The Levi-Civita regularizations

by Alain Albouy

Abstract. When approaching a binary collision of the *n*-body problem, the gravitational acceleration and the velocity tend to infinity. The Levi-Civita regularization transforms this singular motion into a smooth motion where all the variables remain finite, and which continues after the collision. The modern authors refers on this matter to an article of Levi-Civita published in 1920 in Acta Math. They further write that this regularization is restricted to the planar problem, and that one had to wait for the year 1965 and the publication by Kustaanheimo & Stiefel to see a regularization in space. This is not true. Levi-Civita regularizes the problem in space. His planar regularization dates back to 1904. He explains in 1916, then in 1920, that he finally found a 3D regularization. His formulas are simpler than those of Kustaanheimo & Stiefel. They were ignored by all the authors, with these notable exceptions (in turn ignored): Wintner, in his 1941 book, §415, Siegel, in his 1956 book, §7 and Moser, in his 1970 article, p. 615.

Abstract (continuation). Levi-Civita's construction presents a flaw that we will explain and correct. In order to motivate this correction, we ask the question: does there exist a regularization that produces a smooth flow at the neighborhood of a parabolic binary collision?
I thank Giovanni Gronchi for informing me of the subject treated by Levi-Civita in 1920.

1817. A.M. Legendre, *Exercices de calcul intégral sur divers ordres de transcendantes et sur les quadratures*, t.2, Paris, 6e partie
1889. G. Darboux, Remarque sur la Communication précédente, *Comptes Rendus Acad. Sci. Paris*, 108, pp. 449–450
1906. T. Levi-Civita, Sur la résolution qualitative du problème restreint des trois corps, Acta Mathematica 30.1, pp. 305-327
1907. K.F. Sundman, Recherches sur le problème des trois corps, Acta Societatis Scientiarum Fennicæ, 34-6, pp. I–II–1–43
1909. K.F. Sundman, Nouvelles recherches sur le problème des trois corps, Acta Societatis Scientiarum Fennicæ, 35-9, pp. 1–27
1912. K.F. Sundman, Mémoire sur le problème des trois corps, Acta Mathematica, 36, pp. 105–179

1916. T. Levi-Civita, Sopra due trasformazioni canoniche desunte dal moto parabolico, Atti Reale Acc. Lincei, Rendic., s. 5, v. 25

1920. T. Levi-Civita, Sur la régularisation du problème des trois corps, Acta Mathematica, 42, pp. 99–144

1924. T. Levi-Civita, Regolarizzazione del problema dei tre corpi e sua portata, *Questioni di Meccanica Classica e Relativistica*, 1–38

1935. V. Fock, Zur Theorie des Wasserstoffatoms, Zeits. Phys. 98, pp. 145–154

1965. P. Kustaanheimo, E. Stiefel, Perturbation theory of Kepler motion based on spinor regularization, Journal für die reine und angewandte Mathematik, 218, pp. 204–219

1966. H. Bacry, H. Ruegg, J.-M. Souriau, Dynamical groups and spherical potentials in classical mechanics, Co. math. phy., 3, 323–333

1968. G. Györgyi, Kepler's Equation, Fock Variables, Bacry's Generators and Dirac Brackets, Nuovo Cimento, 53A, pp. 717–735

1970. J. Moser, Regularization of Kepler's problem and the averaging method on a manifold, Commun. pure and applied math., 23, 609–636

1981. R. McGehee, Double collisions for a classical particle system with nongravitational interactions, Commen. math. helv., 56, 524–557

Discussion

Levi-Civita obtained in 1904 a regularizing transformation of the planar restricted 3-body problem based on the map $\mathbb{C} \to \mathbb{C}$, $z \mapsto \sqrt{z}$, where the complex plane is the plane of motion. Sundman's results in 1909 indicated that a similar transformation should exist without such restrictions, i.e., for the full 3-body problem in space. Levi-Civita published such a transformation in 1916. It is very different from the previous one, but both have some relation with the Darboux inversion (1889).

A recent work with Andreas Knauf indicates that McGehee power laws may be treated as the Newton law. Most of the above references were collected with Lei Zhao and are described in:
2022. A. Albouy, L. Zhao, Darboux inversions of the Kepler problem, Regular and Chaotic Dynamics, 27, 253–280

Levi-Civita insisted that he did not find this transformation easily. I will try to explain the difficulties, helped by a remark by J. Moser about this transformation (1970).

