An aerial photograph of the University of California, Berkeley campus. In the foreground, the light-colored Sather Tower (the Campanile) rises prominently. Below it, the green lawns and red-roofed buildings of the campus are visible. In the background, the city of Berkeley extends towards the hills, with a bridge visible across the water under a clear blue sky.

# The Chow-Lam Form

Lizzie Pratt

Slides: <https://lizziepratt.com/notes>  
The Chow-Lam Form (w/ Bernd Sturmfels)  
The Segre Determinant arXiv 2505.09204

January 5, 2025

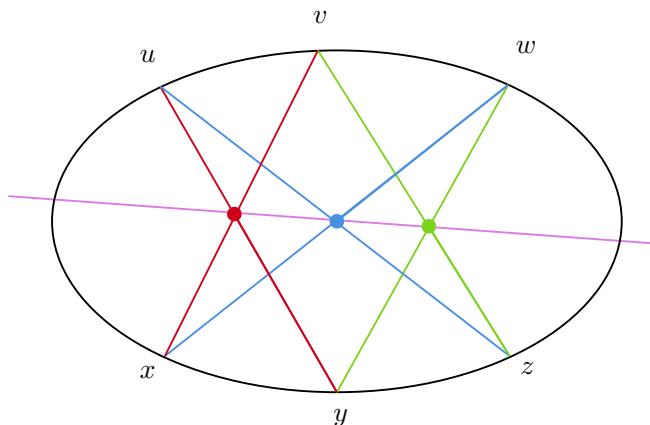
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Example (Pascal 1640)



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### Example

Given  $\begin{bmatrix} u_0 & v_0 & w_0 & x_0 & y_0 & z_0 \\ u_1 & v_1 & w_1 & x_1 & y_1 & z_1 \\ u_2 & v_2 & w_2 & x_2 & y_2 & z_2 \end{bmatrix}$ , when

$$\det \begin{bmatrix} u_0^2 & v_0^2 & w_0^2 & x_0^2 & y_0^2 & z_0^2 \\ u_0u_1 & v_0v_1 & w_0w_1 & x_0x_1 & y_0y_1 & z_0z_1 \\ u_0u_2 & v_0v_2 & w_0w_2 & x_0x_2 & y_0y_2 & z_0z_2 \\ u_1^2 & v_1^2 & w_1^2 & x_1^2 & y_1^2 & z_1^2 \\ u_1u_2 & v_1v_2 & w_1w_2 & x_1x_2 & y_1y_2 & z_1z_2 \\ u_2^2 & v_2^2 & w_2^2 & x_2^2 & y_2^2 & z_2^2 \end{bmatrix} = 0.$$

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### Theorem (First fundamental theorem of invariant theory)

Let  $x_{ij}$  be entries of a  $d \times n$  matrix  $X$ , and let  $SL_d$  act by multiplication on  $X$ . The invariant ring  $\mathbb{C}[x_{ij}]^{SL_d}$  is generated by the  $d \times d$  minors of  $X$ .

When do  $N := \binom{r+d}{d}$  points in  $\mathbb{P}^r$  lie on a degree  $d$  hypersurface?

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In the 20 minors  $q_{ijk}$ , this equals

$$q_{123} q_{145} q_{246} q_{356} - q_{124} q_{135} q_{236} q_{456} = 0.$$

When do  $N := \binom{r+d}{d}$  points in  $\mathbb{P}^r$  lie on a degree  $d$  hypersurface?

### Example

Bruxelles problem (1825): 10 points on a quadratic surface in  $\mathbb{P}^3$

When do  $N := \binom{r+d}{d}$  points in  $\mathbb{P}^r$  lie on a degree  $d$  hypersurface?

## Example

Bruxelles problem (1825): 10 points on a quadratic surface in  $\mathbb{P}^3$

- ▶ Synthetic construction due to [Traves 2024]
- ▶ Condition in the minors of

$$\begin{bmatrix} a_0 & b_0 & c_0 & d_0 & e_0 & f_0 & h_0 & g_0 & i_0 & j_0 \\ a_1 & b_1 & c_1 & d_1 & e_1 & f_1 & h_1 & g_1 & i_1 & j_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 & h_2 & g_2 & i_2 & j_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 & h_3 & g_3 & i_3 & j_3 \end{bmatrix}$$

given by [Turnbull-Young 1927] ; polynomial with 240 terms

- ▶ Straightened to a 148-term polynomial by [White 1990]

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Today: when do  $k\ell$  points lie on a bilinear hypersurface in  $\mathbb{P}^{k-1} \times \mathbb{P}^{\ell-1}$ ?

