ b_{2,4} \end{bmatrix} \in \mathbb{P}^1 \times \mathbb{P}^1$ . Then

$$\text{Seg}_{2,2} = \det \begin{bmatrix} a_{1,1}b_{1,1} & a_{1,2}b_{1,2} & a_{1,3}b_{1,3} & a_{1,4}b_{1,4} \\ a_{1,1}b_{2,1} & a_{1,2}b_{2,2} & a_{1,3}b_{2,3} & a_{1,4}b_{2,4} \\ a_{2,1}b_{1,1} & a_{2,2}b_{1,2} & a_{2,3}b_{1,3} & a_{2,4}b_{1,4} \\ a_{2,1}b_{2,1} & a_{2,2}b_{2,2} & a_{2,3}b_{2,3} & a_{2,4}b_{2,4} \end{bmatrix}.$$

## Lemma

The Segre determinant  $\text{Seg}_{k,\ell}$  is a polynomial of bi-degree  $(\ell, k)$  in the maximal minors of

$$A := \begin{bmatrix} A_1 & \dots & A_n \end{bmatrix}, \quad B := \begin{bmatrix} B_1 & \dots & B_n \end{bmatrix}.$$

## Example

In maximal minors of

$$A := \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \end{bmatrix}, \quad B := \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} \end{bmatrix}$$

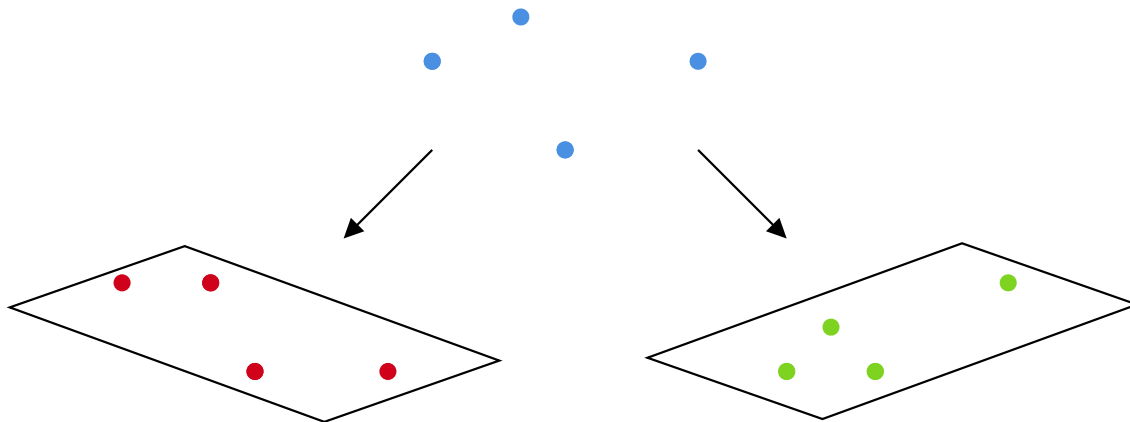
we have

$$\text{Seg}_{2,2} = A_{12}A_{34}B_{13}B_{24} - A_{13}A_{24}B_{12}B_{34}.$$

Vanishes when the *cross-ratios*  $\frac{A_{12}A_{34}}{A_{13}A_{24}}$  and  $\frac{B_{12}B_{34}}{B_{13}B_{24}}$  are equal.

# Computer vision

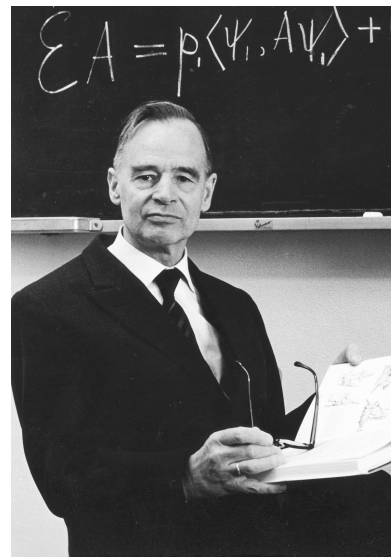
The polynomial  $\text{Seg}_{3,3}$  appears in *algebraic vision* as a necessary condition for two configurations of nine points in  $\mathbb{P}^2$  to be projections of the same configuration in  $\mathbb{P}^3$ .



