8. Random variables and distributions

Principles of Data Science with R

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PSTAT 10

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We did:

- Probability
  - Definitions
  - Rules of Probability: Addition, complement, multiplication
  - Conditional Probability
  - Mutually exclusive events
  - Independent events

Next we will see...

- Random Variables: Discrete or Continuous
- Discrete Random variables (By hand and using R)
    - P.m.f
    - Expectation
    - Variance
    - C.d.f

Mutually exclusive and independent events

- Two events, A and B, are independent if the occurrence of one event does not change the probability of the occurrence of the other event
  - $P(A|B) = P(A)$

- Two events are mutually exclusive if they cannot occur together. ($P(A \cap B) = 0$)
  - $P(A|B) = 0$

For events A and B

- Addition rule, OR rule: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  - For mutually exclusive events : $P(A \cup B) = P(A) + P(B)$
- Multiplication rule, AND rule: $P(B \cap A) = P(A)P(B|A)$
  - For independent events: $P(A \cap B) = P(A)P(B)$
- Law of total probability: $P(A) = P(A \text{ and } B) + P(A \text{ and } B^c)$

Random variable

A RV maps outcomes in sample space S to numbers.

Outcomes arise by chance, so a random variable's value is also dependent on chance.

- We use a capital letter, like X, to denote a random variable
- The values of a random variable are denoted with a lowercase letter, in this case x
- For example, $P(X = x)$

Two types of random variables

- **Discrete RV**, where X can take only a finite (or countably infinite) number of values
  - 'things you count'
  - ex.: number of heads in 4 flips, cars that enter in a parking lot in a given period of time, etc.
- **Continuous RV**, where X can take any value on the real line in a bounded or unbounded interval.
  - 'things you measure'
  - ex.: height of PSTAT 10 students, time till the next bus arrives

Discrete Random variable

Example Flip a fair coin once

$$S = \{H, T\}$$

$$X(H) = 1, X(T) = 0$$

,

$$X = \begin{cases} 1 & \text{if coin lands heads} \\ 0 & \text{if coin lands tails} \end{cases}$$

In words, X = number of heads in one coin flip

Example: Flipping two coins

$$S = \{HH, HT, TH, TT\}$$

$$X(HH) = 2, X(HT) = 1, X(TH) = 1, X(TT) = 0$$

In words, $X =$ number of heads in two independent coin flips

- RV's make it easier to describe events succinctly. For eg, Flipping two coins and getting at most one head can be succinctly written as $X \leq 1$ instead of 'getting either no or one head in two coin flips'

Discrete Probability Distribution or p.m.f

A discrete probability distribution, also known as a **probability mass function** or p.m.f, consists of all of the values a random variable can take, along with the corresponding probabilities of taking those values.

Example Flip a fair coin once

Outcome	H	T
Values: $X = x$	1	0
Probability: $P(X = x)$	1/2	1/2

- Note: The sum of these probabilities must be equal to 1.

Toss a coin twice and record the number of heads.

Outcome	TT	HT	HT	HH
# of Heads	0	1	1	2
Probability	0.25	0.25	0.25	0.25

The resulting pmf is the table

x	0	1	2
$P(X = x)$	0.25	0.5	0.25

$$\sum_{\text{all } x} P(X = x) = P(X = 0) + P(X = 1) + P(X = 2) = 1$$

Expectation

- We are often interested in the average outcome of a random variable.
- We call this the **expected value** (mean, average value or expectation),
- This is the average of all possible values of X, weighted by their probabilities.

Given a random variable X with probability mass function (p.m.f.) $p(x) = P(X = x)$,

Expected Value of X is

$$E(X) = \sum_{i=1}^{k} x_i p(x_i) = \sum_{i=1}^{k} x_i P(X = x_i)$$

Your turn

In a game of cards you win \$1 if you draw a heart, \$5 if you draw an ace (including the ace of hearts), \$10 if you draw the king of spades and nothing for any other card you draw. Write the probability mass function for your winnings, and calculate your expected winning.

- (What is the experiment?)
- What are the outcomes?
- What is the random variable? it's values and probabilities?