Fix vector spaces  $V, W$  of dimensions  $k, \ell$  and write  $n = k\ell$ . The *Segre embedding* is

$$\iota : \mathbb{P}(V) \times \mathbb{P}(W) \hookrightarrow \mathbb{P}(V \otimes W).$$

Let  $A_1 \times B_1, \dots, A_n \times B_n$  denote  $n$  points in  $\mathbb{P}(V) \times \mathbb{P}(W)$ . The *Segre determinant* is the polynomial

$$\text{Seg}_{k,\ell} = \det \begin{bmatrix} & & & \\ \vdots & & & \vdots \\ \iota(A_1 \times B_1) & \dots & \iota(A_n \times B_n) & \\ & & & \vdots \\ \vdots & & & \vdots \end{bmatrix}.$$

### Example

Four points  $\begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix} \times \begin{bmatrix} b_{1,1} \\ b_{2,1} \end{bmatrix}, \dots, \begin{bmatrix} a_{1,4} \\ a_{2,4} \end{bmatrix} \times \begin{bmatrix} b_{1,4} \\ b_{2,4} \end{bmatrix} \in \mathbb{P}^1 \times \mathbb{P}^1$ . Then

$$\text{Seg}_{2,2} = \det \begin{bmatrix} a_{1,1}b_{1,1} & a_{1,2}b_{1,2} & a_{1,3}b_{1,3} & a_{1,4}b_{1,4} \\ a_{1,1}b_{2,1} & a_{1,2}b_{2,2} & a_{1,3}b_{2,3} & a_{1,4}b_{2,4} \\ a_{2,1}b_{1,1} & a_{2,2}b_{1,2} & a_{2,3}b_{1,3} & a_{2,4}b_{1,4} \\ a_{2,1}b_{2,1} & a_{2,2}b_{2,2} & a_{2,3}b_{2,3} & a_{2,4}b_{2,4} \end{bmatrix}.$$

## Lemma

The Segre determinant  $\text{Seg}_{k,\ell}$  is a polynomial of bi-degree  $(\ell, k)$  in the maximal minors of

$$A := [A_1 \ \dots \ A_n], \quad B := [B_1 \ \dots \ B_n].$$

## Example

In maximal minors of

$$A := \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \end{bmatrix}, \quad B := \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} \end{bmatrix}$$

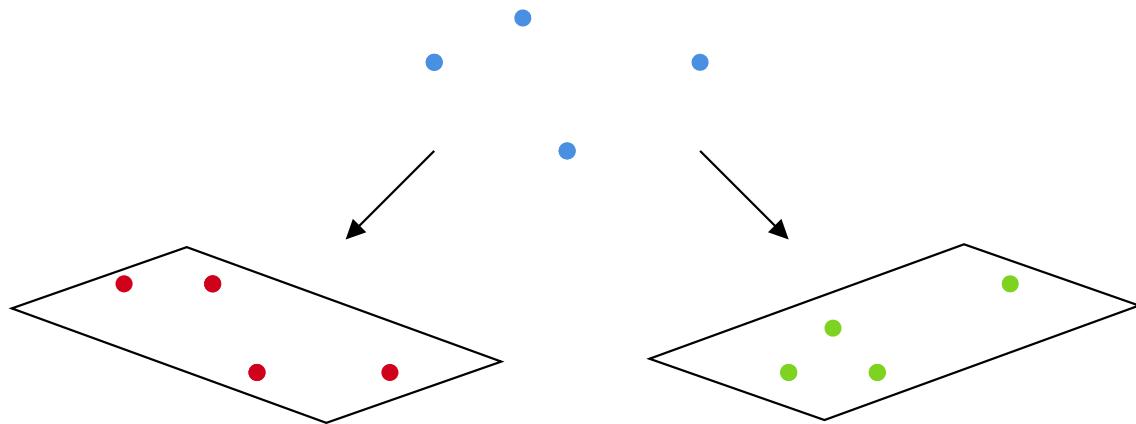
we have

$$\text{Seg}_{2,2} = A_{12}A_{34}B_{13}B_{24} - A_{13}A_{24}B_{12}B_{34}.$$

Vanishes when the *cross-ratios*  $\frac{A_{12}A_{34}}{A_{13}A_{24}}$  and  $\frac{B_{12}B_{34}}{B_{13}B_{24}}$  are equal.

