Semiring Algebraic Structure for Metarouting with Automatic Tunneling

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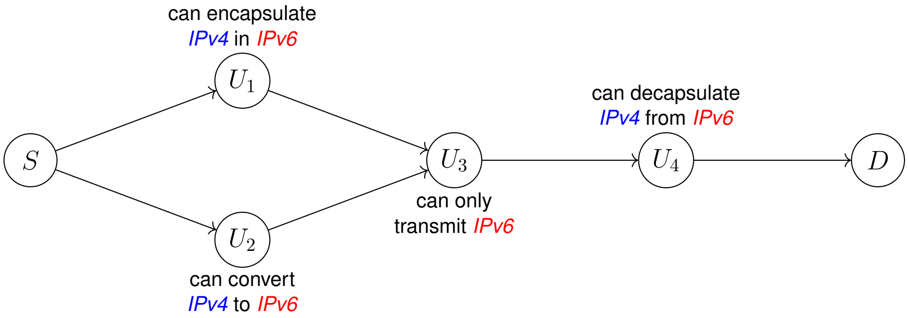
Outline

Routing with Automatic Tunneling



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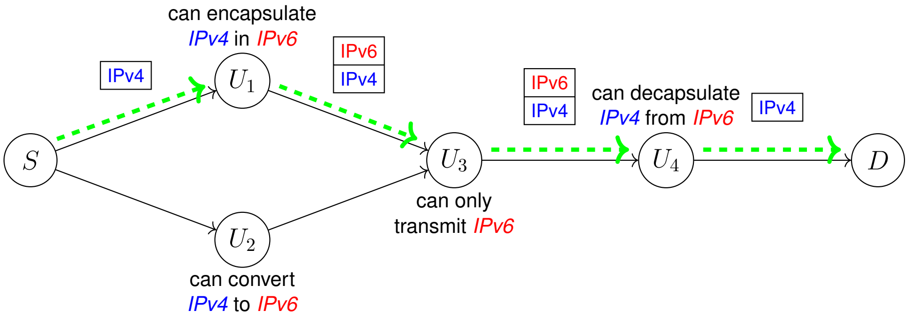
Routing with Automatic Tunneling



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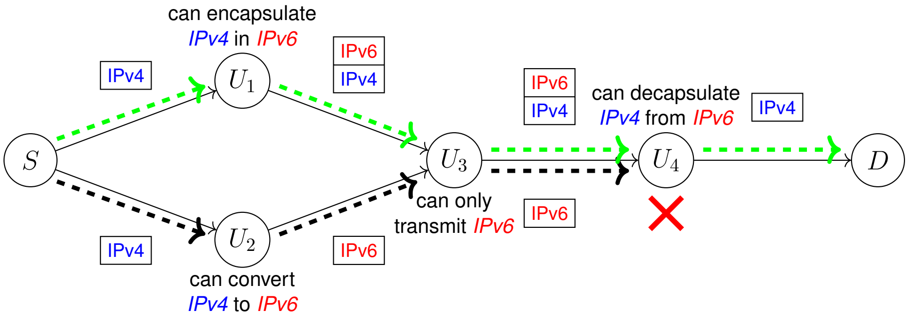

Example of a network encompassing IPv4 and IPv6 protocols.

- The top path (in green) is valid with a tunnel from U_1 to U_4.
- The bottom path (in black) is invalid.

Routing with Automatic Tunneling

- The computation of paths (routing) in networks with tunnels is not yet fully automated. (*e.g.*, Teredo, 6over4, 6to4, ISATAP, etc.)
- The path computation in multi-protocol networks with conversions and encapsulations, cannot be performed by using classical path computation algorithms. (*e.g.*, Dijkstra, Bellman-Ford, etc.)
- There are a few path computation algorithms which take into account encapsulations and conversions. (*e.g.*, Stack-Vector, etc.)
- The valid path problem under bandwidth constraints is NP-hard.

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$$\text{Routing Algebra} = (\underbrace{S}_{\text{set of routes}}, \underbrace{\oplus}_{\text{route selection}}, \underbrace{\otimes}_{\text{route extension}}, \underbrace{\overline{0}}_{\text{initial route}}, \underbrace{\overline{1}}_{\text{invalid route}})$$

- Separates the routing data and algorithm.
  (route exchange and route update.)

- Study the properties of routing protocols.
  (convergence and the set of optimal paths.)

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Our contribution is to generalize the semiring structure for modeling the routing problem with automatic tunneling and to study some algebraic properties of convergence.