Event	X	$P(X)$	$X\ P(X)$
Heart (not ace)	1	$\frac{12}{52}$	$\frac{12}{52}$
Ace	5	$\frac{4}{52}$	$\frac{20}{52}$
King of spades	10	$\frac{1}{52}$	$\frac{10}{52}$
All else	0	$\frac{35}{52}$	0
Total			$E(X) = \frac{42}{52} \approx 0.81$

On average, you expect to make 0.81 in this game.

Note Expected value doesn't need to be one of the values that the variable can take.

Using R

```r
x <- c(0, 1, 5, 10)
p <- c(35/52, 12/52, 4/52, 1/52)
ex <- sum(x*p)
ex
```

```
## [1] 0.8076923
```

Your turn

Simulate a sample of size 100 from the probability distribution of winnings from this game and view the results

- `sample()`: what values and with what probability?
- What type of plot? (hint: Data falls into categories)

```r
values <- c(0,1,5,10)
p <- c(35/52, 12/52, 4/52, 1/52 )
p
```

```
## [1] 0.67307692 0.23076923 0.07692308 0.01923077
```

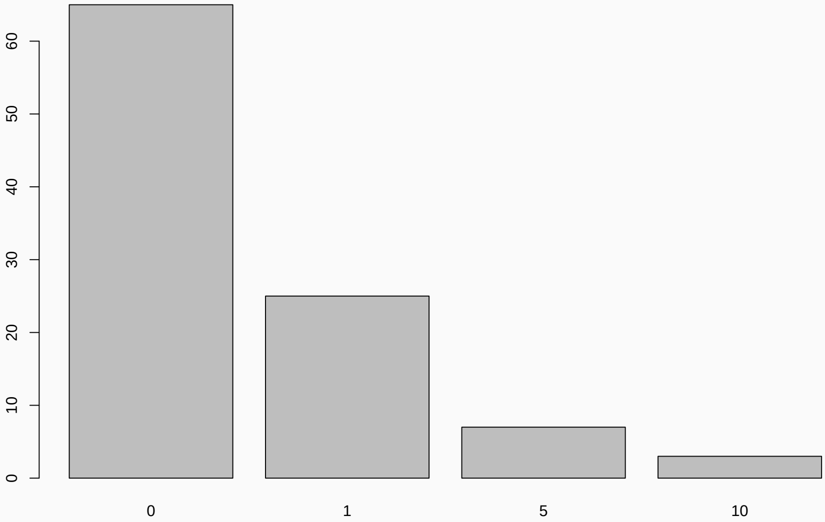
```r
x <- sample( values, 100, replace = TRUE, prob = p)
table(x)
```

```
## x
##  0  1  5 10
## 65 25  7  3
```

```r
prop.table(table(x))
```

```
## x
##    0    1    5   10
## 0.65 0.25 0.07 0.03
```

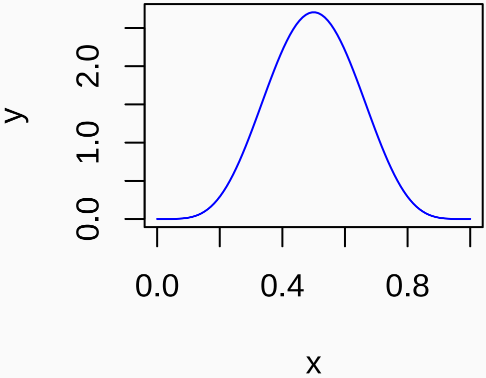
```
barplot(table(x))
```



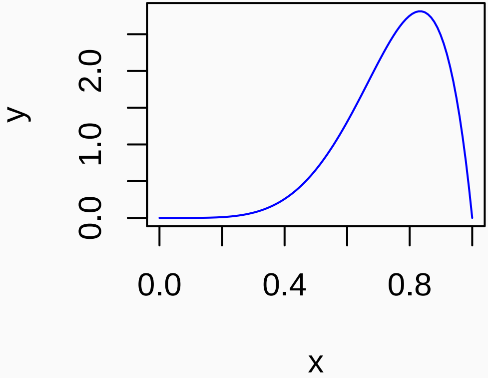
- This is a right skewed distribution since it has a long tail to the right.

