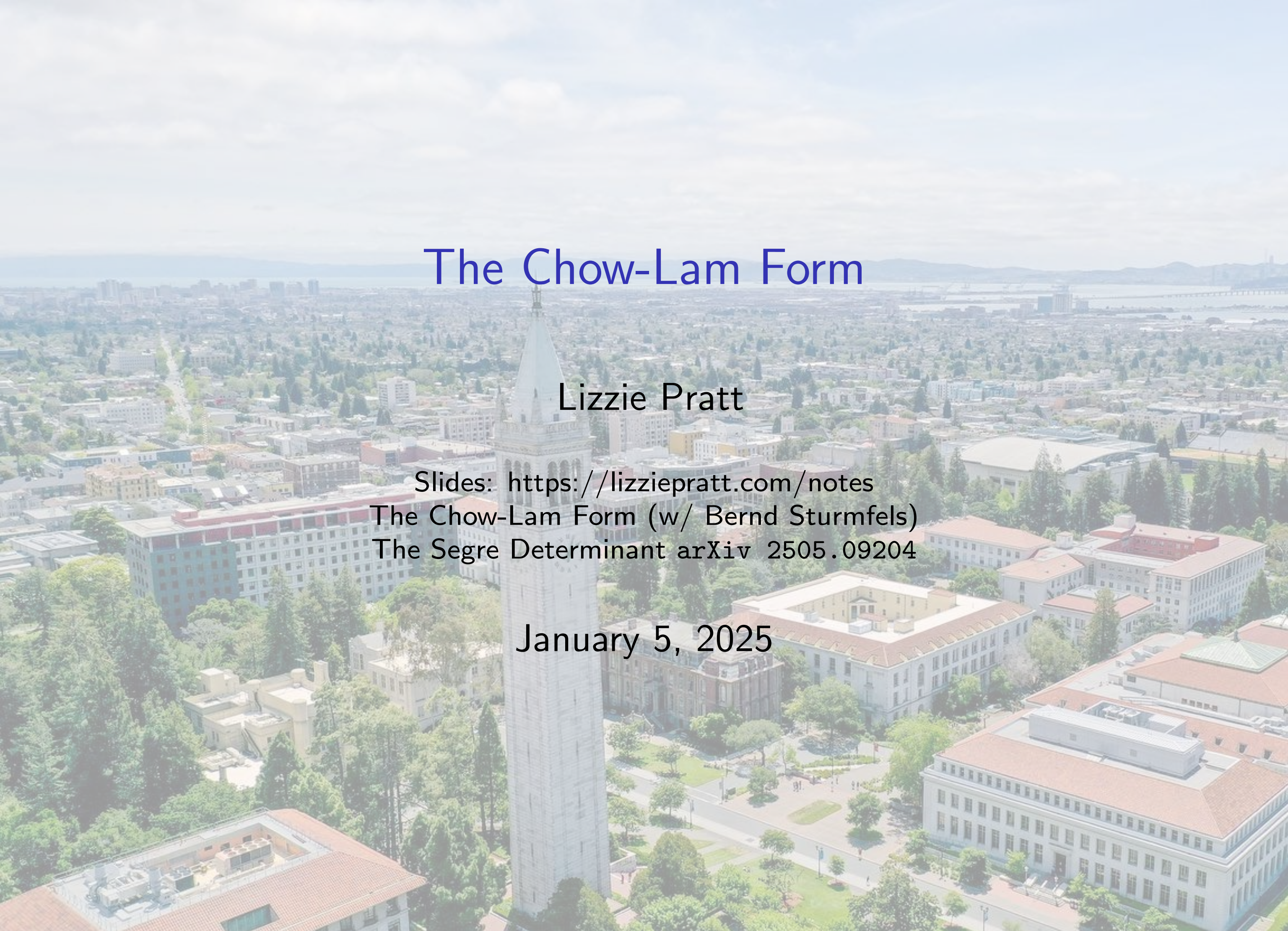

The Chow-Lam Form

Lizzie Pratt

Slides: https://lizziepratt.com/notes
The Chow-Lam Form (w/ Bernd Sturmfels)
The Segre Determinant arXiv 2505.09204

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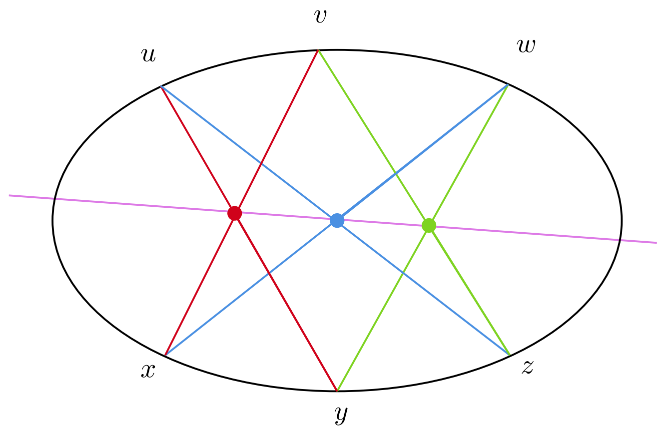
Example (Pascal 1640)



When do $N := \binom{r+d}{d}$ points in $\mathbb{P}^r$ lie on a degree d hypersurface?

