Simulating the Solar System

A Computational Report

This report presents and communicates the results of a simulation of the solar system, which uses the Velocity Verlet Integration Scheme to study the trajectories of celestial bodies. The observables it finds include, the apsides and orbital periods of each astronomical object. These were obtained by finding the maximum and minimum values of the distance between an object and its primary, as well as the time taken between them. The simulation was tested using a toy solar system consisting of the Sun, Mercury, Earth, and Moon, and the ideal time-step parameter was determined to be 0.5 earth days, ensuring a relative error of 0.5% or less for all observables. Both the energy deviation and relative errors of the observables converged, proving the success of the simulation. All the observables found by the simulation had an error of less than 2% with respect to the observational data, apart from the orbital period of Neptune, whose trajectory had a wiggle that the simulation did not account for. Furthermore, Kepler's Third Law was verified using the results of two simulation runs, both with Jupiter's mass being accurate and 20 times heavier.

Background

The objects within the simulated solar system, which include the Sun, all the planets, the Earth's moon, Pluto, and Halley's Comet, are represented as point particles. They interact via gravitational forces, such that the force on any one object is given by:

$$\boldsymbol{F}_i = -Gm_i \sum_j m_\mathrm{j} \frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3}$$

The total energy of the system at time t is given by the kinetic energy $T_i = \frac{1}{2} m_i \boldsymbol{v}_i{}^2$ of each particle plus the potential energy of the system, given by:

$$U = \sum_i \sum_{j>\mathrm{i}} -\frac{Gm_i m_\mathrm{j}}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}$$

Note that the units of the simulation are astronomical units (AU) for lengths, earth days for times, and earth mass for masses.

Velocity Verlet Integration Scheme

The initial conditions that are provided for each astronomical object are their mass, velocity and position. The centre-of-mass velocity is then subtracted from velocities of the particles, to avoid a drift during long simulations due to non-vanishing linear momentum in the initial conditions.

The trajectories of the interacting particles are simulated at each time step, δt, using the velocity Verlet integration scheme. This involves, at each new time step, computing the forces on each object using the equation above, and using these forces to update the positions of each of the objects to the second order:

$$\vec{x}(t + \delta t) = \vec{x}(t) + \vec{v}(t)\delta t + \frac{1}{2}\vec{a}(t)\delta t^2$$

where the accelerations are found using the force divided by the mass of the particle.

Then, the new forces and separations of all the objects are computed, to subsequently update the velocities of each object, using:

$$\vec{v}(t + \delta t) = \vec{v}(t) + \frac{1}{2}[\vec{a}(t) + \vec{a}(t + \delta t)]\delta t$$

Once all the velocities and positions are updated, the simulation moves on to the next time step to repeat the process. The positions of each object, at each time step, are stored then in an array.

Finding the Astrophysical Observables

The simulation was tasked to find the following astrophysical observables of each object: the perihelion and aphelion, which are the smallest and largest distances of an object from the sun, and the orbital period, which is the time for an object to complete one orbit. For the moon specifically, the simulation can find the perigee and apogee, or the minimum and maximum distances from the Earth.

By subtracting the trajectory of the sun from the trajectory of an object, the perihelion and aphelion of the object can be found through the minimum and maximum values in this new trajectory array. Finding the average time between each peak in this new trajectory array, and the average time between each trough, and then finding the average of these, gives a fairly accurate calculation of the orbital period of an object within the simulation. The same can be computed for the Moon with respect to the Earth.

Note that if the length of the simulation means that the object has not completed half an orbit, the simulation returns NaN (not a number) for all the parameters. On the other hand, if the object has completed less than one orbit, but more than half, the period is calculated using the time in between a peak and a trough, multiplied by two.

Results and Discussion

Convergence Tests & Ideal Time-Step Parameter

The convergence of the simulation was deduced using a “toy” solar system consisting of the Sun, Mercury, the Earth and the Moon being simulated for 10 years. The simulation was run multiple times, each time with a different time step parameter, δt, and each time the values of the observables were recorded.