# Computer vision

The polynomial  $\text{Seg}_{3,3}$  appears in *algebraic vision* as a necessary condition for two configurations of nine points in  $\mathbb{P}^2$  to be projections of the same configuration in  $\mathbb{P}^3$ .



See [*On The Existence of Epipolar Matrices*, Agarwal–Lee–Sturmfels–Thomas 2016]

$$\begin{aligned}
\text{Seg}_{3,3} = & [123][456][789](3\langle 123 \rangle \langle 457 \rangle \langle 689 \rangle - \langle 123 \rangle \langle 467 \rangle \langle 589 \rangle + 3\langle 124 \rangle \langle 356 \rangle \langle 789 \rangle - 3\langle 124 \rangle \langle 357 \rangle \langle 689 \rangle + \langle 124 \rangle \langle 367 \rangle \langle 589 \rangle - \\
& \langle 124 \rangle \langle 368 \rangle \langle 579 \rangle - \langle 125 \rangle \langle 346 \rangle \langle 789 \rangle + \langle 125 \rangle \langle 347 \rangle \langle 689 \rangle + \langle 127 \rangle \langle 348 \rangle \langle 569 \rangle - \langle 134 \rangle \langle 258 \rangle \langle 679 \rangle - \langle 135 \rangle \langle 247 \rangle \langle 689 \rangle + \\
& \langle 145 \rangle \langle 267 \rangle \langle 389 \rangle + \langle 147 \rangle \langle 258 \rangle \langle 369 \rangle) + [123][457][689](-3\langle 123 \rangle \langle 456 \rangle \langle 789 \rangle + \langle 124 \rangle \langle 368 \rangle \langle 579 \rangle - \langle 126 \rangle \langle 348 \rangle \langle 579 \rangle + \\
& \langle 135 \rangle \langle 246 \rangle \langle 789 \rangle - \langle 146 \rangle \langle 258 \rangle \langle 379 \rangle) + [123][458][679](-\langle 124 \rangle \langle 367 \rangle \langle 589 \rangle - \langle 125 \rangle \langle 346 \rangle \langle 789 \rangle + \langle 126 \rangle \langle 347 \rangle \langle 589 \rangle + \\
& \langle 146 \rangle \langle 257 \rangle \langle 389 \rangle) + [123][467][589](\langle 123 \rangle \langle 456 \rangle \langle 789 \rangle - \langle 124 \rangle \langle 358 \rangle \langle 679 \rangle + \langle 125 \rangle \langle 348 \rangle \langle 679 \rangle + \langle 134 \rangle \langle 256 \rangle \langle 789 \rangle - \\
& \langle 135 \rangle \langle 246 \rangle \langle 789 \rangle + \langle 145 \rangle \langle 268 \rangle \langle 379 \rangle) + [124][356][789](-3\langle 123 \rangle \langle 456 \rangle \langle 789 \rangle + \langle 123 \rangle \langle 468 \rangle \langle 579 \rangle + \langle 135 \rangle \langle 247 \rangle \langle 689 \rangle - \\
& \langle 135 \rangle \langle 267 \rangle \langle 489 \rangle - \langle 137 \rangle \langle 258 \rangle \langle 469 \rangle) + [124][357][689](3\langle 123 \rangle \langle 456 \rangle \langle 789 \rangle - \langle 123 \rangle \langle 468 \rangle \langle 579 \rangle - \langle 135 \rangle \langle 246 \rangle \langle 789 \rangle + \\
& \langle 136 \rangle \langle 258 \rangle \langle 479 \rangle) + [124][358][679](\langle 123 \rangle \langle 467 \rangle \langle 589 \rangle - \langle 136 \rangle \langle 257 \rangle \langle 489 \rangle) + [124][367][589](-\langle 123 \rangle \langle 456 \rangle \langle 789 \rangle + \\
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& \langle 135 \rangle \langle 267 \rangle \langle 489 \rangle) + [125][346][789](\langle 123 \rangle \langle 456 \rangle \langle 789 \rangle + \langle 123 \rangle \langle 458 \rangle \langle 679 \rangle - \langle 123 \rangle \langle 468 \rangle \langle 579 \rangle - \langle 134 \rangle \langle 257 \rangle \langle 689 \rangle + \\
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\end{aligned}$$