Example

Given $\begin{bmatrix} u_0 & v_0 & w_0 & x_0 & y_0 & z_0 \\ u_1 & v_1 & w_1 & x_1 & y_1 & z_1 \\ u_2 & v_2 & w_2 & x_2 & y_2 & z_2 \end{bmatrix}$, when

$$\det \begin{bmatrix} u_0^2 & v_0^2 & w_0^2 & x_0^2 & y_0^2 & z_0^2 \\ u_0u_1 & v_0v_1 & w_0w_1 & x_0x_1 & y_0y_1 & z_0z_1 \\ u_0u_2 & v_0v_2 & w_0w_2 & x_0x_2 & y_0y_2 & z_0z_2 \\ u_1^2 & v_1^2 & w_1^2 & x_1^2 & y_1^2 & z_1^2 \\ u_1u_2 & v_1v_2 & w_1w_2 & x_1x_2 & y_1y_2 & z_1z_2 \\ u_2^2 & v_2^2 & w_2^2 & x_2^2 & y_2^2 & z_2^2 \end{bmatrix} = 0.$$

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Theorem (First fundamental theorem of invariant theory)

Let x_{ij} be entries of a $d \times n$ matrix X, and let SL_d act by multiplication on X. The invariant ring $\mathbb{C}[x_{ij}]^{SL_d}$ is generated by the $d \times d$ minors of X.

When do $N := \binom{r+d}{d}$ points in $\mathbb{P}^r$ lie on a degree d hypersurface?

Example

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In the 20 minors q_{ijk}, this equals

$$q_{123}\, q_{145}\, q_{246}\, q_{356} - q_{124}\, q_{135}\, q_{236}\, q_{456} = 0.$$

When do $N := \binom{r+d}{d}$ points in $\mathbb{P}^r$ lie on a degree d hypersurface?

Example

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Today: when do $k\ell$ points lie on a bilinear hypersurface in $\mathbb{P}^{k-1} \times \mathbb{P}^{\ell-1}$?

Fix vector spaces V, W of dimensions k, ℓ and write $n = k\ell$. The *Segre embedding* is

$$\iota : \mathbb{P}(V) \times \mathbb{P}(W) \hookrightarrow \mathbb{P}(V \otimes W).$$

Let $A_1 \times B_1, \ldots, A_n \times B_n$ denote n points in $\mathbb{P}(V) \times \mathbb{P}(W)$. The *Segre determinant* is the polynomial

$$\mathsf{Seg}_{k,\ell} = \det \begin{bmatrix} \vdots & & \vdots \\ \iota(A_1 \times B_1) & \ldots & \iota(A_n \times B_n) \\ \vdots & & \vdots \end{bmatrix}.$$

Example

Four points $\begin{bmatrix} a_{1,1} \\ a_{2,1} \end{bmatrix} \times \begin{bmatrix} b_{1,1} \\ b_{2,1} \end{bmatrix}, \ldots, \begin{bmatrix} a_{1,4} \\ a_{2,4} \end{bmatrix} \times \begin{bmatrix} b_{1,4} \\ b_{2,4} \end{bmatrix} \in \mathbb{P}^1 \times \mathbb{P}^1$. Then

$$\mathsf{Seg}_{2,2} = \det \begin{bmatrix} a_{1,1}b_{1,1} & a_{1,2}b_{1,2} & a_{1,3}b_{1,3} & a_{1,4}b_{1,4} \\ a_{1,1}b_{2,1} & a_{1,2}b_{2,2} & a_{1,3}b_{2,3} & a_{1,4}b_{2,4} \\ a_{2,1}b_{1,1} & a_{2,2}b_{1,2} & a_{2,3}b_{1,3} & a_{2,4}b_{1,4} \\ a_{2,1}b_{2,1} & a_{2,2}b_{2,2} & a_{2,3}b_{2,3} & a_{2,4}b_{2,4} \end{bmatrix}.$$

Lemma

The Segre determinant $\mathrm{Seg}_{k,\ell}$ *is a polynomial of bi-degree* (ℓ, k) *in the maximal minors of*

$$A := \begin{bmatrix} A_1 & \dots & A_n \end{bmatrix}, \qquad B := \begin{bmatrix} B_1 & \dots & B_n \end{bmatrix}.$$

Example

In maximal minors of

$$A := \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \end{bmatrix}, \qquad B := \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} \end{bmatrix}$$