Plotted below are two examples of how the observables in the “toy” solar system converged. On the y-axis are the relative errors of the observables, given by:

$$relative\ error\ at\ \delta t\ = \frac{|observable\ found\ using\ \delta t - expected\ value\ of\ observable|}{|expected\ value\ of\ observable|}$$

In this case, the best estimate for the expected value of the observable is given by the observable at the smallest δt.

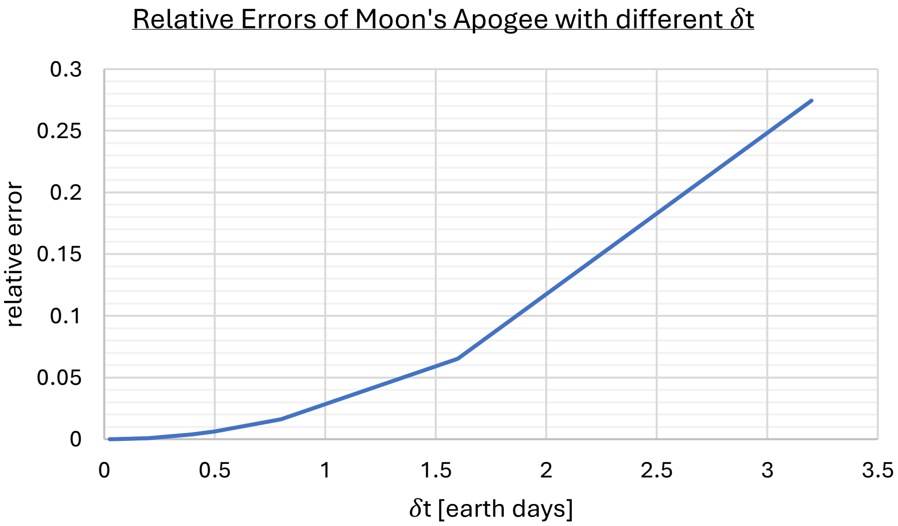

Figure 1

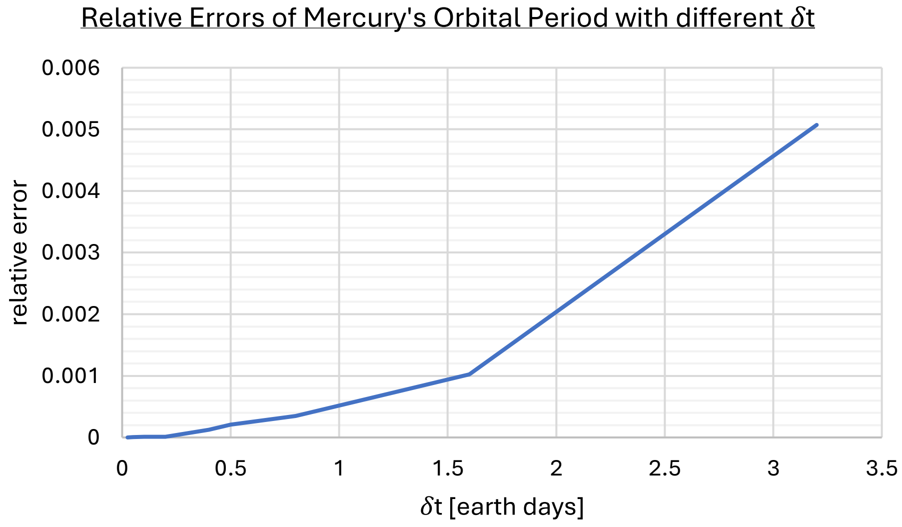

Figure 2

As is evident, the errors of the observables are decreasing as dt gets smaller. Thus, it seems that as $\delta t \rightarrow 0$, the $relative\ error\ \rightarrow 0$, meaning the observable tends towards the expected value of the observable. This implies that the simulation is converging towards a specific value.

The largest error in the simulated toy-solar system at $\delta t = 0.5$ earth days is approximately 0.005. This occurs for the Moon's apogee as seen in Figure 1. Thus, the ideal time-step parameter has been chosen by estimating the maximum time step which will allow all observables to have a relative error smaller than or equal to 0.001 = 0.5%.