See [*On The Existence of Epipolar Matrices*, Agarwal–Lee–Sturmfels–Thomas 2016]

$$\begin{aligned}
\text{Seg}_{3,3} = & [123][456][789](3\langle 123 \rangle \langle 457 \rangle \langle 689 \rangle - \langle 123 \rangle \langle 467 \rangle \langle 589 \rangle + 3\langle 124 \rangle \langle 356 \rangle \langle 789 \rangle - 3\langle 124 \rangle \langle 357 \rangle \langle 689 \rangle + \langle 124 \rangle \langle 367 \rangle \langle 589 \rangle - \\
& \langle 124 \rangle \langle 368 \rangle \langle 579 \rangle - \langle 125 \rangle \langle 346 \rangle \langle 789 \rangle + \langle 125 \rangle \langle 347 \rangle \langle 689 \rangle + \langle 127 \rangle \langle 348 \rangle \langle 569 \rangle - \langle 134 \rangle \langle 258 \rangle \langle 679 \rangle - \langle 135 \rangle \langle 247 \rangle \langle 689 \rangle + \\
& \langle 145 \rangle \langle 267 \rangle \langle 389 \rangle + \langle 147 \rangle \langle 258 \rangle \langle 369 \rangle) + [123][457][689](-3\langle 123 \rangle \langle 456 \rangle \langle 789 \rangle + \langle 124 \rangle \langle 368 \rangle \langle 579 \rangle - \langle 126 \rangle \langle 348 \rangle \langle 579 \rangle + \\
& \langle 135 \rangle \langle 246 \rangle \langle 789 \rangle - \langle 146 \rangle \langle 258 \rangle \langle 379 \rangle) + [123][458][679](-\langle 124 \rangle \langle 367 \rangle \langle 589 \rangle - \langle 125 \rangle \langle 346 \rangle \langle 789 \rangle + \langle 126 \rangle \langle 347 \rangle \langle 589 \rangle + \\
& \langle 146 \rangle \langle 257 \rangle \langle 389 \rangle) + [123][467][589](\langle 123 \rangle \langle 456 \rangle \langle 789 \rangle - \langle 124 \rangle \langle 358 \rangle \langle 679 \rangle + \langle 125 \rangle \langle 348 \rangle \langle 679 \rangle + \langle 134 \rangle \langle 256 \rangle \langle 789 \rangle - \\
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& \langle 135 \rangle \langle 267 \rangle \langle 489 \rangle - \langle 137 \rangle \langle 258 \rangle \langle 469 \rangle) + [124][357][689](3\langle 123 \rangle \langle 456 \rangle \langle 789 \rangle - \langle 123 \rangle \langle 468 \rangle \langle 579 \rangle - \langle 135 \rangle \langle 246 \rangle \langle 789 \rangle + \\
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& [147][258][369](\langle 123 \rangle \langle 456 \rangle \langle 789 \rangle).
\end{aligned}$$

# The Chow-Lam form



- ▶ The Chow form (Chow–van der Waerden 1937): Assigns to  $\mathcal{V} \subset \mathbb{P}^{n-1}$  a polynomial  $C_{\mathcal{V}}$
- ▶ Chow-Lam form (P–Sturmfels 2025): Assigns to  $\mathcal{V} \subset \text{Gr}(k, n)$  a polynomial  $\text{CL}_{\mathcal{V}}$

