Анализ модельного риска
дз2
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1

Let $\Delta_i W := W_{t_i} - W_{t_{i-1}}$ with independent Gaussian increments $\Delta_i W \sim N(0, \Delta t_i)$

Set $X_i := |\Delta_i W|^3$, also we know that $Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i)$ because of independence. With $\Delta_i W = \sqrt{\Delta t_i} Z$, where $Z \sim N(0,1)$

$Var(X_i) = Var(|\Delta_i W|^3) = Var(|\sqrt{\Delta t_i} Z|^3) = \Delta t_i^3 \cdot Var(|Z|^3) = (\Delta t_i)^3(\mathbb{E}|Z|^6 - (|Z|^3)^2)$

We know that $Z \sim N(0,1)$, so we know the formula for the p-th moment: $\mathbb{E}|Z|^p = 2^{p/2}\frac{\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}$

So we can easily calculate the 3rd moment: $\mathbb{E}|Z|^3 = 2^{3/2}\frac{\Gamma(\frac{3+1}{2})}{\sqrt{\pi}} = \frac{2\sqrt{2}\Gamma(2)}{\sqrt{\pi}} = \frac{2\sqrt{2}}{\sqrt{\pi}}$

The same for the 6th moment: $\mathbb{E}|Z|^6 = 2^{6/2}\frac{\Gamma(\frac{6+1}{2})}{\sqrt{\pi}} = \frac{8\Gamma(3.5)}{\sqrt{\pi}} = \frac{8 \cdot \frac{15}{8}\Gamma(\frac{1}{2})}{\sqrt{\pi}} = 15$

So we have: $\mathbb{E}|Z|^6 - (\mathbb{E}(|Z|^3))^2 = 15 - (\frac{2\sqrt{2}}{\sqrt{\pi}})^2 = 15 - \frac{8}{\pi}$

And so we get, that our variance $= \sum_{i=1}^{n} |W_{t_i} - W_{t_{i-1}}|^3 = \sum_{i=1}^{n}((\Delta t_i)(15 - \frac{8}{\pi}) = (15 - \frac{8}{\pi})\sum_{i=1}^{n}(\Delta t_i)^3$ and for an equally spaced partition this equals to $(15 - \frac{8}{\pi})\sum_{i=1}^{n}(\frac{iT}{n})^3 = \frac{T^3}{n^2}(15 - \frac{8}{\pi})$

2

We can use Ito's formula here. For X_t solving the Ornstein–Uhlenbeck $dX_t = -\kappa X_t dt + \sigma dW_t$ and we'll take $f(t, X) = sin(x)$, $f'_t(sin(x)) = 0$

So we have $df(X_t) = \frac{1}{2}f''_{xx}(t, X_t)(dX_t)^2 + f'_x(t, X_t)dX_t = -\frac{1}{2}sin(X_t)\sigma^2 dt + cos(X_t)(-\kappa X_t dt + \sigma dW_t) = [-\frac{\sigma^2}{2}sin(X_t) - \kappa cos(X_t)X_t]dt + \sigma cos(X_t)dW_t$, the first part is the drift part, it's the finite variation part. The second part with dW_t is the martingale part, and we know that the quadratic variation comes only from the martingale part. So $d\langle sin(X_t)\rangle_t = (\sigma cos(X_t))^2 dt \implies \langle sin(X_t)\rangle_t = \int_0^t (\sigma cos(X_s))^2 ds$

3

a) Let $Y_t = (T - t)W_t$, so $Y_T = (T - T)W_T = 0$ and Brownian paths are continuous, so $\frac{Y_t - Y_T}{t - T} = \frac{(T - t)W_t}{(t - T)} = -W_t \overset{t \to T}{\to} -W_T$ a.s.

So Y_t is differentiable at $t = T$ with the derivative $= -W_T$ a.s.

q.e.d

b) For any $s \in [0, T)$, $t \to s$, let's look at $Y'_s = \lim_{t \to s}\frac{Y_t - Y_s}{t - s} = \lim_{t \to s}\frac{(T - t)W_t - (T - s)W_s}{t - s} = \lim_{t \to s}\frac{(T - t)W_t - (T - t + (t - s))W_s}{t - s} = \lim_{t \to s}\frac{(T - t)(W_t - W_s) - (t - s)W_s}{t - s} = (T - t)\lim_{t \to s}(\frac{W_t - W_s}{t - s}) - W_s$, so if we suggest that the process Y_t is differentiable in any point a.s besides t=T, we need the process W_t to be differentiable a.s, but it's a Brownian motion, and as we know it's not a differentiable process in any of it's points, so the limit does not exist, and so the there are no other points, besides t=T, in which the process Y_t is differentiable a.s.