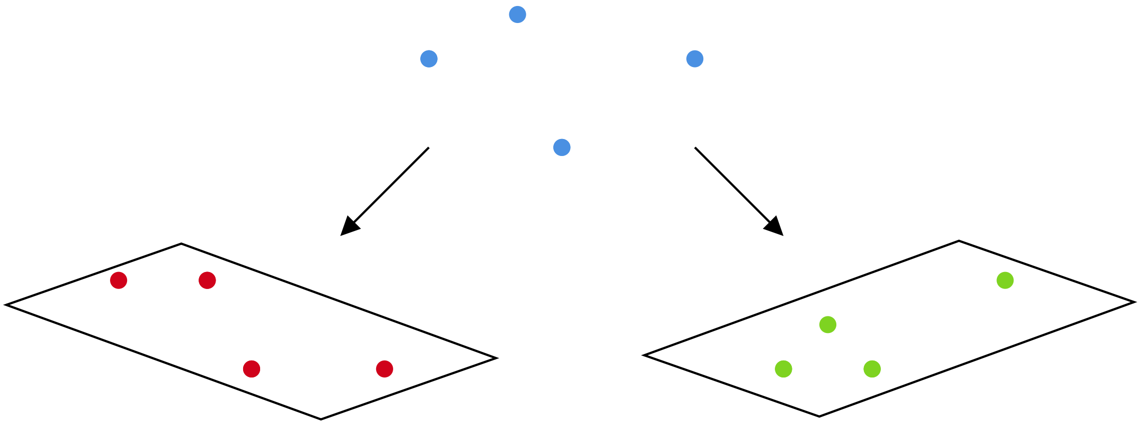
we have

$$\mathrm{Seg}_{2,2} = A_{12}A_{34}B_{13}B_{24} - A_{13}A_{24}B_{12}B_{34}.$$

Vanishes when the *cross-ratios* $\frac{A_{12}A_{34}}{A_{13}A_{24}}$ and $\frac{B_{12}B_{34}}{B_{13}B_{24}}$ are equal.

Computer vision

The polynomial $\mathrm{Seg}_{3,3}$ appears in *algebraic vision* as a necessary condition for two configurations of nine points in $\mathbb{P}^2$ to be projections of the same configuration in $\mathbb{P}^3$.



See [*On The Existence of Epipolar Matrices*, Agarwal–Lee––Sturmfels–Thomas 2016]