# The Chow-Lam form



- ▶ The Chow form (Chow–van der Waerden 1937): Assigns to  $\mathcal{V} \subset \mathbb{P}^{n-1}$  a polynomial  $C_{\mathcal{V}}$
- ▶ Chow-Lam form (P-Sturmfels 2025): Assigns to  $\mathcal{V} \subset \text{Gr}(k, n)$  a polynomial  $\text{CL}_{\mathcal{V}}$

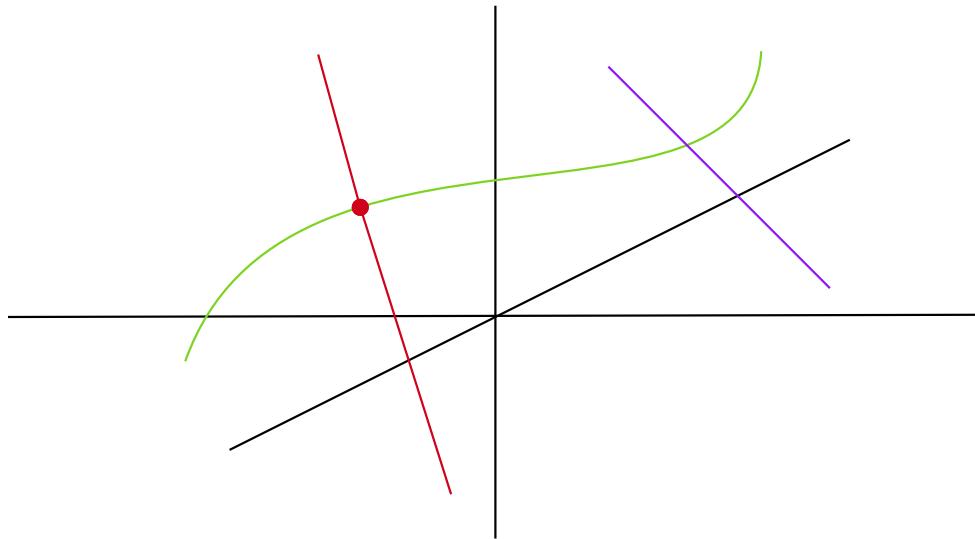
Thomas Lam studied  $\text{CL}_{\mathcal{V}}$  where  $\mathcal{V}$  is a positroid variety.

## Definition (Chow form)

Let  $\mathcal{V} \subset \mathbb{P}^{n-1}$  be a  $d$ -dimensional projective variety. The *Chow locus* of  $\mathcal{V}$  is

$$\mathcal{C}_{\mathcal{V}} = \{L \in \mathrm{Gr}(n-d-1, n) : \mathcal{V} \cap L \neq \emptyset\}.$$

The *Chow form*  $C_{\mathcal{V}}$  is the defining equation of  $\mathcal{C}_{\mathcal{V}}$ .



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Examples:

- ▶ The Chow form of a hypersurface  $V(F)$  is  $F$ .
- ▶ The Chow locus of a linear space is a Schubert divisor.

A linear space  $L$  can be represented multiple ways.

- ▶ **Primal** : as the kernel of an  $(n - k) \times n$  matrix
- ▶ **Dual** : as the rowspan of a  $k \times n$  matrix

The primal and dual **Plücker coordinates** are the maximal minors of these matrices.

### Example (Coordinates on $\text{Gr}(3, 5)$ )

$$\begin{array}{cccccccccc} p_{12} & p_{13} & p_{14} & p_{15} & p_{23} & p_{24} & p_{25} & p_{34} & p_{35} & p_{45} \\ q_{345} & -q_{245} & q_{235} & -q_{234} & q_{145} & -q_{135} & q_{134} & q_{125} & -q_{124} & q_{123} \end{array}$$

# Lines meeting the twisted cubic

Consider the closure of

$$t \mapsto [1 : t : t^2 : t^3] \in \mathbb{P}^3.$$

The Chow form is the determinant of the *Bézout matrix* :

$$C_V = \det \begin{bmatrix} p_{12} & p_{13} & p_{14} \\ p_{13} & p_{14} + p_{23} & p_{24} \\ p_{14} & p_{24} & p_{34} \end{bmatrix}.$$

Its expansion is

$$-p_{14}^3 - p_{14}^2 p_{23} + 2p_{13}p_{14}p_{24} - p_{12}p_{24}^2 - p_{13}^2 p_{34} + p_{12}p_{14}p_{34} + p_{12}p_{23}p_{34}.$$

