Анализ модельного риска
дз4
Петрова Елизавета

1

$\dfrac{\partial V}{\partial r}$ - ?

For a call option $V = C = S_0N(d_1) - Ke^{-rT}N(d_2)$, $d_1 = \dfrac{\ln(\frac{S_0}{K}) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$

So, $\dfrac{\partial C}{\partial r} = \dfrac{\partial(S_0N(d_1) - Ke^{-rT}N(d_2))}{\partial r} = S_0n(d_1)\dfrac{\sqrt{T}}{\sigma} - K(-Te^{-rT}N(d_2) + e^{-rT}n(d_2)\dfrac{\sqrt{T}}{\sigma}) = S_0n(d_1)\dfrac{\sqrt{T}}{\sigma} + KTe^{-rT}N(d_2) - Ke^{-rT}n(d_2)\dfrac{\sqrt{T}}{\sigma}) = KTe^{-rT}N(d_2)$, because in hw1 we got an equation $S_0N(d_1) = Ke^{-rT}N(d_2)$, so $S_0n(d_1)\frac{\sqrt{T}}{\sigma} = Ke^{-rT}n(d_2)\frac{\sqrt{T}}{\sigma}$, and we get $\boxed{\dfrac{\partial V}{\partial C} = KTe^{-rT}N(d_2)}$

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$\dfrac{\partial^2 V}{\partial S\partial\sigma}$ - ?

First, I'll calculate the $\Delta = \dfrac{\partial V}{\partial S}$

$\Delta = \dfrac{\partial V}{\partial S} = \dfrac{\partial C}{\partial S} = \dfrac{\partial(S_0N(d_1) - Ke^{-rT}N(d_2))}{\partial S} = N(d_1) + S_0n(d_1)\dfrac{1}{\sigma\sqrt{T}} \cdot \dfrac{K}{S_0} \cdot \dfrac{1}{K} - Ke^{-rT}n(d_2) \cdot \dfrac{1}{\sigma\sqrt{T}S_0} = N(d_1) + S_0n(d_1) \cdot \dfrac{1}{S_0\sigma\sqrt{T}} - Ke^{-rT}n(d_2) \cdot \dfrac{1}{\sigma\sqrt{T}S_0} = N(d_1)$,

because of the equation from hw1 $S_0N(d_1) = Ke^{-rT}N(d_2)$, so $\dfrac{\partial V}{\partial S} = N(d_1)$

Now we need to calculate $\Delta'_\sigma = \dfrac{\partial^2 V}{\partial S\partial\sigma} = (N(d_1))'_\sigma = n(d_1) \cdot \dfrac{\partial d_1}{\partial\sigma}$

$\dfrac{\partial d_1}{\partial\sigma} = (\dfrac{\ln(\frac{S_0}{K})}{\sigma\sqrt{T}})'_\sigma + (\dfrac{rT}{\sigma\sqrt{T}})'_\sigma + (\dfrac{\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}})'_\sigma = -\dfrac{\ln(\frac{S_0}{K})}{\sqrt{T}} \cdot \dfrac{1}{\sigma^2} - \dfrac{r\sqrt{T}}{\sigma^2} + \dfrac{\sqrt{T}}{2} = -\dfrac{1}{\sigma}(\dfrac{\ln(\frac{S_0}{K})}{\sqrt{T}} \cdot \dfrac{1}{\sigma} - \dfrac{r\sqrt{T}}{\sigma} + \dfrac{\sqrt{T}\sigma}{2}) = -\dfrac{1}{\sigma}(d_1 - \sigma\sqrt{T}) = -\dfrac{1}{\sigma}d_2 = -\dfrac{d_2}{\sigma}$

So, $\boxed{\dfrac{\partial^2 V}{\partial S\partial\sigma} = -n(d_1) \cdot \dfrac{d_2}{\sigma}}$

3

$\dfrac{\partial^2 V}{\partial\sigma^2}$ - ?

$Vega = \dfrac{\partial V}{\partial\sigma} = S_0n(d_1) \cdot \dfrac{\partial d_1}{\partial\sigma} - Ke^{-rT}n(d_2) \cdot \dfrac{\partial d_2}{\partial\sigma} = -S_0n(d_1)\dfrac{d_2}{\sigma} - Ke^{-rT}n(d_2)(\dfrac{\partial d_1}{\partial\sigma} - \dfrac{\partial(\sigma\sqrt{T})}{\partial\sigma}) = -S_0n(d_1)\dfrac{d_2}{\sigma} + Ke^{-rT}n(d_2)\dfrac{d_2}{\sigma} + Ke^{-rT}n(d_2)\sqrt{T} = S_0n(d_1)\sqrt{T}$, because of the equation from hw1 $S_0N(d_1) = Ke^{-rT}N(d_2)$

$\dfrac{\partial^2 V}{\partial \sigma^2} = (S_0 n(d_1)\sqrt{T})'_\sigma = S_0\sqrt{T}(n(d_1))'_\sigma \cdot \dfrac{\partial d_1}{\partial \sigma} = -S_0\sqrt{T}d_1 n(d_1)\cdot(-\dfrac{d_2}{\sigma}) = S_0\sqrt{T}n(d_1)\dfrac{d_1 d_2}{\sigma} = Vega \cdot \dfrac{d_1 d_2}{\sigma}$

$$\boxed{\frac{\partial^2 V}{\partial \sigma^2} = S_0\sqrt{T}n(d_1)\frac{d_1 d_2}{\sigma} = Vega \cdot \frac{d_1 d_2}{\sigma}}$$

4

$\dfrac{\partial V}{\partial t}$ - ?