$$
\begin{aligned}
\mathrm{Seg}_{3,3} = {} & [123][456][789](3\langle 123\rangle\langle 457\rangle\langle 689\rangle-\langle 123\rangle\langle 467\rangle\langle 589\rangle+3\langle 124\rangle\langle 356\rangle\langle 789\rangle-3\langle 124\rangle\langle 357\rangle\langle 689\rangle+\langle 124\rangle\langle 367\rangle\langle 589\rangle- \\
& \langle 124\rangle\langle 368\rangle\langle 579\rangle-\langle 125\rangle\langle 346\rangle\langle 789\rangle+\langle 125\rangle\langle 347\rangle\langle 689\rangle+\langle 127\rangle\langle 348\rangle\langle 569\rangle-\langle 134\rangle\langle 258\rangle\langle 679\rangle-\langle 135\rangle\langle 247\rangle\langle 689\rangle+ \\
& \langle 145\rangle\langle 267\rangle\langle 389\rangle+\langle 147\rangle\langle 258\rangle\langle 369\rangle)+[123][457][689](-3\langle 123\rangle\langle 456\rangle\langle 789\rangle+\langle 124\rangle\langle 368\rangle\langle 579\rangle-\langle 126\rangle\langle 348\rangle\langle 579\rangle+ \\
& \langle 135\rangle\langle 246\rangle\langle 789\rangle-\langle 146\rangle\langle 258\rangle\langle 379\rangle)+[123][458][679](-\langle 124\rangle\langle 367\rangle\langle 589\rangle-\langle 125\rangle\langle 346\rangle\langle 789\rangle+\langle 126\rangle\langle 347\rangle\langle 589\rangle+ \\
& \langle 146\rangle\langle 257\rangle\langle 389\rangle)+[123][467][589](\langle 123\rangle\langle 456\rangle\langle 789\rangle-\langle 124\rangle\langle 358\rangle\langle 679\rangle+\langle 125\rangle\langle 348\rangle\langle 679\rangle+\langle 134\rangle\langle 256\rangle\langle 789\rangle- \\
& \langle 135\rangle\langle 246\rangle\langle 789\rangle+\langle 145\rangle\langle 268\rangle\langle 379\rangle)+[124][356][789](-3\langle 123\rangle\langle 456\rangle\langle 789\rangle+\langle 123\rangle\langle 468\rangle\langle 579\rangle+\langle 135\rangle\langle 247\rangle\langle 689\rangle- \\
& \langle 135\rangle\langle 267\rangle\langle 489\rangle-\langle 137\rangle\langle 258\rangle\langle 469\rangle)+[124][357][689](3\langle 123\rangle\langle 456\rangle\langle 789\rangle-\langle 123\rangle\langle 468\rangle\langle 579\rangle-\langle 135\rangle\langle 246\rangle\langle 789\rangle+ \\
& \langle 136\rangle\langle 258\rangle\langle 479\rangle)+[124][358][679](\langle 123\rangle\langle 467\rangle\langle 589\rangle-\langle 136\rangle\langle 257\rangle\langle 489\rangle)+[124][367][589](-\langle 123\rangle\langle 456\rangle\langle 789\rangle+ \\
& \langle 123\rangle\langle 458\rangle\langle 679\rangle+\langle 135\rangle\langle 246\rangle\langle 789\rangle-\langle 135\rangle\langle 268\rangle\langle 479\rangle)+[124][368][579](\langle 123\rangle\langle 456\rangle\langle 789\rangle-\langle 123\rangle\langle 457\rangle\langle 689\rangle+ \\
& \langle 135\rangle\langle 267\rangle\langle 489\rangle)+[125][346][789](\langle 123\rangle\langle 456\rangle\langle 789\rangle+\langle 123\rangle\langle 458\rangle\langle 679\rangle-\langle 123\rangle\langle 468\rangle\langle 579\rangle-\langle 134\rangle\langle 257\rangle\langle 689\rangle+ \\
& \langle 134\rangle\langle 267\rangle\langle 589\rangle+\langle 137\rangle\langle 248\rangle\langle 569\rangle)+[125][347][689](-\langle 123\rangle\langle 456\rangle\langle 789\rangle+\langle 123\rangle\langle 468\rangle\langle 579\rangle+\langle 134\rangle\langle 256\rangle\langle 789\rangle- \\
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& [127][348][569](-\langle 123\rangle\langle 456\rangle\langle 789\rangle+\langle 135\rangle\langle 246\rangle\langle 789\rangle)-[127][358][469]\langle 134\rangle\langle 256\rangle\langle 789\rangle+[134][256][789](-\langle 123\rangle\langle 467\rangle\langle 589\rangle- \\
& \langle 125\rangle\langle 347\rangle\langle 689\rangle+\langle 125\rangle\langle 367\rangle\langle 489\rangle+\langle 127\rangle\langle 358\rangle\langle 469\rangle)+[134][257][689](\langle 125\rangle\langle 346\rangle\langle 789\rangle-\langle 126\rangle\langle 358\rangle\langle 479\rangle)+ \\
& [134][258][679](\langle 123\rangle\langle 456\rangle\langle 789\rangle+\langle 126\rangle\langle 357\rangle\langle 489\rangle)+[134][267][589](-\langle 125\rangle\langle 346\rangle\langle 789\rangle+\langle 125\rangle\langle 368\rangle\langle 479\rangle)- \\
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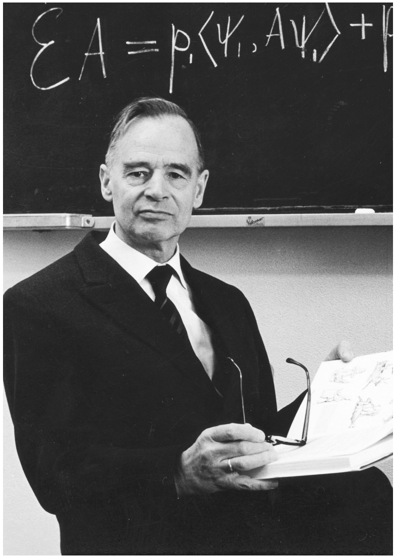
& [147][258][369]\langle 123\rangle\langle 456\rangle\langle 789\rangle.
\end{aligned}
$$

The Chow-Lam form



- The Chow form (Chow–van der Waerden 1937): Assigns to $\mathcal{V} \subset \mathbb{P}^{n-1}$ a polynomial $\mathrm{C}_{\mathcal{V}}$
- Chow-Lam form (P–Sturmfels 2025): Assigns to $\mathcal{V} \subset \mathrm{Gr}(k, n)$ a polynomial $\mathrm{CL}_{\mathcal{V}}$