Network Model

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- A set of adaptation functions, $\mathcal{F}$ of type:
  - $(x \rightarrow x)$ is the retransmission of the protocol x
  - $(x \rightarrow y)$ is the conversion of the protocol x to y
  - $(x \rightarrow xy)$ is the encapsulation of the protocol x in y
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- A weight function, $\omega : \mathcal{V} \times \mathcal{F} \times \mathcal{V} \to \mathcal{R}^+$

Protocol Stack and Functions Composition

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- The set of all stacks starting with the sub-stack xy is $H_{xy} \in \mathcal{H}$.
- An adaptation function $f = (x \rightarrow xy)$ is defined as $f : H_x \rightarrow H_{xy}$.

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And

$$H''_a = \left\{ h \in H_a \mid f(h) \in H'_a \right\} \qquad H''_b = \left\{ f'(h) \mid h \in (H_b \cap H'_a) \right\}$$

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- The set of all identity functions (classical retransmission) is:

$$\mathcal{F}_{id} = \Big\{ x \to x, y \to y, \ldots \Big\}$$

Path Validity

- A path $p = h_i v_i f_i v_{i+1} f_{i+1} \dots v_{j-1} f_{j-1} h_j v_j$ from v_i to v_j is a mixed sequence of nodes and adaptation functions and it is valid iff:

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- The weight of a valid path p from v_i to v_j is the sum of the weights of its links and its adaptation functions,

$$\omega(p) \overset{def}{=} \sum_{k=i}^{j-1} \omega(v_k, f_i, v_{k+1})$$

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Problem	S	$\oplus$	$\otimes$	$\overline{0}$	$\overline{1}$
shortest paths (SM_{sp})	$\mathbb{N}^{\infty}$	min	+	$+\infty$	0
widest paths	$\mathbb{N}^{\infty}$	max	min	0	$+\infty$
most reliable paths	$[0, 1]$	max	$\times$	0	1
accessible paths	{0,1}	max	min	0	1

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If $\oplus$ is idempotent then the relation $\leq_\oplus$ is a partial order over S:

$$(a \leq_\oplus b) \equiv (a = a \oplus b)$$
$$(a <_\oplus b) \equiv (a = a \oplus b \neq b)$$

Note that this order is total when the operation $\oplus$ is selective

Matrix Semirings

Given a semiring $(S, \oplus, \otimes, \overline{0}, \overline{1})$, the semiring of $n \times n$ matrix is $\big(\mathbf{M}_n(S), \oplus, \otimes, \mathbf{N}, \mathbf{I}\big)$, where:

- $\mathbf{N}_{i,j} = \overline{0}$
- $\mathbf{I}_{i,j} = \begin{cases} \overline{1} & \text{if } (i = j) \\ \overline{0} & \text{otherwise} \end{cases}$

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And for any two matrices $\mathbf{X}, \mathbf{Y} \in \mathbf{M}_n(S)$, the two operations $\oplus$ and $\otimes$ are defined as follow:

- $(\mathbf{X} \oplus \mathbf{Y})_{i,j} = \mathbf{X}_{i,j} \oplus \mathbf{Y}_{i,j}$
- $(\mathbf{X} \otimes \mathbf{Y})_{i,j} = \bigoplus_{k=1}^{n} \mathbf{X}_{i,k} \otimes \mathbf{Y}_{k,j}$

The General Path Computation Problem

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- The weight of a path $p = v_0, v_1, v_2, ..., v_{k-1}, v_k$ is:

$$\omega(p) \;=\; \omega_{0,1} \otimes \omega_{1,2} \otimes ... \otimes \omega_{k-1,k} \;=\; \bigotimes_{i=0}^{k-1} \omega_{i,i+1}$$

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- The weighted adjacency matrix $\mathbf{A} \in \mathbf{M}_n(S)$ of the graph $\mathcal{G}$ is:

$$\mathbf{A}_{i,j} = \begin{cases} \omega_{i,j} & \text{if } e_{i,j} \in \mathcal{E} \\ \overline{0} & \text{otherwise} \end{cases}$$

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- The power of a matrix $\mathbf{A} \in \mathbf{M}_n(S)$ is:

$$\mathbf{A}^{\mathbf{k}} = \begin{cases} \mathbf{I} & \text{if } k = 0 \\ \mathbf{A} \otimes \mathbf{A}^{\mathbf{k-1}} & \text{otherwise} \end{cases}$$

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- The weight of an optimal path from v_i to v_j of size k is:

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- The global optimal solution for the generalized path computation problem consists in finding (if it exists) the matrix $\mathbf{A}^{*}$,

$$\mathbf{A}^{*} = \bigoplus_{k \geq 0} A^{(k)} = \bigoplus_{p \in \mathcal{P}^{(k)}} \omega(p)$$

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$$SM_{vp} = \left(\mathcal{P}(\hat{\mathcal{F}}), \cup, \odot, \emptyset, \mathcal{F}_{id}\right) \qquad (valid\ paths\ semiring)$$

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Where:

- $\mathcal{P}(\hat{\mathcal{F}})$ is the power set of all functions closed under compositions.
- $\odot$ is the composition operation of subsets in $\mathcal{P}(\hat{\mathcal{F}})$:

$$\hat{F}_1 \odot \hat{F}_2 = \left\{\hat{f}_1 \odot \hat{f}_2 \,|\, \hat{f}_1 \in \hat{F}_1 \text{ and } \hat{f}_2 \in \hat{F}_2\right\}$$

- $\mathcal{F}_{id}$ is the set of identity adaptation functions $\{x \to x, y \to y, \dots\}$.