Our terminology

Consider for example the 3-body problem. We will speak only of the regularization of the binary collisions. Here is our terminology.

A binary collision is a singularity of the dynamics: some velocity tends to infinity when $t \to t_0$. We want a **singular map** such that that the **image dynamics is smooth**. In particular the trajectory passes the binary collision smoothly. Optional requirements are:

- nearby initial conditions are sent onto nearby conditions,
- in a neighborhood of the collisions, the map is smooth except at the collisions,
- the **image dynamics is ODE**.

We mean that locally the image dynamics is the dynamics of a smooth Ordinary Differential Equation. This requirement is not in Sundman's work. We can state Sundman's result as:

Theorem. The change of time $t \to s$ with $ds = Udt$, where U is the potential, is such that the positions (and new velocities) of the 3 bodies are analytic functions of s when passing a binary collision.

Indeed, the use of the 3-body potential U is due to Levi-Civita.

Three “conceptually irrelevant” restrictions

Sundman’s regularization is just a change of time. But does he prove that nearby initial conditions are sent on nearby conditions? He adds this claim to his work in 1912, without proof. A deep analysis on this claim may be found in McGehee’s paper (1981), who changed the Newtonian potential and found systems which are **branch regularizable** but not **block regularizable**. As many authors, McGehee

- changes the time,
- restricts his blocks to an energy hypersurface,
- restricts his study to the planar case.

These three “restrictions” seem irrelevant. Note that

- Barcy, Ruegg and Souriau in 1966 regularize the 2-body problem without changing the time (the energy is fixed, $\neq 0$).
- Levi-Civita 1904 regularization does not need a fixed energy (but yes a nonzero energy).
- Levi-Civita 1920 does not have the planar restriction (but has the energy restriction).

1915. La restrizione che si tratti di moto piano sembra concettualmente irrilevante, e si è tratti a presumere che analoga regolarizzazione possa raggiungersi anche per il problema generale. Ho incontrato finora qualche difficoltà nella costruzione delle trasformazioni regolarizzanti; ma non dispero di superarla con studio ulteriore.
1916. In questa fiducia, dopo aver infruttuosamente saggiato parecchie trasformazioni di coordinate, pensai di ricorrere ad uno spediente di calcolo alquanto più penetrante, cioè ad una trasformazione canonica di contatto (anzichè semplicemente puntuale), la quale abbia carattere regolarizzante per il problema elementare dei due corpi.

Scopo principale della presente Nota è la deduzione di questa trasformazione dai moti centrali di tipo parabolico e l'analisi delle sue eleganti proprietà geometrico-cinematiche.

Mostrerò prossimamente come essa conduca alla desiderata regolarizzazione canonica del problema dei tre corpi. Qui ne ho tratto occasione per far conoscere una seconda trasformazione canonica, che introduce *elementi osculatori parabolici* riattaccandosi ad un'altra mia ricerca.

1915. The restriction to the planar motion seems conceptually irrelevant, and one is led to believe that a similar regularization may be obtained for the general problem as well. I met some difficulty in constructing the regularizing transformations, but I am confident I will be successful after more study.

$$q \mapsto |p|^2 q - 2\langle p, q\rangle p, \qquad p \mapsto \frac{p}{|p|^2}.$$

1916. In this hope, after unsuccessfully trying many coordinate transformations, I came to use a much more effective computational trick, consisting of a contact canonical transformation (rather than a point transformation), which is also regularizing the elementary two-body problem. The main purpose of this Note is to deduce this transformation from the parabolic central motions and to analyse its elegant geometric and kinematic features. I shall soon show how this leads to the desired canonical regularization of the three-body problem. Here I have taken the opportunity to present a second canonical transformation introducing the *parabolic osculating elements*, thus connecting to another research of mine.