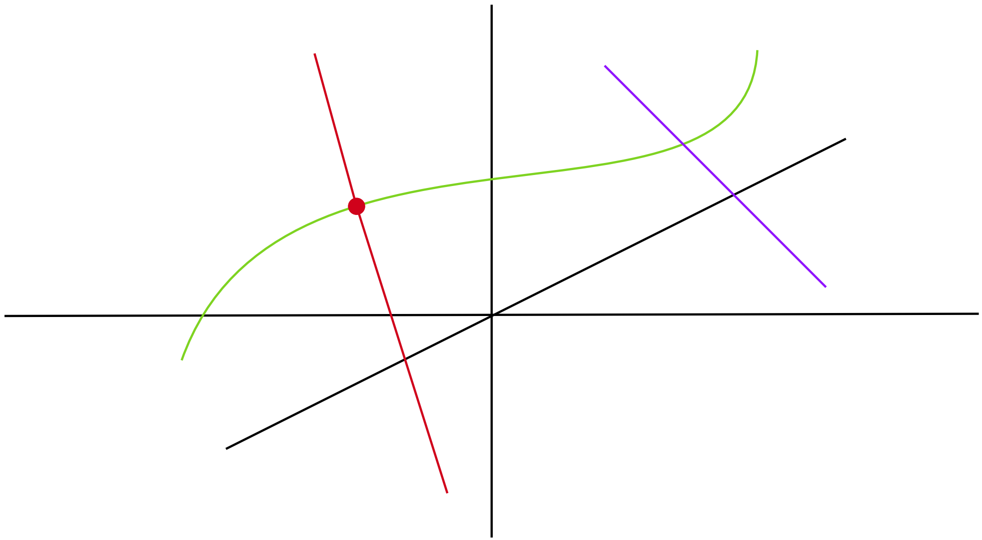
Thomas Lam studied $\mathrm{CL}_{\mathcal{V}}$ where $\mathcal{V}$ is a positroid variety.

Definition (Chow form)

Let $\mathcal{V} \subset \mathbb{P}^{n-1}$ be a d-dimensional projective variety. The *Chow locus* of $\mathcal{V}$ is

$$\mathcal{C}_{\mathcal{V}} = \{L \in \mathrm{Gr}(n-d-1, n) : \mathcal{V} \cap L \neq \varnothing\}.$$

The *Chow form* $\mathrm{C}_{\mathcal{V}}$ is the defining equation of $\mathcal{C}_{\mathcal{V}}$.



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Examples:

- The Chow form of a hypersurface $V(F)$ is F.
- The Chow locus of a linear space is a Schubert divisor.

A linear space L can represented multiple ways.

- Primal : as the kernel of an $(n-k) \times n$ matrix
- Dual : as the rowspan of a $k \times n$ matrix

The primal and dual Plücker coordinates are the maximal minors of these matrices.

Example (Coordinates on Gr$(3,5)$)

$$\begin{array}{cccccccccc} p_{12} & p_{13} & p_{14} & p_{15} & p_{23} & p_{24} & p_{25} & p_{34} & p_{35} & p_{45} \\ q_{345} & -q_{245} & q_{235} & -q_{234} & q_{145} & -q_{135} & q_{134} & q_{125} & -q_{124} & q_{123} \end{array}$$

Lines meeting the twisted cubic

Consider the closure of

$$t \mapsto [1 : t : t^2 : t^3] \ \in \ \mathbb{P}^3.$$

The Chow form is the determinant of the *Bézout matrix* :

$$\mathrm{C}_{\mathcal{V}} = \det \begin{bmatrix} p_{12} & p_{13} & p_{14} \\ p_{13} & p_{14} + p_{23} & p_{24} \\ p_{14} & p_{24} & p_{34} \end{bmatrix}.$$

Its expansion is

$$-p_{14}^3 - p_{14}^2 p_{23} + 2p_{13}p_{14}p_{24} - p_{12}p_{24}^2 - p_{13}^2 p_{34} + p_{12}p_{14}p_{34} + p_{12}p_{23}p_{34}.$$

If p_{ij} are maximal minors of $\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{bmatrix}$, then

$$\mathrm{C}_{\mathcal{V}} = 0 \quad \iff \quad \begin{cases} a_3 t^3 + a_2 t^2 + a_1 t + a_0 \\ b_3 t^3 + b_2 t^2 + b_1 t + b_0 \end{cases}$$ share a root.

Definition (Chow-Lam form)

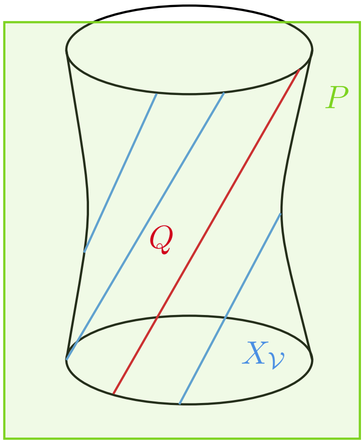
Let $\mathcal{V} \subset \mathrm{Gr}(k,n)$ be a variety of dimension $k(r-k)-1$ for some $k < r \leq n$. The *Chow-Lam locus* of $\mathcal{V}$ is

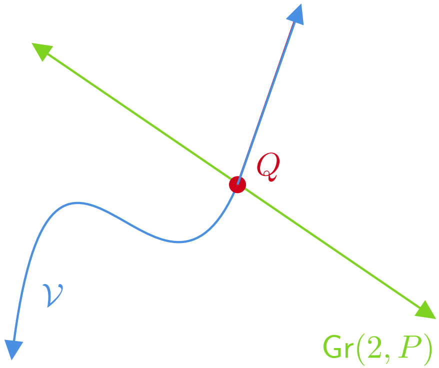
$$\mathcal{CL}_{\mathcal{V}} = \{P \in \mathrm{Gr}(n+k-r,n) : \mathcal{V} \cap \mathrm{Gr}(k,P) \neq \varnothing\}.$$

When $\mathcal{CL}_{\mathcal{V}}$ has codimension 1, its defining equation is the *Chow-Lam form* $\mathrm{CL}_{\mathcal{V}}$. Otherwise, we set $\mathrm{CL}_{\mathcal{V}} := 1$.