Valid Shortest Paths Semiring

- The valid shortest paths semiring is the semi-directed product:

$$SM_{vsp} = SM_{vp} \rtimes SM_{sp} = \Big(\mathcal{P}\big(\hat{\mathcal{F}} \times \mathbb{N}^{\infty}\big), \underset{min}{\cup}, (\odot \times +), \emptyset, (\mathcal{F}_{id} \times 0)\Big)$$

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$$SM_{vsp} = SM_{vp} \rtimes SM_{sp} = \Big(\mathcal{P}\big(\hat{\mathcal{F}} \times \mathbb{N}^{\infty}\big), \underset{min}{\cup}, (\odot \times +), \emptyset, (\mathcal{F}_{id} \times 0)\Big)$$

- The extension operator $(\odot \times +)$ is a direct product.
- The selection operator $(\underset{min}{\cup})$ is an union-min product:

$$S_1 \underset{min}{\cup} S_2 = \Big\{(\hat{f}, \omega_{\hat{f}}) \mid (1) \vee (2)\Big\} \qquad \forall S_1, S_2 \in \mathcal{P}\big(\hat{\mathcal{F}} \times \mathbb{N}^{\infty}\big)$$

$$(\hat{f}, \omega_{\hat{f}}) \in S_1 \wedge \forall\,(\hat{g}, \omega_{\hat{g}}) \in S_2,\ (\hat{f} = \hat{g}) \Rightarrow \omega_{\hat{f}} = min[\omega_{\hat{f}}, \omega_{\hat{g}}] \qquad (1)$$

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- The partial order relation over elements of $\mathcal{P}\big(\hat{\mathcal{F}} \times \mathbb{N}^{\infty}\big)$:

$$S_1 \subseteq S_2 \equiv \forall\, (\hat{f}, \omega_{\hat{f}}) \in S_1 \Rightarrow \exists\, (\hat{g}, \omega_{\hat{g}}) \in S_2,\ (\hat{f} = \hat{g}) \wedge (\omega_{\hat{f}} \leq \omega_{\hat{g}})$$

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Proposition

The direct product operator $(\odot \times +)$ over the power set $\mathcal{P}(\hat{\mathcal{F}} \times \mathbb{N}^{\infty})$ is isotonic and not monotonic.

Iterative Convergence Theorem

Theorem [B. A. Carré, 71]

In a free network (without absorbing circuits) we have:

$$\mathbf{A}^* = \mathbf{A}^{(n-1)} = \mathbf{I} \oplus \mathbf{A} \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^{n-1}$$

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- A multilayer circuit is a valid path $p = h_i v_i f_i \ldots f_{j-1} h_j v_j$ where:
    - The node v_i is the same node v_j, *i.e.*, $v_i = v_j$
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- A multilayer elementary path is a valid path in which its circuits (if it exists) are non multilayer circuits.
- A free multilayer network is a network in which all of its multilayer circuits have positive weights.

Iterative Convergence Theorem

Theorem

In a free multilayer network $\mathcal{N}$ we have:

$$\mathbf{A}^* = \mathbf{A}^{(k)} = \mathbf{I} \oplus \mathbf{A} \oplus \mathbf{A}^2 \oplus \cdots \oplus \mathbf{A}^k$$

Where k is the maximum length of the multilayer elementary paths in $\mathcal{N}$, and it is equal to $2^{(\lambda+1)\lambda^2 n^2} - 1$.

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Proposition [M. L. Lamali, S. Lassourreuille, S. Kunne, and J. Cohen, 19]

For any multilayer network $\mathcal{N}$, the valid shortest path (if any) between two nodes is upper bounded by $2^{(\lambda+1)\lambda^2 n^2}$.

Conclusion and Future Work

- New routing algebra based on semirings for path computation with automatic tunneling: an isotonic and non-monotonic algebra with a partial order.
- A generalization of the iterative convergence theorem for the optimal solution of the valid shortest paths problem.

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- New routing algebra based on semirings for path computation with automatic tunneling: an isotonic and non-monotonic algebra with a partial order.
- A generalization of the iterative convergence theorem for the optimal solution of the valid shortest paths problem.
- The adaptation of the other existing algebraic structures (algebra of endomorphisms and Sobrinho's algebra) to the valid shortest paths problem.
- The generalization of the asynchronous convergence theorem for the stack-vector protocol.

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