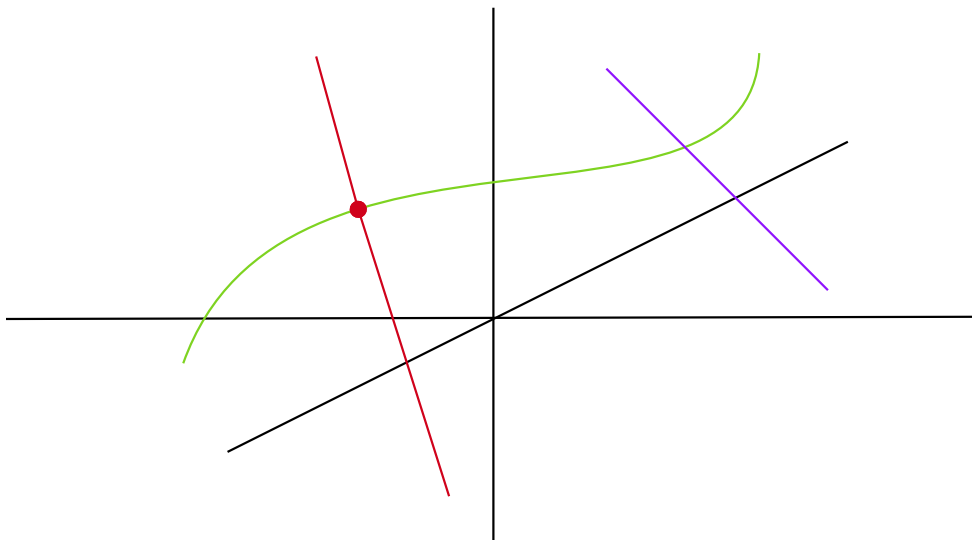
Thomas Lam studied  $\text{CL}_{\mathcal{V}}$  where  $\mathcal{V}$  is a positroid variety.

## Definition (Chow form)

Let  $\mathcal{V} \subset \mathbb{P}^{n-1}$  be a  $d$ -dimensional projective variety. The *Chow locus* of  $\mathcal{V}$  is

$$\mathcal{C}_{\mathcal{V}} = \{L \in \text{Gr}(n - d - 1, n) : \mathcal{V} \cap L \neq \emptyset\}.$$

The *Chow form*  $C_{\mathcal{V}}$  is the defining equation of  $\mathcal{C}_{\mathcal{V}}$ .





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Examples:

- ▶ The Chow form of a hypersurface  $V(F)$  is  $F$ .
- ▶ The Chow locus of a linear space is a Schubert divisor.

A linear space  $L$  can be represented multiple ways.

► **Primal** : as the kernel of an  $(n - k) \times n$  matrix

► **Dual** : as the rowspan of a  $k \times n$  matrix

The primal and dual **Plücker coordinates** are the maximal minors of these matrices.

Example (Coordinates on  $\text{Gr}(3, 5)$ )

$$\begin{array}{cccccccccc} p_{12} & p_{13} & p_{14} & p_{15} & p_{23} & p_{24} & p_{25} & p_{34} & p_{35} & p_{45} \\ q_{345} & -q_{245} & q_{235} & -q_{234} & q_{145} & -q_{135} & q_{134} & q_{125} & -q_{124} & q_{123} \end{array}$$

# Lines meeting the twisted cubic

Consider the closure of

$$t \mapsto [1 : t : t^2 : t^3] \in \mathbb{P}^3.$$

The Chow form is the determinant of the *Bézout matrix* :

$$C_V = \det \begin{bmatrix} p_{12} & p_{13} & p_{14} \\ p_{13} & p_{14} + p_{23} & p_{24} \\ p_{14} & p_{24} & p_{34} \end{bmatrix}.$$

Its expansion is

$$-p_{14}^3 - p_{14}^2 p_{23} + 2p_{13} p_{14} p_{24} - p_{12} p_{24}^2 - p_{13}^2 p_{34} + p_{12} p_{14} p_{34} + p_{12} p_{23} p_{34}.$$

If  $p_{ij}$  are maximal minors of  $\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{bmatrix}$ , then

$$C_V = 0 \quad \Longleftrightarrow \quad \begin{cases} a_3 t^3 + a_2 t^2 + a_1 t + a_0 \\ b_3 t^3 + b_2 t^2 + b_1 t + b_0 \end{cases} \text{ share a root.}$$