Common Distribution Shapes

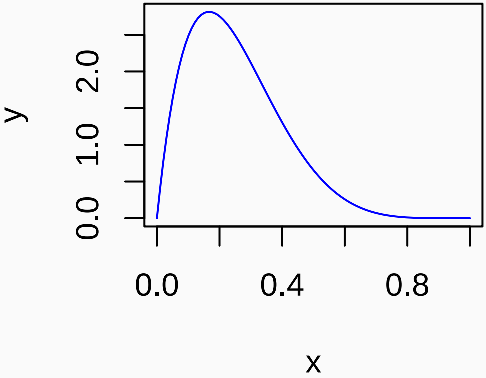
- **Symmetric**:



- **Left-Skewed**:



- **Right-Skewed**:



Variability

We are also often interested in the variability in the values of a random variable.

$$\sigma^2 = Var(X) = \sum_{i=1}^{k} (x_i - E(X))^2 P(X = x_i)$$

$$\sigma = SD(X) = \sqrt{Var(X)}$$

Variability of a discrete random variable

For the previous card game example, how much is the variability in the winnings?

X	$P(X)$	$X\ P(X)$	$(X-E(X))^2$	$(X-E(X))^2 P(X$
1	$\frac{12}{52}$	$1 \cdot \frac{12}{52} = \frac{12}{52}$	$(1-0.81)^2 = 0.0361$	$\frac{12}{52} \cdot 0.0361 = 0.0$
5	$\frac{4}{52}$	$5 \cdot \frac{4}{52} = \frac{20}{52}$	$(5-0.81)^2 = 17.5561$	$\frac{4}{52} \cdot 17.5561 = 1.$
10	$\frac{1}{52}$	$10 \cdot \frac{1}{52} = \frac{10}{52}$	$(10-0.81)^2 = 84.4561$	$\frac{1}{52} \cdot 84.0889 = 1.$
0	$\frac{35}{52}$	$0 \cdot \frac{35}{52} = 0$	$(0-0.81)^2 = 0.6561$	$\frac{35}{52} \cdot 0.6561 = 0.4$
		$E(X) = 0.81$		$V(X) = 3.4246$
				$SD(X) = \sqrt{3.424}$
				$SD(X) = 1.85$

Using R

```
x <- c(0, 1, 5, 10)
p <- c(35/52, 12/52, 4/52, 1/52 )
ex <- sum(x*p)
ex
```

```
## [1] 0.8076923
```

```
varx <-  sum(((x-ex)^2)*p)
varx
```

```
## [1] 3.424556
```

```
sdx <- sqrt(varx)
sdx
```

```
## [1] 1.850556
```

Cumulative distribution function (c.d.f)

The cumulative distribution function (or CDF) ,$F(k)$ is the probability that the random variable X is at most, as big as some particular value k ie $P(X \leq k)$

For a discrete random variable X, this is given by,

$$F(k) = P(X \leq k) = \sum_{x \leq k} P(X = x)$$

- `cumsum()` function in R executes a cumulative summation element by element

For the previous card game example,

x	0	1	5	10
$P(X = x)$	$35/52$	$12/52$	$4/52$	$1/52$
$P(X \leq x)$	$35/52$	$35/52 + 12/52 = 47/52$	$47/52 + 4/52 = 51/52$	$51/52 + 1/52 = 52/52 = 1$

```r
values <- c(0,1,5,10)
p <- c(35/52, 12/52, 4/52, 1/52 )
cp <- cumsum(p)
cp
```

```
## [1] 0.6730769 0.9038462 0.9807692 1.0000000
```

- What is $P(X \leq 7)$? ie F(7)

```r
cp[3]
```

```
## [1] 0.9807692
```

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we did:

- Random Variables: Discrete or Continuous
- Discrete Random variables (By hand and using R)
    - P.m.f
    - E(X)
    - V(X)
    - C.d.f