If  $p_{ij}$  are maximal minors of  $\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{bmatrix}$ , then

$$C_V = 0 \iff \begin{cases} a_3t^3 + a_2t^2 + a_1t + a_0 \\ b_3t^3 + b_2t^2 + b_1t + b_0 \end{cases} \text{ share a root.}$$

## Definition (Chow-Lam form)

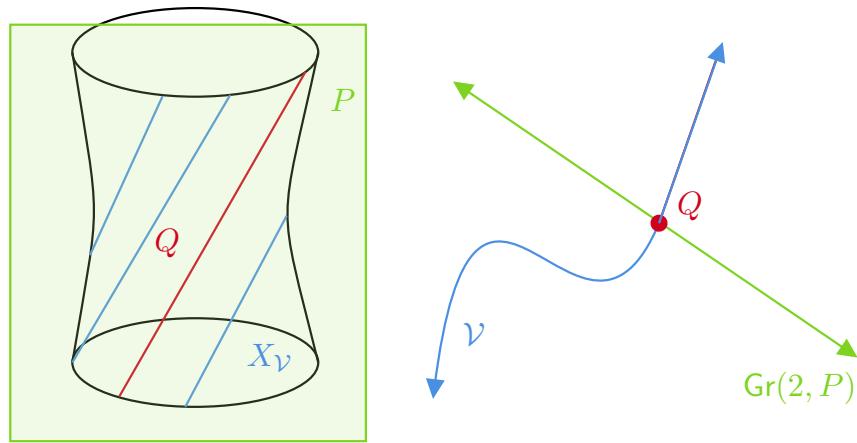
Let  $\mathcal{V} \subset \mathrm{Gr}(k, n)$  be a variety of dimension  $k(r - k) - 1$  for some  $k < r \leq n$ . The *Chow-Lam locus* of  $\mathcal{V}$  is

$$\mathcal{CL}_{\mathcal{V}} = \{P \in \mathrm{Gr}(n + k - r, n) : \mathcal{V} \cap \mathrm{Gr}(k, P) \neq \emptyset\}.$$

When  $\mathcal{CL}_{\mathcal{V}}$  has codimension 1, its defining equation is the *Chow-Lam form*  $\mathrm{CL}_{\mathcal{V}}$ . Otherwise, we set  $\mathrm{CL}_{\mathcal{V}} := 1$ .

## An example

Let  $\mathcal{V}$  be a curve in  $\text{Gr}(2, 4)$ , so  $k = 2, r = 3, n = 4$ . Then  $\mathcal{CL}_{\mathcal{V}}$  is planes  $P$  containing a line  $Q$  in  $\mathbb{P}^3$ , with  $Q$  on  $\mathcal{V}$ .



Let  $X_{\mathcal{V}}$  be the surface in  $\mathbb{P}^3$  swept out by all of the lines in  $\mathcal{V}$ . Then  $\mathcal{CL}_{\mathcal{V}}$  equals the dual variety  $X_{\mathcal{V}}^{\vee}$ .

The torus  $T = (\mathbb{C}^*)^n$  acts on  $\text{Gr}(k, n)$  via

$$t \cdot \begin{bmatrix} \vdots & & \vdots \\ A_1 & \dots & A_n \\ \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & & \vdots \\ t_1 A_1 & \dots & t_n A_n \\ \vdots & & \vdots \end{bmatrix}.$$

We denote the orbit closure of a point  $A$  in  $\text{Gr}(k, n)$  by

$$\mathcal{T}_A := \overline{T \cdot A} \subset \text{Gr}(k, n).$$

If  $A$  is general, then  $\dim \mathcal{T}_A = n - 1$ .

## Theorem (P. 2025)

Suppose  $n = k\ell$  with  $k, \ell \geq 2$  and that  $A \in \text{Gr}(k, n)$  has nonzero Plücker coordinates. Then

$$\mathcal{CL}_{T_A} \subset \text{Gr}(n - \ell, n).$$

The Chow-Lam form of  $T_A$  in primal Plücker coordinates  $B_I$  on  $\text{Gr}(n - \ell, n)$  is the Segre determinant  $\text{Seg}_{k,\ell}(A, B)$ .

UPSHOT: the Segre determinant computes the Chow-Lam form of a general torus orbit closure.