## Definition (Chow-Lam form)

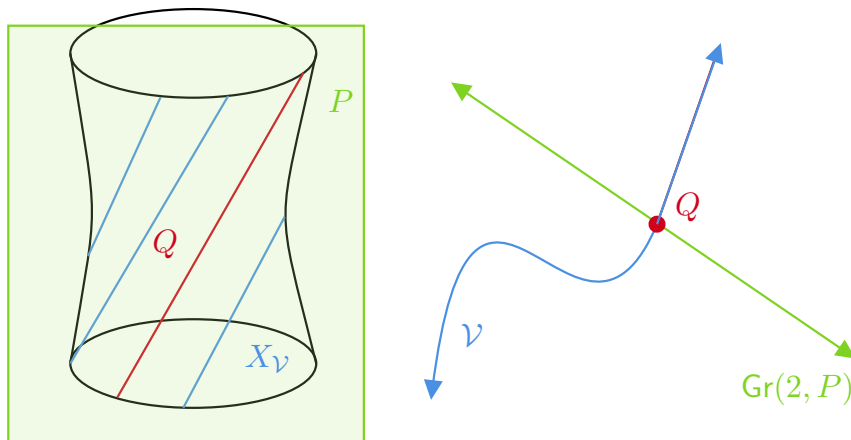
Let  $\mathcal{V} \subset \mathrm{Gr}(k, n)$  be a variety of dimension  $k(r - k) - 1$  for some  $k < r \leq n$ . The *Chow-Lam locus* of  $\mathcal{V}$  is

$$\mathcal{CL}_{\mathcal{V}} = \{P \in \mathrm{Gr}(n + k - r, n) : \mathcal{V} \cap \mathrm{Gr}(k, P) \neq \emptyset\}.$$

When  $\mathcal{CL}_{\mathcal{V}}$  has codimension 1, its defining equation is the *Chow-Lam form*  $\mathrm{CL}_{\mathcal{V}}$ . Otherwise, we set  $\mathrm{CL}_{\mathcal{V}} := 1$ .

# An example

Let  $\mathcal{V}$  be a curve in  $\text{Gr}(2, 4)$ , so  $k = 2, r = 3, n = 4$ . Then  $\mathcal{CL}_{\mathcal{V}}$  is planes  $P$  containing a line  $Q$  in  $\mathbb{P}^3$ , with  $Q$  on  $\mathcal{V}$ .



Let  $X_{\mathcal{V}}$  be the surface in  $\mathbb{P}^3$  swept out by all of the lines in  $\mathcal{V}$ . Then  $\mathcal{CL}_{\mathcal{V}}$  equals the dual variety  $X_{\mathcal{V}}^{\vee}$ .

The torus  $T = (\mathbb{C}^*)^n$  acts on  $\mathrm{Gr}(k, n)$  via

$$t \cdot \begin{bmatrix} \vdots & & \vdots \\ A_1 & \dots & A_n \\ \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & & \vdots \\ t_1 A_1 & \dots & t_n A_n \\ \vdots & & \vdots \end{bmatrix}.$$

We denote the orbit closure of a point  $A$  in  $\mathrm{Gr}(k, n)$  by

$$\mathcal{T}_A := \overline{T \cdot A} \subset \mathrm{Gr}(k, n).$$

If  $A$  is general, then  $\dim \mathcal{T}_A = n - 1$ .

## Theorem (P. 2025)

*Suppose  $n = k\ell$  with  $k, \ell \geq 2$  and that  $A \in \text{Gr}(k, n)$  has nonzero Plücker coordinates. Then*

$$\mathcal{CL}_{\mathcal{T}_A} \subset \text{Gr}(n - \ell, n).$$

*The Chow-Lam form of  $\mathcal{T}_A$  in primal Plücker coordinates  $B_I$  on  $\text{Gr}(n - \ell, n)$  is the Segre determinant  $\text{Seg}_{k,\ell}(A, B)$ .*

UPSHOT: the Segre determinant computes the Chow-Lam form of a general torus orbit closure.

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The proof involves two steps.