An example

Let $\mathcal{V}$ be a curve in Gr$(2,4)$, so $k=2, r=3, n=4$. Then $\mathcal{CL}_{\mathcal{V}}$ is planes P containing a line Q in $\mathbb{P}^3$, with Q on $\mathcal{V}$.





Let $X_{\mathcal{V}}$ be the surface in $\mathbb{P}^3$ swept out by all of the lines in $\mathcal{V}$. Then $\mathcal{CL}_{\mathcal{V}}$ equals the dual variety $X_{\mathcal{V}}^{\vee}$.

The torus $T = (\mathbb{C}^*)^n$ acts on $\mathsf{Gr}(k,n)$ via

$$t \cdot \begin{bmatrix} \vdots & & \vdots \\ A_1 & \dots & A_n \\ \vdots & & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & & \vdots \\ t_1 A_1 & \dots & t_n A_n \\ \vdots & & \vdots \end{bmatrix}.$$

We denote the orbit closure of a point A in $\mathsf{Gr}(k,n)$ by

$$\mathcal{T}_A := \overline{T \cdot A} \subset \mathsf{Gr}(k,n).$$

If A is general, then $\dim \mathcal{T}_A = n - 1$.

Theorem (P. 2025)

Suppose $n = k\ell$ *with* $k, \ell \geq 2$ *and that* $A \in \mathit{Gr}(k,n)$ *has nonzero Plücker coordinates. Then*

$$\mathcal{CL}_{\mathcal{T}_A} \subset \mathit{Gr}(n-\ell, n).$$

The Chow-Lam form of $\mathcal{T}_A$ *in primal Plücker coordinates* B_I *on* $\mathit{Gr}(n-\ell,n)$ *is the Segre determinant* $\mathit{Seg}_{k,\ell}(A,B)$*.*

UPSHOT: the Segre determinant computes the Chow-Lam form of a general torus orbit closure.

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UPSHOT: the Segre determinant computes the Chow-Lam form of a general torus orbit closure.

The proof involves two steps.

1. Show the CL form divides the Segre determinant
2. Show they have the same degree

An example

Fix a general point

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \end{bmatrix} \in \ \mathsf{Gr}(2,6).$$

Then $\mathcal{T}_A$ is a toric variety with polytope

$$\Delta(2,6) = \mathsf{conv}\{110000, 101000, \ldots\} \subset \mathbb{R}^6.$$

Its Chow-Lam form is

$$\mathsf{Seg}_{2,3} = (A_{12}A_{34}A_{56} + A_{14}A_{25}A_{36})\, B_{123}B_{456} - A_{13}A_{25}A_{46}\, B_{124}B_{356}$$
$$+A_{12}A_{35}A_{46}\, B_{134}B_{256} - A_{12}A_{34}A_{56}\, B_{135}B_{246} + A_{13}A_{24}A_{56}\, B_{125}B_{346}.$$

Suppose $\dim \mathcal{V} = k(r-k) - 1$ and write

$$[\mathcal{V}] = \sum_{\lambda \subset k \times (n-k)} c_\lambda(\mathcal{V}) \cdot [\Omega_\lambda] \ \in \ H^*(\mathsf{Gr}(k,n), \mathbb{Z})$$

where

$$\Omega_\lambda = \{L \ : \ L \cap E_i \geq n - k + \lambda_i - i\}.$$

The *Chow-Lam degree* $\alpha(\mathcal{V})$ is the unique coefficient with

$$\lambda = (n - r + 1, n - r, \ldots, n - r).$$

Proposition (P-Sturmfels 2025)

The Chow-Lam form $CL_{\mathcal{V}}$ is irreducible and has degree $\alpha(\mathcal{V})$.

For example, from $[\mathcal{T}_A] = 4\Omega_3 + 2\Omega_{2,1}$ we get $\alpha(\mathcal{T}_A) = 2$.

The variety $\mathcal{T}_A$ depends only on the *matroid* M of A, i.e.

$$\{I \,:\, A_I \neq 0\} \subset \binom{[n]}{k}.$$

The numbers $c_\lambda(M) := c_\lambda(A)$ are the *Schubert coefficients of M.*

Proposition (Klyachko 85)

Let λ be a partition fitting in a $k \times (n-k)$ rectangle. Then the coefficient $c_\lambda(U_{k,n})$ is

$$c_\lambda(U_{k,n}) \;=\; \sum_{i=0}^{k} (-1)^i \binom{n}{i} \dim \mathbb{S}_{\lambda^c}(\mathbb{C}^{k-i}). \tag{1}$$

Corollary

The Chow-Lam degree $\alpha(\mathcal{T}_A)$ is k for A generic.