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The proof involves two steps.

1. Show the CL form divides the Segre determinant
2. Show they have the same degree

## An example

Fix a general point

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \end{bmatrix} \in \mathrm{Gr}(2, 6).$$

Then  $\mathcal{T}_A$  is a **toric variety** with polytope

$$\Delta(2, 6) = \mathrm{conv}\{110000, 101000, \dots\} \subset \mathbb{R}^6.$$

Its Chow-Lam form is

$$\begin{aligned} \mathrm{Seg}_{2,3} = & (A_{12}A_{34}A_{56} + A_{14}A_{25}A_{36})B_{123}B_{456} - A_{13}A_{25}A_{46}B_{124}B_{356} \\ & + A_{12}A_{35}A_{46}B_{134}B_{256} - A_{12}A_{34}A_{56}B_{135}B_{246} + A_{13}A_{24}A_{56}B_{125}B_{346}. \end{aligned}$$

Suppose  $\dim \mathcal{V} = k(r - k) - 1$  and write

$$[\mathcal{V}] = \sum_{\lambda \subset k \times (n-k)} c_\lambda(\mathcal{V}) \cdot [\Omega_\lambda] \in H^*(\mathrm{Gr}(k, n), \mathbb{Z})$$

where

$$\Omega_\lambda = \{L : L \cap E_i \geq n - k + \lambda_i - i\}.$$

The *Chow-Lam degree*  $\alpha(\mathcal{V})$  is the unique coefficient with

$$\lambda = (n - r + 1, n - r, \dots, n - r).$$

### Proposition (P-Sturmfels 2025)

*The Chow-Lam form  $CL_{\mathcal{V}}$  is irreducible and has degree  $\alpha(\mathcal{V})$ .*

For example, from  $[\mathcal{T}_A] = 4\Omega_3 + 2\Omega_{2,1}$  we get  $\alpha(\mathcal{T}_A) = 2$ .

The variety  $\mathcal{T}_A$  depends only on the *matroid*  $M$  of  $A$ , i.e.

$$\{I : A_I \neq 0\} \subset \binom{[n]}{k}.$$

The numbers  $c_\lambda(M) := c_\lambda(A)$  are the *Schubert coefficients of  $M$* .

### Proposition (Klyachko 85)

Let  $\lambda$  be a partition fitting in a  $k \times (n - k)$  rectangle. Then the coefficient  $c_\lambda(U_{k,n})$  is

$$c_\lambda(U_{k,n}) = \sum_{i=0}^k (-1)^i \binom{n}{i} \dim \mathbb{S}_{\lambda^c}(\mathbb{C}^{k-i}). \quad (1)$$

### Corollary

The Chow-Lam degree  $\alpha(\mathcal{T}_A)$  is  $k$  for  $A$  generic.

# Chow varieties

Let  $\mathcal{C}_r(\mathbb{P}(V), d)$  denote the set of dimension  $r$  irreducible subvarieties of  $\mathbb{P}(V)$  with degree  $d$ . Write  $n = \dim V$ . Then

$$\begin{aligned}\varphi: \mathcal{C}_r(\mathbb{P}(V), d) &\hookrightarrow \mathbb{P} \left( \text{Sym}^d \left( \bigwedge^{n-r-1} V \right) \right) \\ \mathcal{V} &\longmapsto C_{\mathcal{V}}.\end{aligned}$$

## Definition

We call  $\overline{\varphi(\mathcal{C}_r(\mathbb{P}(V), d))}$  the *Chow variety of  $r$ -cycles with degree  $d$* .

Consider the map

$$\begin{aligned}\pi : \mathrm{Gr}(k, k\ell)^\circ &\rightarrow \mathbb{P}(H^0(\mathcal{O}_{\mathrm{Gr}(\ell, k\ell)}(k))) \\ A &\mapsto \mathrm{Seg}(A, B).\end{aligned}$$

We call the image the *Segre coefficient variety*.