One can also use the energy deviation, $\Delta E\ = max(E) - min(E)$, to determine the accuracy of a simulation. The energy deviation of a simulation is due to inevitable inaccuracies, and in theory, a "perfect" simulation, meaning one that perfectly reflects real life, would have an energy deviation of zero due to conservation of energy.

Below is the energy deviation of the simulation plotted with respect to δt.

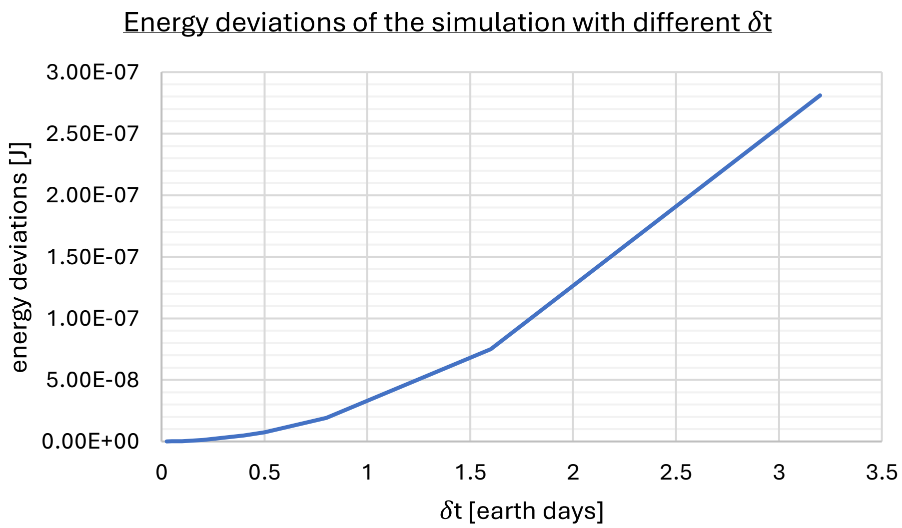

Figure 3

The energy deviation tends towards zero as the time step parameter decrease, meaning a smaller δt provides a more accurate simulation with respect to energy, and the simulation converges towards a conservation of energy.

Verification of the Code

Below are the apsides and periods of the objects in the solar system from the simulation, which was ran for 15,000 days with a time step of 0.5 days, as well as the observed data[1][2], and the relative error between the two values:

	Orbital Period [Earth Days]			**Aphelion [AU]**			**Perihelion [AU]**		
Planet	**Simulated**	**Observed**	**Error**	**Simulation**	**Observations**	**Relative Error**	**Simulation**	**Observations**	**Error**
Mercury	88.04	88	0.000417	0.467	0.4666	0.000933	0.3078	0.307491	0.000948
Venus	224.68	224.7	0.000075	0.728	0.7280	0.000134	0.7186	0.718593	0.000006
Earth	365.64	365.2	0.001203	1.017	1.0167	0.000219	0.9845	0.983303	0.001183
Moon	27.59	27.3	0.010552	0.003	0.0027	0.008015	0.0024	0.002427	0.017930
Mars	686.41	687	0.000858	1.667	1.6665	0.000099	1.3789	1.381704	0.002014
Jupiter	4344.36	4331	0.003086	5.476	5.4573	0.003519	4.9461	4.950605	0.000905
Saturn	10747.96	10747	0.000089	10.053	10.0703	0.001738	9.0141	9.074995	0.006707
Uranus	30808.31	30589	0.007170	20.115	20.0631	0.002590	18.3770	18.266971	0.006021
Neptune	23266.05	59800	0.610936	30.265	30.4744	0.006883	29.6690	29.887457	0.007309
Pluto	90362.00	90560	0.002186	49.181	49.3048	0.002517	29.5744	29.658176	0.002826
1P/Halley	27298.16	27740	0.015928	35.305	35.2500	0.001561	0.5929	0.587100	0.009873

Table 1

All the errors above, except for one discussed below, are less than 0.02 = 2%, which is very successful.

The orbital period of Neptune has an error of 0.611 (3sf) which is extremely large, implying this observable is an anomaly and there is an error in the simulation. The reason for this is that the simulation is not able to account for the wiggle in Neptune's trajectory, as visible below in Figure 4 and 'zoomed in' in Figure 5.