Chow varieties

Let $\mathcal{C}_r(\mathbb{P}(V), d)$ denote the set of dimension r irreducible subvarieties of $\mathbb{P}(V)$ with degree d. Write $n = \dim V$. Then

$$\begin{aligned} \varphi\colon \mathcal{C}_r(\mathbb{P}(V), d) &\hookrightarrow \mathbb{P}\left(\mathsf{Sym}^d\left(\bigwedge^{n-r-1} V\right)\right) \\ \mathcal{V} &\longmapsto C_{\mathcal{V}}. \end{aligned}$$

Definition

We call $\overline{\varphi(\mathcal{C}_r(\mathbb{P}(V), d))}$ the *Chow variety of* r*-cycles with degree* d.

Consider the map

$$\pi : \mathsf{Gr}(k, k\ell)^\circ \to \mathbb{P}(H^0(\mathcal{O}_{\mathsf{Gr}(\ell,k\ell)}(k))$$
$$A \mapsto \mathsf{Seg}(A, B).$$

We call the image the *Segre coefficient variety* .

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$$\mathsf{Gr}(2,6)^\circ \to \mathbb{P}^4$$
$$A \mapsto (A_{12}A_{34}A_{56}+A_{14}A_{25}A_{36})\, B_{123}B_{456}-A_{13}A_{25}A_{46}\, B_{124}B_{356}$$
$$+A_{12}A_{35}A_{46}\, B_{134}B_{256}-A_{12}A_{34}A_{56}\, B_{135}B_{246}+A_{13}A_{24}A_{56}\, B_{125}B_{346}.$$

The image is the Segre threefold cut out by

$$x_0x_1x_3 - x_1x_2x_3 - x_0x_2x_4 - x_1x_2x_4 - x_1x_3x_4 - x_2x_3x_4.$$

It is the unique (up to isomorphism) cubic hypersurface in $\mathbb{P}^4$ with the max number of ordinary double points, namely 10 [Kalker 86] .

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Theorem (P. 2025)

When $k = 2$, the Segre coefficient variety is isomorphic to the GIT quotient $(\mathbb{P}^1)^{2\ell}//_w\, SL_2$, where $w := 1^{2\ell}$ is the linearization.

Warning for $k = 3$

Let $k = 3$ and $n = 6$. The Segre coefficient map is given by

$$\begin{aligned}\mathrm{Gr}(3,6)^\circ &\to \mathbb{P}^4 \\ A &\mapsto (B_{12}B_{34}B_{56}+B_{14}B_{25}B_{36})\,A_{123}A_{456}-B_{13}B_{25}B_{46}\,A_{124}A_{356} \\ &\quad +B_{12}B_{35}B_{46}\,A_{134}A_{256}-B_{12}B_{34}B_{56}\,A_{135}A_{246}+B_{13}B_{24}B_{56}\,A_{125}A_{346}.\end{aligned}$$

In this case there is a $2:1$ map from the GIT quotient $(\mathbb{P}^2)^6//_{1^6}\mathrm{SL}_3$ to the Segre coefficient variety, whose ramification locus consists of co-conic points.

The proof boils down to showing the following polynomial cannot be written as the sum of products of Segre determinants:

$$B_{123}B_{145}B_{246}B_{356} - B_{124}B_{135}B_{236}B_{456}.$$

See also *Point sets in projective spaces and theta functions*, Example 3 (Dolgachev–Ortland 88)

Thank you for listening!

Chow varieties

Let $\mathcal{C}_r(\mathbb{P}(V), d)$ denote the set of dimension r cycles in $\mathbb{P}(V)$ with degree d. Then

$$\begin{aligned} \mathcal{C}_r(\mathbb{P}(V), d) &\xrightarrow{\varphi} \mathbb{P}(\mathsf{Sym}^d(\wedge^{\dim V - r - 1} V)) \\ \mathcal{V} &\mapsto C_{\mathcal{V}}. \end{aligned}$$

Definition

We call $\overline{\varphi(\mathcal{C}_r(\mathbb{P}(V), d))}$ the *Chow variety of r-cycles with degree d.*

Non-generic torus orbits

Theorem (P- 2025)

Fix a point A in the Grassmannian Gr(k, n) such that $\dim \mathcal{T}_A = k(r-k) - 1 < n$. Then the Chow-Lam form of $\mathcal{T}_A$ in primal Plücker coordinates B_I on Gr$(n-r+k, n)$ divides the Segre determinant Seg$_{k,\ell}(A, B)$

Example

Suppose that A is in Gr$(2, 4)$ and $A_{12} = 0$. Then the 4×4 Segre matrix has determinant $A_{13}A_{34}B_{12}B_{34}$, but the Chow-Lam form is B_{12}.