1. Show the CL form divides the Segre determinant
2. Show they have the same degree



## An example

Fix a general point

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \end{bmatrix} \in \operatorname{Gr}(2, 6).$$

Then  $\mathcal{T}_A$  is a **toric variety** with polytope

$$\Delta(2, 6) = \operatorname{conv}\{110000, 101000, \dots\} \subset \mathbb{R}^6.$$

Its Chow-Lam form is

$$\begin{aligned} \operatorname{Seg}_{2,3} = & (A_{12}A_{34}A_{56} + A_{14}A_{25}A_{36}) B_{123}B_{456} - A_{13}A_{25}A_{46} B_{124}B_{356} \\ & + A_{12}A_{35}A_{46} B_{134}B_{256} - A_{12}A_{34}A_{56} B_{135}B_{246} + A_{13}A_{24}A_{56} B_{125}B_{346}. \end{aligned}$$

Suppose  $\dim \mathcal{V} = k(r - k) - 1$  and write

$$[\mathcal{V}] = \sum_{\lambda \subset k \times (n-k)} c_{\lambda}(\mathcal{V}) \cdot [\Omega_{\lambda}] \in H^*(\mathrm{Gr}(k, n), \mathbb{Z})$$

where

$$\Omega_{\lambda} = \{L : L \cap E_i \geq n - k + \lambda_i - i\}.$$

The *Chow-Lam degree*  $\alpha(\mathcal{V})$  is the unique coefficient with

$$\lambda = (n - r + 1, n - r, \dots, n - r).$$

Proposition (P-Sturmfels 2025)

*The Chow-Lam form  $CL_{\mathcal{V}}$  is irreducible and has degree  $\alpha(\mathcal{V})$ .*

For example, from  $[\mathcal{T}_A] = 4\Omega_3 + 2\Omega_{2,1}$  we get  $\alpha(\mathcal{T}_A) = 2$ .

The variety  $\mathcal{T}_A$  depends only on the *matroid*  $M$  of  $A$ , i.e.

$$\{I : A_I \neq 0\} \subset \binom{[n]}{k}.$$

The numbers  $c_\lambda(M) := c_\lambda(A)$  are the *Schubert coefficients of  $M$* .

### Proposition (Klyachko 85)

*Let  $\lambda$  be a partition fitting in a  $k \times (n - k)$  rectangle. Then the coefficient  $c_\lambda(U_{k,n})$  is*

$$c_\lambda(U_{k,n}) = \sum_{i=0}^k (-1)^i \binom{n}{i} \dim \mathbb{S}_{\lambda^c}(\mathbb{C}^{k-i}). \quad (1)$$

### Corollary

*The Chow-Lam degree  $\alpha(\mathcal{T}_A)$  is  $k$  for  $A$  generic.*

# Chow varieties

Let  $\mathcal{C}_r(\mathbb{P}(V), d)$  denote the set of dimension  $r$  irreducible subvarieties of  $\mathbb{P}(V)$  with degree  $d$ . Write  $n = \dim V$ . Then

$$\begin{aligned}\varphi: \mathcal{C}_r(\mathbb{P}(V), d) &\hookrightarrow \mathbb{P}\left(\mathrm{Sym}^d\left(\bigwedge^{n-r-1} V\right)\right) \\ \mathcal{V} &\longmapsto C_{\mathcal{V}}.\end{aligned}$$

## Definition

We call  $\overline{\varphi(\mathcal{C}_r(\mathbb{P}(V), d))}$  the *Chow variety of  $r$ -cycles with degree  $d$* .

Consider the map

$$\begin{aligned}\pi : \mathrm{Gr}(k, k\ell)^\circ &\rightarrow \mathbb{P}(H^0(\mathcal{O}_{\mathrm{Gr}(\ell, k\ell)}(k))) \\ A &\mapsto \mathrm{Seg}(A, B).\end{aligned}$$

We call the image the *Segre coefficient variety* .

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We call the image the *Segre coefficient variety*. For example,

$$\begin{aligned}\operatorname{Gr}(2, 6)^\circ &\rightarrow \mathbb{P}^4 \\ A &\mapsto (A_{12}A_{34}A_{56} + A_{14}A_{25}A_{36}) B_{123}B_{456} - A_{13}A_{25}A_{46} B_{124}B_{356} \\ &\quad + A_{12}A_{35}A_{46} B_{134}B_{256} - A_{12}A_{34}A_{56} B_{135}B_{246} + A_{13}A_{24}A_{56} B_{125}B_{346}.\end{aligned}$$

The image is the Segre threefold cut out by

$$x_0x_1x_3 - x_1x_2x_3 - x_0x_2x_4 - x_1x_2x_4 - x_1x_3x_4 - x_2x_3x_4.$$

It is the unique (up to isomorphism) cubic hypersurface in  $\mathbb{P}^4$  with the max number of ordinary double points, namely 10 [Kalker 86].