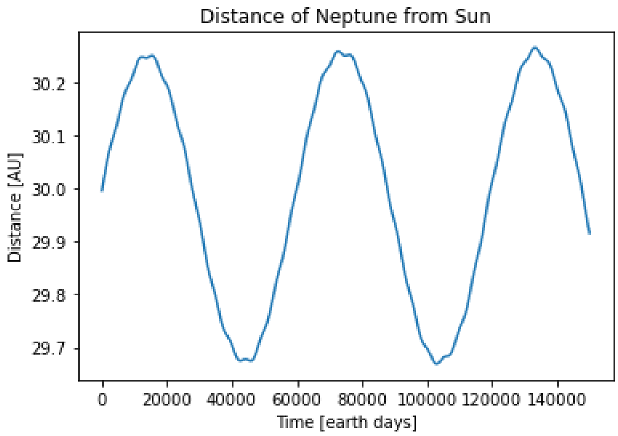

Figure 4

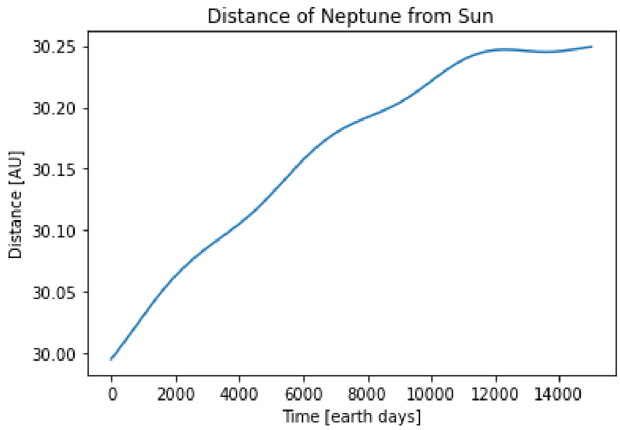

Figure 5

When attempting to find the orbital period as the time between peak locations, the simulation was not able to differentiate a local maximum which occurred due to the wiggle, and an absolute maximum which occurred due to the orbit. This resulted in the error. Due to the discontinuous nature of the data set, there was no "exact" value for the absolute maximum at each peak. So, while it is easy to visualise the difference between the two, it is harder to distinguish between them in the simulation.

A possible solution to this would be to have the simulation only identify peaks within a certain range of the maximum distance, meaning any maxima not occurring within a certain distance of the aphelion would not be used.

Mini-Task: Verify Kepler's Third Law

Kepler's Third Law states that the ratio of the square of an object's orbital period with the cube of the semi-major axis is the same for all objects orbiting the same primary [3]. This can be expressed as:

$$T^2 = ka^3 \text{ where } \frac{1}{k} \approx 7.496 \times 10^{-6} \frac{\mathrm{AU}^3}{\mathrm{days}^2} \text{ is constant}$$

T is the orbital period and a is the semi-major axis, which can be found using the average of the perihelion and aphelion.

Rearranging the equation gives: $\log_{10} T^2 = \log_{10} k + 3\log_{10} a$, which is a linear equation in the form of $y = mx + c$. Thus, plotting corresponding simulation values of $\log_{10} T^2$ versus $\log_{10} a$ for each planet orbiting the sun should form a linear graph with a gradient of 3 and y-intercept of $\log_{10} k$, verifying Kepler's Third Law.