$|y|^{-1} G_y = H_y$, $|y|^{-1} G_x = H_x$ with

$$H = |y|^{-1} G - \tfrac{1}{2} = |y|^{-1} (\sqrt{2F} - 1) - \tfrac{1}{2} = \tfrac{1}{2} |x|^2 - \frac{1}{|y|}, \tag{2.9}$$

the resulting system is Hamiltonian, with H as its Hamilton function, on $H = -\frac{1}{2}$. Therefore, if we set $p = -x$, $q = y$, the system goes over into the Kepler problem on the energy surface

$$H = \tfrac{1}{2} |p|^2 - \frac{1}{|q|} = -\tfrac{1}{2}.$$

d. Summarizing, we have shown that the transformation (2.2), (2.2*), together with (2.8), maps the unit tangent bundle $T_1(\hat{S}^n)$ into the $2n$-dimensional phase space and the geodesic circles into the Kepler orbits on the energy surface $H = -\frac{1}{2}$. So far, we have excluded the north pole $\xi = e_0 = (1, 0, \cdots, 0)$. Now we include this point thereby compactifying the energy surface. The so compactified energy surface is equivalent to the unit tangent bundle and the geodesics through the north pole are transformed into collision orbits.*

If one wants to describe the flow near the collision states, $q = 0$, which correspond to the north pole, one merely has to employ a reflection:

$$\xi_0 \to -\xi_0, \qquad \eta_0 \to -\eta_0,$$
$$\xi_k \to \xi_k, \qquad \eta_k \to \eta_k, \qquad\qquad k = 1, 2, \cdots, n,$$

taking the north pole into the south pole. Under our transformation, this mapping goes over into

$$q \to |p|^2 q - 2(p, q)p, \qquad p \to \frac{p}{|p|^2}, \tag{2.10}$$

a transformation already used in the Sundman theory. Since the above reflection preserves the geodesic flow, this latter transformation maps Kepler orbits into Kepler orbits. Moreover, the collision states $|p| = \infty$, $q = 0$ are transformed into $p = 0$, $|q| = 2$ (since $H = -\frac{1}{2}$); hence the collisions can easily be investigated.

So far we have discussed the equivalence of the Kepler problem with the geodesic flow only for $H = -\frac{1}{2}$. The general case of a negative energy surface $H = -1/2\rho^2$ is easily reduced to this case by a scaling of variables according to $q \to \rho^2 q$, $p \to \rho^{-1} p$, $t \to \rho^3 t$.

* *Added in proof:* The transformation of the Kepler problem to the geodesic flow on the unit sphere is ultimately related to Fock's treatment of the quantum theoretical problem for the hydrogen atom (V. Fock, Zeitschrift für Physik, 98, 1935, p. 145). He also applies the stereographic projection to the momentum variables to transform the Schrödinger equation and shows the invariance of the resulting equation under the 4-dimensional rotation group O_4. In our case this corresponds to the action of O_4 on the geodesic flow on S^3. For this reference I am indebted to Professor Asim O. Barut.

The Second Levi-Civita regularization

This page by Jürgen Moser in 1970 gives some astonishing information. The stereographic projection by Fock in 1935, in a paper about the foundations of quantum mechanics, suggests exactly the same formulas as the second Levi-Civita regularization, namely

$$q \mapsto |p|^2 q - 2\langle p, q\rangle p, \qquad p \mapsto \frac{p}{|p|^2}.$$

There is a difference in the energy domain: Fock cannot be understood for zero energy, while the construction by Levi-Civita works for any energy.

The formulas themselves are astonishing. If we differentiate the second formula in the direction of q, we get the first formula up to a factor. It is strange to differentiate the velocity with respect to the position. Or maybe, we indeed differentiate with respect to the acceleration which central, i.e., proportional to the position.

Let us look at the series and see the flaw

In the Kepler problem on the line $\mathrm{O}x$, $x > 0$, $\ddot{x} = -x^{-2}$, $\dot{x} = v$, $2H = v^2 - 2x^{-1}$. We have

$$x = \sqrt[3]{\frac{9}{2}}\; t^{2/3} + \frac{3\sqrt[3]{6}}{10} H t^{4/3} - \frac{27}{350} H^2 t^2 + \cdots$$

$$v = \sqrt[3]{\frac{4}{3}}\; t^{-1/3} + \frac{2\sqrt[3]{6}}{5} H t^{1/3} - \frac{27}{175} H^2 t + \cdots$$

The even powers of $t^{1/3}$ in x show the extension of the motion by bouncing.

The one-dimensional problem gives us two simple options: either we take $\sqrt{x}$ and change time to $s = t^{1/3}$ or we begin with changing v into $1/v$ in order to deal with $v \to \infty$. Extending these options we get both Levi-Civita regularizations, 1906 and 1920.