Let's say $\tau := T - t$, so $\frac{\partial V}{\partial \tau} = -\frac{\partial V}{\partial t}$, I'll be calculating $\dfrac{\partial V}{\partial \tau} = S_0 n(d_1)\dfrac{\partial d_1}{\partial \tau} + Kre^{-r\tau}N(d_2) - Ke^{-r\tau}n(d_2)\dfrac{\partial d_2}{\partial \tau} = S_0 n(d_1)\dfrac{\partial d_1}{\partial \tau} + Kre^{-r\tau}N(d_2) - Ke^{-r\tau}n(d_2)\dfrac{\partial d_1}{\partial \tau} + Ke^{-r\tau}n(d_2)\dfrac{\partial(\sigma\sqrt{\tau})}{\partial \tau} = Kre^{-r\tau}N(d_2) + Ke^{-r\tau}n(d_2)\dfrac{\partial(\sigma\sqrt{\tau})}{\partial \tau} = Kre^{-r\tau}N(d_2) + S_0 n(d_1)\cdot\dfrac{\sigma}{2\sqrt{\tau}}$, because of the equation from hw1 $S_0 N(d_1) = Ke^{-rT}N(d_2)$

And, knowing that $\dfrac{\partial V}{\partial \tau} = -\dfrac{\partial V}{\partial t}$, we get: $\boxed{\dfrac{\partial V}{\partial t} = -Kre^{-rt}N(d_2) - S_0 n(d_1)\cdot\dfrac{\sigma}{2\sqrt{t}}}$

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$[W_t^1, W_t^2] = \rho t \iff d[W_t^1, W_t^2]_t = \rho dt$

For continuous semimartingales X, Y we have $d(X_tY_t) = X_t dY_t + Y_t dX_t + d[X,Y]_t$, and we can apply this to $X = W_t^1,\ Y = W_t^2$, $d(W_t^1 W_t^2) = W_t^1 dW_t^2 + W_t^2 dW_t^1 + d[W^1, W^2]_t$, now we can integrate both parts, and we get $W_t^1 W_t^2 = \displaystyle\int_0^t W_s^1 dW_s^2 + \int_0^t W_s^2 dW_s^2 + [W^1, W^2]_t = \int_0^t W_s^1 dW_s^2 + \int_0^t W_s^2 dW_s^2 + \rho t$

Now we take expectation from both sides, so we now have $\mathbb{E}(W_t^1 W_t^2) = \mathbb{E}(\displaystyle\int_0^t W_s^1 dW_s^2) + \mathbb{E}(\int_0^t W_s^2 dW_s^2) + \rho t$, and we know that if H_s is adapted and square-integrable, the Ito integral $\int_0^t H_s dW_s$ is a martingale starting at 0, hence $\mathbb{E}(\int_0^t H_s dW_s) = 0$, so in our case, since W^1 and W^2 satisfy the conditions, we have $\mathbb{E}(W_t^1 W_t^2) = \mathbb{E}(\displaystyle\int_0^t W_s^1 dW_s^2) + \mathbb{E}(\int_0^t W_s^2 dW_s^2) + \rho t = 0 + 0 + \rho t$, so we got $\mathbb{E}(W_t^1 W_t^2) = \rho t$

Also, knowing that $\mathbb{E}(W_t^1 W_t^2) - \mathbb{E}(W_t^1)\mathbb{E}(W_t^2) = cov(W_t^1, W_t^2)$, since W_t^1, W_t^2 are standard brownian motions, $\mathbb{E}(W_t^1) = 0, \mathbb{E}(W_t^2) = 0$, so we have $\mathbb{E}(W_t^1 W_t^2) = \rho t = cov(W_t^1, W_t^2)$
qed

6

Since Y_t is a GBM, we got $dY_t = \mu Y_t dt + \sigma Y_t dW_t$, $Z_t = \dfrac{1}{Y_t}$
Let's apply Ito's Lemma to the function $f(y) = \frac{1}{y}$, $f' = -\frac{1}{y^2}$, $f'' = \frac{2}{y^3}$

So, $dZ_t = df(Y_t) = f_t'dt + f_y'dY_t + \frac{1}{2}f_{yy}''(dY_t)^2 = -\frac{1}{Y_t^2}dY_t + \frac{1}{Y_t^3}\sigma^2 Y_t^2 dt = -\frac{1}{Y_t^2}(\mu Y_t dt + \sigma Y_t dW_t) + \frac{1}{Y_t^3}\sigma^2 Y_t^2 dt = (-\frac{\mu}{Y_t} + \frac{\sigma^2}{Y_t})dt - \frac{\sigma}{Y_t}dW_t = Z_t(-\mu + \sigma^2)dt - Z_t\sigma dW_t$

Now, $d[Y,Z]_t = (\sigma Y_t)(-Z_t\sigma)dt = -\sigma^2 dt \implies \boxed{[Y,Z]_t = -\sigma^2 t}$