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It is the unique (up to isomorphism) cubic hypersurface in  $\mathbb{P}^4$  with the max number of ordinary double points, namely 10 [Kalker 86].

**Theorem (P. 2025)**

*When  $k = 2$ , the Segre coefficient variety is isomorphic to the GIT quotient  $(\mathbb{P}^1)^{2\ell} //_w SL_2$ , where  $w := 1^{2\ell}$  is the linearization.*

## Warning for $k = 3$

Let  $k = 3$  and  $n = 6$ . The Segre coefficient map is given by

$$\mathrm{Gr}(3, 6)^\circ \rightarrow \mathbb{P}^4$$

$$\begin{aligned} A \mapsto & (B_{12}B_{34}B_{56} + B_{14}B_{25}B_{36}) A_{123}A_{456} - B_{13}B_{25}B_{46} A_{124}A_{356} \\ & + B_{12}B_{35}B_{46} A_{134}A_{256} - B_{12}B_{34}B_{56} A_{135}A_{246} + B_{13}B_{24}B_{56} A_{125}A_{346}. \end{aligned}$$

In this case the image is  $\mathbb{P}^4$ , and there is a  $2 : 1$  map from the GIT quotient  $(\mathbb{P}^2)^6 //_{1^6} \mathrm{SL}_3$  to the Segre coefficient variety, whose ramification locus consists of co-conic points.

The proof boils down to showing the following polynomial cannot be written as the sum of products of Segre determinants:

$$B_{123}B_{145}B_{246}B_{356} - B_{124}B_{135}B_{236}B_{456}.$$

See also *Point sets in projective spaces and theta functions*,  
Example 3 (Dolgachev–Ortland 88)





Thank you for listening!

# Chow varieties

Let  $\mathcal{C}_r(\mathbb{P}(V), d)$  denote the set of dimension  $r$  cycles in  $\mathbb{P}(V)$  with degree  $d$ . Then

$$\begin{aligned}\mathcal{C}_r(\mathbb{P}(V), d) &\xrightarrow{\varphi} \mathbb{P}(\mathrm{Sym}^d(\wedge^{\dim V - r - 1} V)) \\ \mathcal{V} &\mapsto C_{\mathcal{V}}.\end{aligned}$$

## Definition

We call  $\overline{\varphi(\mathcal{C}_r(\mathbb{P}(V), d))}$  the *Chow variety of  $r$ -cycles with degree  $d$* .

# Non-generic torus orbits

## Theorem (P- 2025)

*Fix a point  $A$  in the Grassmannian  $Gr(k, n)$  such that  $\dim \mathcal{T}_A = k(r - k) - 1 < n$ . Then the Chow-Lam form of  $\mathcal{T}_A$  in primal Plücker coordinates  $B_I$  on  $Gr(n - r + k, n)$  divides the Segre determinant  $\text{Seg}_{k,\ell}(A, B)$*

## Example

Suppose that  $A$  is in  $Gr(2, 4)$  and  $A_{12} = 0$ . Then the  $4 \times 4$  Segre matrix has determinant  $A_{13}A_{34}B_{12}B_{34}$ , but the Chow-Lam form is  $B_{12}$ .

## Warning for $k = 3$

The Segre coefficient map is linearly equivalent to the map

$$\mathrm{Gr}(3, 6)^\circ \rightarrow \mathbb{P}^4$$

$$A \mapsto ([123][456], [124][356], [125][346], [134][256], [135][246]).$$

The following matrices have nonzero Plücker coordinates, which each go to the point  $(-2, 2, 9, 2, 8)$  under the map  $\pi'$ :

$$p = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 & 3 & 5 \end{bmatrix}, \quad q := \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -4 & -2 \\ 0 & 0 & 1 & 1 & -3 & -1 \end{bmatrix}.$$

However, their torus orbit closures are different. We may see this by noting that  $[123][145][246][356]$  evaluates to  $-8$  for the first point and 4 for the second in the affine chart given by  $[123] = 1$ .