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We call the image the *Segre coefficient variety*. For example,

$$\mathrm{Gr}(2, 6)^\circ \rightarrow \mathbb{P}^4$$

$$\begin{aligned}A &\mapsto (A_{12}A_{34}A_{56} + A_{14}A_{25}A_{36})B_{123}B_{456} - A_{13}A_{25}A_{46}B_{124}B_{356} \\ &\quad + A_{12}A_{35}A_{46}B_{134}B_{256} - A_{12}A_{34}A_{56}B_{135}B_{246} + A_{13}A_{24}A_{56}B_{125}B_{346}.\end{aligned}$$

The image is the Segre threefold cut out by

$$x_0x_1x_3 - x_1x_2x_3 - x_0x_2x_4 - x_1x_2x_4 - x_1x_3x_4 - x_2x_3x_4.$$

It is the unique (up to isomorphism) cubic hypersurface in  $\mathbb{P}^4$  with the max number of ordinary double points, namely 10 [Kalker 86].

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### Theorem (P. 2025)

When  $k = 2$ , the Segre coefficient variety is isomorphic to the GIT quotient  $(\mathbb{P}^1)^{2\ell} //_w SL_2$ , where  $w := 1^{2\ell}$  is the linearization.

## Warning for $k = 3$

Let  $k = 3$  and  $n = 6$ . The Segre coefficient map is given by

$$\mathrm{Gr}(3, 6)^\circ \rightarrow \mathbb{P}^4$$

$$\begin{aligned} A \mapsto & (B_{12}B_{34}B_{56} + B_{14}B_{25}B_{36})A_{123}A_{456} - B_{13}B_{25}B_{46}A_{124}A_{356} \\ & + B_{12}B_{35}B_{46}A_{134}A_{256} - B_{12}B_{34}B_{56}A_{135}A_{246} + B_{13}B_{24}B_{56}A_{125}A_{346}. \end{aligned}$$

In this case the image is  $\mathbb{P}^4$ , and there is a  $2 : 1$  map from the GIT quotient  $(\mathbb{P}^2)^6 // {}_{1^6}\mathrm{SL}_3$  to the Segre coefficient variety, whose ramification locus consists of co-conic points.

The proof boils down to showing the following polynomial cannot be written as the sum of products of Segre determinants:

$$B_{123}B_{145}B_{246}B_{356} - B_{124}B_{135}B_{236}B_{456}.$$

See also *Point sets in projective spaces and theta functions*,  
Example 3 (Dolgachev–Ortland 88)

An aerial photograph of the University of California Berkeley campus. In the center foreground stands the iconic Sather Tower, also known as the Campanile. The surrounding campus is filled with various buildings, including the Doe Library and the Sproul Hall, all nestled among numerous green trees. Beyond the campus, the city of Berkeley and the San Francisco Bay area are visible under a clear blue sky.

Thank you for listening!

# Chow varieties

Let  $\mathcal{C}_r(\mathbb{P}(V), d)$  denote the set of dimension  $r$  cycles in  $\mathbb{P}(V)$  with degree  $d$ . Then

$$\begin{aligned}\mathcal{C}_r(\mathbb{P}(V), d) &\xrightarrow{\varphi} \mathbb{P}(\mathrm{Sym}^d(\wedge^{\dim V - r - 1} V)) \\ \mathcal{V} &\mapsto C_{\mathcal{V}}.\end{aligned}$$

## Definition

We call  $\overline{\varphi(\mathcal{C}_r(\mathbb{P}(V), d))}$  the *Chow variety of  $r$ -cycles with degree  $d$* .

# Non-generic torus orbits

## Theorem (P- 2025)

*Fix a point  $A$  in the Grassmannian  $\text{Gr}(k, n)$  such that  $\dim \mathcal{T}_A = k(r - k) - 1 < n$ . Then the Chow-Lam form of  $\mathcal{T}_A$  in primal Plücker coordinates  $B_I$  on  $\text{Gr}(n - r + k, n)$  divides the Segre determinant  $\text{Seg}_{k,\ell}(A, B)$*

## Example

Suppose that  $A$  is in  $\text{Gr}(2, 4)$  and  $A_{12} = 0$ . Then the  $4 \times 4$  Segre matrix has determinant  $A_{13}A_{34}B_{12}B_{34}$ , but the Chow-Lam form is  $B_{12}$ .

## Warning for $k = 3$

The Segre coefficient map is linearly equivalent to the map

$$\mathrm{Gr}(3,6)^\circ \rightarrow \mathbb{P}^4$$

$$A \mapsto ([123][456], [124][356], [125][346], [134][256], [135][246]).$$

The following matrices have nonzero Plücker coordinates, which each go to the point  $(-2, 2, 9, 2, 8)$  under the map  $\pi'$ :

$$p = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 & 3 & 5 \end{bmatrix}, \quad q := \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -4 & -2 \\ 0 & 0 & 1 & 1 & -3 & -1 \end{bmatrix}.$$

However, their torus orbit closures are different. We may see this by noting that  $[123][145][246][356]$  evaluates to  $-8$  for the first point and  $4$  for the second in the affine chart given by  $[123] = 1$ .