However, since H does not appear at the first order of x, these maps send all the collisions with same direction but different energy to the same state of the regularized system.

What was difficult for Levi-Civita? What is difficult?

We want an image dynamics that is smooth and ODE. This is also what Levi-Civita wants in 1920 but I understand that he does not reach this goal because of the above objection. But the regularization is ODE on a given energy level. If we fix the energy of the 3 bodies, this fixes the energy of the binary at the collision. Any sign of the energy may happen. **The only regularization I know which is ODE at the neighborhood of a parabolic collision is in Andreas Knauf's book of 2019**. This is a regularization without change of time.

To fix the energy is usually considered as a requirement for having symplectic (or contact) ODE. Indeed, if we don't fix the energy we don't even have a smooth ODE as the image of the map.

How to fix the flaw

$$A = |p|^2 q - 2\langle p, q\rangle p, \quad B = p/|p|^2.$$

Comme $E = q/r + \langle p, q\rangle p - |p|^2 q$, $2E + A = -2Hq$. En $q = 0$, qui entraîne $|E| = 1$, on a $A = -2E$, $B = 0$, pour toute énergie, donc il n'y a pas d'équation différentielle en A, B, parce que pas d'unicité.

On ajoute la variable H qui satisfait $H = |B|^{-2}(1/2 - |A|^{-1})$ d'après l'expression $|A| = |p|^2 r$. On préfère écrire cette contrainte sous la forme $2H|B|^2 = 1 - 2|A|^{-1}$. Le gradient en A de cette équation est non nul à la collision. L'équation différentielle est $\dot{A} = -2Hp$, $\dot{B} = -Ar^{-3}|p|^{-4}$ ou par changement de temps

$$A' = -2Hrp = -2H|A|B, \quad B' = -A|A|^{-2}.$$

On rechange le temps autrement en divisant par $|A|$ et on confirme que A décrit une ellipse képlérienne autour de zéro. Il y a en fait échange des foyers. Bref on régularise en échangeant les foyers, et selon une remarque de Tait en 1865, le nouveau temps est l'action de Maupertuis.

The first Levi-Civita regularization

Hooke problem (harmonic oscillator in the plane) and *Kepler problem in the plane* are related by the conformal map, written $z \mapsto z^2$ in the complex coordinate z of the plane. The change of time depends on the position **and on the energy**. The energy of Kepler is restricted to be negative. Energy k of Hooke is sent on energy $h = -1/k$ of Kepler. Here in the figure from V.I. Arnold's book of 1990.

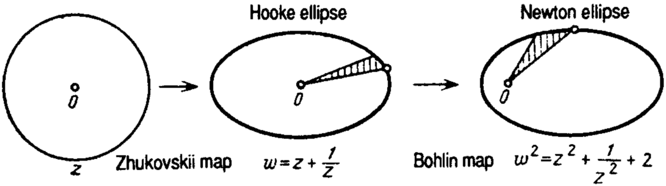

Fig. 35.
The ellipses of Hooke and Newton

The velocity of the image is the initial velocity such that the chosen curve will be described. It is unique, since the force is nonzero (for a geodesic flow only the direction of the velocity is unique).

About the first Levi-Civita regularization

- $z \mapsto z^2$ sending Hooke onto Kepler (Maclaurin 1742)
- $z \mapsto z^n$ sending a central force onto the dual one (Maclaurin)
- $z \mapsto f(z)$, general conformal map in the plane (Goursat 1889)
- Take $(\mathcal{M}, g, U)$, change g into Ug, U into $\frac{1}{U}$ (Darboux 1889)
- $z \mapsto z^2$ to regularize the planar R3BP (Levi-Civita 1904)
- change of time $\mathrm{d}s = U\mathrm{d}t$ of Darboux inversion is written and used to regularize general 3BP in space (Levi-Civita 1915/16)
- $z \mapsto z^n$ to regularize other central forces (McGehee 1981)

Remark. The Darboux inversion extends to nonnatural "connection-force" systems: $\nabla_{\dot{q}}\dot{q} = f(q)$ where the connection ∇ is not Levi-Civita, and the force f is not derived from a potential. This may suggest impossibility results for a first Levi-Civita regularization in 3D.