# Chow varieties

Setup and notation:

- ▶ A nonsingular projective variety  $X$
- ▶ The set  $\mathcal{C}_r(X, \delta)$  of dimension  $r$  cycles with class  $\delta$  in singular homology

Choose an embedding  $\iota : X \hookrightarrow \mathbb{P}(V)$  and let  $d := \iota_* \delta$  in  $H_{2r}(\mathbb{P}(V), \mathbb{Z}) \cong \mathbb{Z}$ . Then

$$\begin{aligned} \mathcal{C}_r(X, \delta) &\subset \mathcal{C}_r(\mathbb{P}(V), d) \xrightarrow{\varphi} \mathbb{P}(\mathrm{Sym}^d(\wedge^{\dim V - r - 1} V)) \\ \mathcal{V} &\mapsto C_{\mathcal{V}}. \end{aligned}$$

## Definition

We call  $\overline{\varphi(\mathcal{C}_r(X, \delta))}$  the *Chow variety of  $r$ -cycles with class  $\delta$* .

Independent of embedding  $\iota$  by [Barlet 1975] .

# Chow quotients

Setup:

- ▶ A nonsingular projective variety  $X$
- ▶ An algebraic group  $H$  acting on  $X$
- ▶ A  $H$ -stable subset  $U \subset X$  where the dimension and cohomology class of  $\overline{H \cdot x}$  are constant

Idea: create parameter space for  $H$ -orbits which is a projective variety.

## Definition

The *Chow quotient*  $X//H$  is the closure of the image of

$$\begin{aligned} U &\rightarrow \mathcal{C}_r(X, \delta) \\ x &\mapsto \overline{H \cdot x}. \end{aligned}$$

## Proof idea

Fix  $A \in \text{Gr}(2, 6)$ . Parameterize the CL locus of  $\mathcal{T}_A$  as  $3 \times 6$  matrices  $B$  such that, for some  $t \in (\mathbb{C}^*)^6$ , we have

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} \end{bmatrix} \cdot \text{diag}(t_1, \dots, t_6) \cdot \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \\ a_{14} & a_{24} \\ a_{15} & a_{25} \\ a_{16} & a_{26} \end{bmatrix} = 0.$$

Re-arranging, we obtain the expression

$$t_1 \begin{bmatrix} a_{11}b_{11} \\ a_{11}b_{21} \\ a_{11}b_{31} \\ a_{21}b_{11} \\ a_{21}b_{21} \\ a_{21}b_{31} \end{bmatrix} + t_2 \begin{bmatrix} a_{12}b_{12} \\ a_{12}b_{22} \\ a_{12}b_{32} \\ a_{22}b_{12} \\ a_{22}b_{22} \\ a_{22}b_{32} \end{bmatrix} + \dots + t_6 \begin{bmatrix} a_{16}b_{16} \\ a_{16}b_{26} \\ a_{16}b_{36} \\ a_{26}b_{16} \\ a_{26}b_{26} \\ a_{26}b_{36} \end{bmatrix} = \sum_{i=1}^6 t_i A_i \otimes B_i = 0.$$

When do  $N := \binom{r+d}{d}$  points in  $\mathbb{P}^r$  lie on a degree  $d$  hypersurface?



When do  $N := \binom{r+d}{d}$  points in  $\mathbb{P}^r$  lie on a degree  $d$  hypersurface?

## Example

Bruxelles problem (1825): 10 points on a quadratic surface in  $\mathbb{P}^3$

- ▶ Synthetic construction due to [Traves 2024]
- ▶ Condition in the minors of

$$\begin{bmatrix} a_0 & b_0 & c_0 & d_0 & e_0 & f_0 & h_0 & g_0 & i_0 & j_0 \\ a_1 & b_1 & c_1 & d_1 & e_1 & f_1 & h_1 & g_1 & i_1 & j_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 & h_2 & g_2 & i_2 & j_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 & h_3 & g_3 & i_3 & j_3 \end{bmatrix}$$

given by [Turnbull-Young 1927] ; polynomial with 240 terms

- ▶ Straightened to a 148-term polynomial by [White 1990]