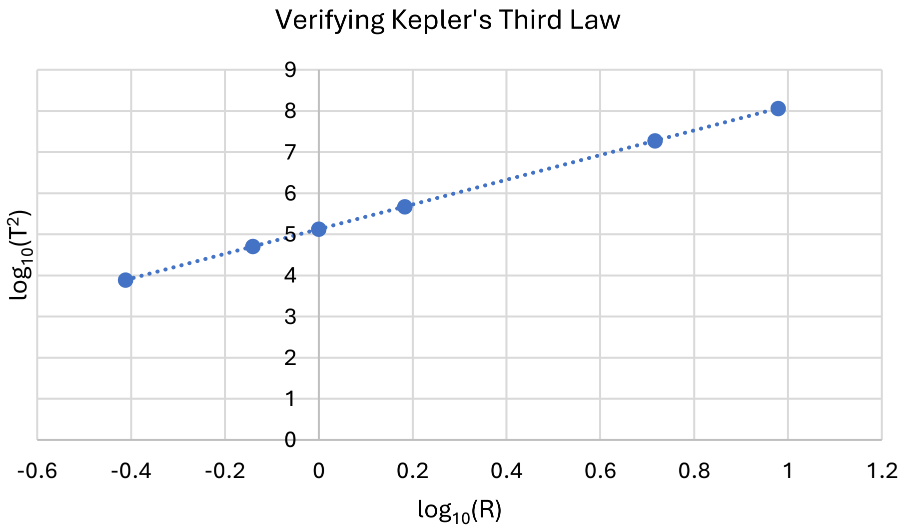

Figure 6

Note that for the mini-task, the simulation was run for 20 years at a time step of 0.5 earth days. This means that not all the planets had completed half an orbit, explaining why there are only six data points.

A least-squares fit was performed to determine values and uncertainties for the gradient, m, and y-intercept, c:

$$m = 3.0012 \pm 0.0004 \text{ and } c = 5.1253 \pm 0.0002$$

$$\frac{1}{k} = 10^{-c} = (7.493 \pm 0.004) \times 10^{-6} \frac{\mathrm{AU}^3}{\mathrm{days}^2}.$$

Both m and $\frac{1}{k}$ are sufficiently close to their expected values. Furthermore, the correlation coefficient was found to be $r = 1.000$ (3dp) which means there is an almost perfect correlation between the two variables $\log_{10} T^2$ and $\log_{10} a$. Therefore, the fit is good, and Kepler's Third Law has been verified.

Super-Jupiter

Performing the same analysis, but replacing Jupiter with a “super-Jupiter” which is 20 times heavier, gives the following results:

$$m = 3.05 \pm 0.03 \text{ and } \frac{1}{k} = (7.41908 \pm 0.03) \times 10^{-6} \, \frac{\mathrm{AU}^3}{\mathrm{days}^2}$$

The correlation coefficient was also found to be $r = 1.000$ (3dp).

m and $\frac{1}{k}$, whilst still being close to their expected values, stray further from them when Jupiter is 20 times heavier. This is most likely because the constant of proportionality $\frac{1}{k}$ is not actually constant, and is instead given by [3]:

$$\frac{1}{k} = \frac{a^3}{T^2} = \frac{G(M+m)}{4\pi^2} \approx \frac{GM}{4\pi^2}$$

where the mass of the planet, m, is so much smaller than the mass of the sun, M, that it can be ignored, allowing $\frac{1}{k}$, to be a constant.

However, this approximation to a constant will begin to break down as m increases, meaning the ratio of a^3 and T^2 will no longer appear equal for all planets. This explains why replacing Jupiter with “super-Jupiter” leads to Kepler’s Third Law being less accurate.

Conclusion

Based on the results presented in this report, it can be concluded that the Velocity Verlet Integration Scheme is a suitable method for simulating the trajectories of celestial bodies in the solar system. The observables obtained through the simulation, which were apsides and orbital periods, had a high level of accuracy with a relative error of 2% or less compared to their respective observational values, except for the orbital period of Neptune. The verification of Kepler's Third Law using two different simulation runs provides additional evidence for the accuracy of the simulation.

The convergence of both the energy deviation and relative errors of the observables is also evidence of the simulation’s success. The determination of an optimal time-step parameter ensures a relative error of 0.5% or less for all observables, further demonstrating the simulation's suitability.

Despite the simulation's success, it is worth noting that the wiggle in Neptune's trajectory was not accounted for. This led to an anomaly in the value of its orbital period. If the simulation were to be rewritten, it should only identify orbital peaks within a certain distance of the aphelion, so that any maxima not occurring within this range would be discounted.

Overall, this project provides valuable insight into the simulation of the solar system and its suitability.

References

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