# Chow varieties

Setup and notation:

- ▶ A nonsingular projective variety  $X$
- ▶ The set  $\mathcal{C}_r(X, \delta)$  of dimension  $r$  cycles with class  $\delta$  in singular homology

Choose an embedding  $\iota : X \hookrightarrow \mathbb{P}(V)$  and let  $d := \iota_*\delta$  in  $H_{2r}(\mathbb{P}(V), \mathbb{Z}) \cong \mathbb{Z}$ . Then

$$\begin{aligned}\mathcal{C}_r(X, \delta) &\subset \mathcal{C}_r(\mathbb{P}(V), d) \xrightarrow{\varphi} \mathbb{P}(\text{Sym}^d(\wedge^{\dim V - r - 1} V)) \\ \mathcal{V} &\mapsto C_{\mathcal{V}}.\end{aligned}$$

## Definition

We call  $\overline{\varphi(\mathcal{C}_r(X, \delta))}$  the *Chow variety of  $r$ -cycles with class  $\delta$* .

Independent of embedding  $\iota$  by [Barlet 1975].

# Chow quotients

Setup:

- ▶ A nonsingular projective variety  $X$
- ▶ An algebraic group  $H$  acting on  $X$
- ▶ A  $H$ -stable subset  $U \subset X$  where the dimension and cohomology class of  $\overline{H \cdot x}$  are constant

Idea: create parameter space for  $H$ -orbits which is a projective variety.

## Definition

The *Chow quotient*  $X//H$  is the closure of the image of

$$\begin{aligned} U &\rightarrow \mathcal{C}_r(X, \delta) \\ x &\mapsto \overline{H \cdot x}. \end{aligned}$$

## Proof idea

Fix  $A \in \mathrm{Gr}(2, 6)$ . Parameterize the CL locus of  $\mathcal{T}_A$  as  $3 \times 6$  matrices  $B$  such that, for some  $t \in (\mathbb{C}^*)^6$ , we have

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} \end{bmatrix} \cdot \mathrm{diag}(t_1, \dots, t_6) \cdot \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \\ a_{14} & a_{24} \\ a_{15} & a_{25} \\ a_{16} & a_{26} \end{bmatrix} = 0.$$

Re-arranging, we obtain the expression

$$t_1 \begin{bmatrix} a_{11}b_{11} \\ a_{11}b_{21} \\ a_{11}b_{31} \\ a_{21}b_{11} \\ a_{21}b_{21} \\ a_{21}b_{31} \end{bmatrix} + t_2 \begin{bmatrix} a_{12}b_{12} \\ a_{12}b_{22} \\ a_{12}b_{32} \\ a_{22}b_{12} \\ a_{22}b_{22} \\ a_{22}b_{32} \end{bmatrix} + \dots + t_6 \begin{bmatrix} a_{16}b_{16} \\ a_{16}b_{26} \\ a_{16}b_{36} \\ a_{26}b_{16} \\ a_{26}b_{26} \\ a_{26}b_{36} \end{bmatrix} = \sum_{i=1}^6 t_i A_i \otimes B_i = 0.$$

When do  $N := \binom{r+d}{d}$  points in  $\mathbb{P}^r$  lie on a degree  $d$  hypersurface?

When do  $N := \binom{r+d}{d}$  points in  $\mathbb{P}^r$  lie on a degree  $d$  hypersurface?

## Example

Bruxelles problem (1825): 10 points on a quadratic surface in  $\mathbb{P}^3$

- ▶ Synthetic construction due to [Traves 2024]
- ▶ Condition in the minors of

$$\begin{bmatrix} a_0 & b_0 & c_0 & d_0 & e_0 & f_0 & h_0 & g_0 & i_0 & j_0 \\ a_1 & b_1 & c_1 & d_1 & e_1 & f_1 & h_1 & g_1 & i_1 & j_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 & h_2 & g_2 & i_2 & j_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 & h_3 & g_3 & i_3 & j_3 \end{bmatrix}$$

given by [Turnbull-Young 1927] ; polynomial with 240 terms

- ▶ Straightened to a 148-term polynomial by [White 1990]