Warning for $k = 3$

The Segre coefficient map is linearly equivalent to the map

$$\mathrm{Gr}(3,6)^\circ \to \mathbb{P}^4$$
$$A \mapsto ([123][456],\ [124][356],\ [125][346],\ [134][256],\ [135][246]).$$

The following matrices have nonzero Plücker coordinates, which each go to the point $(-2, 2, 9, 2, 8)$ under the map π':

$$p = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 & 3 & 5 \end{bmatrix}, \qquad q := \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -4 & -2 \\ 0 & 0 & 1 & 1 & -3 & -1 \end{bmatrix}.$$

However, their torus orbit closures are different. We may see this by noting that $[123][145][246][356]$ evaluates to -8 for the first point and 4 for the second in the affine chart given by $[123] = 1$.

Chow varieties

Setup and notation:

- A nonsingular projective variety X
- The set $\mathcal{C}_r(X,\delta)$ of dimension r cycles with class δ in singular homology

Choose an embedding $\iota : X \hookrightarrow \mathbb{P}(V)$ and let $d := \iota_*\delta$ in $H_{2r}(\mathbb{P}(V),\mathbb{Z}) \cong \mathbb{Z}$. Then

$$\begin{aligned} \mathcal{C}_r(X,\delta) \ \subset \ \mathcal{C}_r(\mathbb{P}(V),d) \ &\xrightarrow{\varphi} \mathbb{P}(\mathsf{Sym}^d(\wedge^{\dim V - r - 1} V)) \\ \mathcal{V} \ &\mapsto C_{\mathcal{V}}. \end{aligned}$$

Definition

We call $\overline{\varphi(\mathcal{C}_r(X,\delta))}$ the *Chow variety of r-cycles with class δ*.

Independent of embedding ι by [Barlet 1975] .

Chow quotients

Setup:

- A nonsingular projective variety X
- An algebraic group H acting on X
- A H-stable subset $U \subset X$ where the dimension and cohomology class of $\overline{H \cdot x}$ are constant

Idea: create parameter space for H-orbits which is a projective variety.

Definition

The *Chow quotient* $X//H$ is the closure of the image of

$$\begin{aligned} U &\to \mathcal{C}_r(X, \delta) \\ x &\mapsto \overline{H \cdot x}. \end{aligned}$$

Proof idea

Fix $A \in \mathrm{Gr}(2,6)$. Parameterize the CL locus of $\mathcal{T}_A$ as 3×6 matrices B such that, for some $t \in (\mathbb{C}^*)^6$, we have

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} \end{bmatrix} \cdot \mathrm{diag}(t_1, ..., t_6) \cdot \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \\ a_{14} & a_{24} \\ a_{15} & a_{25} \\ a_{16} & a_{26} \end{bmatrix} = 0.$$

Re-arranging, we obtain the expression

$$t_1 \begin{bmatrix} a_{11}b_{11} \\ a_{11}b_{21} \\ a_{11}b_{31} \\ a_{21}b_{11} \\ a_{21}b_{21} \\ a_{21}b_{31} \end{bmatrix} + t_2 \begin{bmatrix} a_{12}b_{12} \\ a_{12}b_{22} \\ a_{12}b_{32} \\ a_{22}b_{12} \\ a_{22}b_{22} \\ a_{22}b_{32} \end{bmatrix} + \ldots + t_6 \begin{bmatrix} a_{16}b_{16} \\ a_{16}b_{26} \\ a_{16}b_{36} \\ a_{26}b_{16} \\ a_{26}b_{26} \\ a_{26}b_{36} \end{bmatrix} = \sum_{i=1}^{6} t_i A_i \otimes B_i = 0.$$

When do $N := \binom{r+d}{d}$ points in $\mathbb{P}^r$ lie on a degree d hypersurface?

Example

Bruxelles problem (1825): 10 points on a quadratic surface in $\mathbb{P}^3$

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- Synthetic construction due to [Traves 2024]

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Example

Bruxelles problem (1825): 10 points on a quadratic surface in $\mathbb{P}^3$

- Synthetic construction due to [Traves 2024]
- Condition in the minors of

$$\begin{bmatrix} a_0 & b_0 & c_0 & d_0 & e_0 & f_0 & h_0 & g_0 & i_0 & j_0 \\ a_1 & b_1 & c_1 & d_1 & e_1 & f_1 & h_1 & g_1 & i_1 & j_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 & h_2 & g_2 & i_2 & j_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 & f_3 & h_3 & g_3 & i_3 & j_3 \end{bmatrix}$$

  given by [Turnbull-Young 1927] ; polynomial with 240 terms
- Straightened to a 148-term polynomial by [White 1990]