Анализ модельного риска
дз1
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1

Let $f, g : [a, b] \to \mathbb{R}$ be of bounded first order variation

$\forall$ partition $P = \{a = x_0 < ... < x_n = b\}$ we have $V(f+g, P) = \sum_{i=1}^{n} |(f+g)(x_i) - (f+g)(x_{i-1})| = \sum_{i=1}^{n} |[f(x_i) - f(x_{i-1})] + [g(x_i) - g(x_{i-1})]|$

Now by using the triangle inequality we get: $V(f+g, P) \le \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| + \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| = V(f, P) + V(g, P)$

So, we have $V(f+g) \le V(f) + V(g) \implies$ since f, g are finite $f+g$ is also finite $\implies$ the sum of 2 funcs of bounded first order variation also has bounded first order variation. qed

2

Let $f, g : [a, b] \to \mathbb{R}$ be of bounded first order variation

$\forall$ partition $P = \{a = x_0 < ... < x_n = b\}$ we have $V(fg, P) = \sum_{i=1}^{n} |f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1})| = \sum_{i=1}^{n} |f(x_i)(g(x_i) - g(x_{i-1})) + g(x_{i-1})(f(x_i) - f(x_{i-1}))| \le \sum_{i=1}^{n} |f(x_i)||g(x_i) - g(x_{i-1})| + |g(x_{i-1})||f(x_i) - f(x_{i-1})| \le (\sup_{[a,b]} |f|)V(g, P) + (\sup_{[a,b]} |g|)V(f, P)$

Taking the supremum over all partitions P we get $V(fg) \le (\sup_{[a,b]} |f|)V(g, P) + (\sup_{[a,b]} |g|)V(f, P)$
A function of bounded variation on $[a, b]$ is bounded. Also the sup is finite; since $V(f), V(g)$ are finite by assumption, the rhs is also finite $\implies$ fg has bounded variation $\implies$ the product of 2 funcs of bounded first order variation also has bounded first order variation. qed

3

$$f(x) = \begin{cases} xsin(\frac{1}{x}), & x \in (0, 1] \\ 0, & x = 0 \end{cases}$$

quadratic variation on $[0, 1]$: $V_2(f, [0, 1]) = \sup \sum_{i=1}^{n} (f(x_i) - (f(x_{i-1})))^2$ Where the sup is over all partitions $P = \{0 = x_0 < ... < x_n = 1\}$

Set the zeros $z_n = \dfrac{1}{\pi n}$, so that $f(z_n) = 0$ and the unique extremum in each interval $I_n = [z_{n+1}, z_n]$ occurs at $e_n = \dfrac{1}{\pi n + \frac{\pi}{2}}$ with $|f(e_n)| = \dfrac{1}{\pi n + \frac{\pi}{2}} \le \dfrac{1}{\pi n}$

On I_n the graph goes $0 \to \pm|f(e_n)| \to 0$

For any partition P of I_n the sum of squared increments is maximized by using the points z_{n+1}, e_n, z_n which gives $(f(e_n) - 0)^2 + (0 - f(e_n))^2 = 2|f(e_n)|^2 \ leq \dfrac{2}{(\pi n)^2}$

Hence, the quadratic variation over $\bigcup_{n \ge N} I_n = (0, z_n]$ satisfies $V_2(f, (0, z_n]) \le \sum_{n=N}^{\infty} \dfrac{2}{(\pi n)^2} = \dfrac{2}{\pi^2} \sum_{n=N}^{\infty} \dfrac{1}{n^2} \le \dfrac{2}{\pi^2} \cdot \dfrac{\pi^2}{6} = \dfrac{1}{3}$

On the remaining compact interval $[z_n, 1]$ the function is C^1 so it's Lipshits with a constant L

$\forall$ partition of $[z_n, 1]$ we have $\sum(\Delta f)^2 \le L^2 \sum(\Delta x)^2 \le L^2(1 - z_n)^2$

since $\sum(\Delta x)^2$ is maximized by using a single interval of length $1 - z_n$

Combining $V_2(f, [0, 1]) \le \frac{1}{3} + L^2(1 - z_n)^2 < \infty$ therefore the quadratic variation of $f(x)$ is bounded.

qed

4

We must show that $C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$ satisfies the BSM equation, where $d_1 = \dfrac{ln(\frac{S}{K}) + (r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$, $d_2 = d_1 - \sigma\sqrt{\tau}$
I suggest, $\tau := T - t$
N is the standard normal cdf, $n = N'$
$n(x) = \dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}$ normal density
1) let's calculate $\dfrac{n(d_1)}{n(d_2)} = \dfrac{e^{-d_1^2/2}}{e^{-d_2^2/2}} = e^{-(d_1^2-d_2^2)/2}$
Since $d_2 = d_1 - \sigma\sqrt{\tau}$, we get $d_1^2 - d_2^2 = (d_1 - d_2)(d_1 + d_2) = \sigma\sqrt{\tau}(2d_1 - \sigma\sqrt{\tau})$ So, considering $d_1 = \dfrac{ln(\frac{S}{K}) + (r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$ we multiply both sides by 2 and subtract $\sigma\sqrt{\tau}$ from both sides, so we get
$2d_1 - \sigma\sqrt{\tau} = 2\dfrac{\ln(S/K) + r\tau}{\sigma\sqrt{\tau}}$
So, $d_1^2 - d_2^2 = 2(\ln(S/K) + r\tau)$ and $e^{-(d_1^2-d_2^2)/2} = e^{\ln(S/K)+r\tau} = \dfrac{K}{S}e^{-r\tau}$ and so we get $\boxed{Sn(d_1) = Kn(d_2)e^{-r\tau}}$
Now calculation of the derivatives $(dt = -d\tau)$

$$\frac{\partial C}{\partial S} = N(d_1) + Sn(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r\tau}n(d_2)\frac{\partial d_2}{\partial S} = N(d_1) + Sn(d_1)\frac{\partial d_1}{\partial S} - Sn(d_1)\frac{\partial d_1}{\partial S} = N(d_1)$$

$$\frac{\partial C}{\partial \tau} = Sn(d_1)\frac{\partial d_1}{\partial \tau} + rKe^{-r\tau}N(d_2) - Ke^{-r\tau}n(d_2)\frac{\partial d_2}{\partial \tau} = -\frac{\partial C}{\partial t}$$

$$\frac{\partial^2 C}{\partial S^2} = n(d_1)\frac{\partial d_1}{\partial S}$$

So, we get $\dfrac{\partial C}{\partial t} + \dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2 C}{\partial S^2} + rS\dfrac{\partial C}{\partial S} - rC = -\dfrac{\partial C}{\partial \tau} + \dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2 C}{\partial S^2} + rS\dfrac{\partial C}{\partial S} - r(SN(d_1) - Ke^{-r\tau}N(d_2)) = -Sn(d_1)\dfrac{\partial d_1}{\partial \tau} - rKe^{-r\tau}N(d_2) + Ke^{-r\tau}n(d_2)\dfrac{\partial d_2}{\partial \tau} + \dfrac{1}{2}\sigma^2 S^2 n(d_1)\dfrac{\partial d_1}{\partial S} + rSN(d_1) - rSN(d_1) + rKe^{-r\tau}N(d_2)$
We know that $\dfrac{\partial d_2}{\partial \tau} = \dfrac{\partial d_1}{\partial \tau} - \dfrac{\sigma}{2\sqrt{\tau}}$
and also $\dfrac{\partial d_1}{\partial S} = \dfrac{\partial d_2}{\partial S}$
So, having all of that and using the boxed equation we get the lhs to equal 0. qed