Control of collision orbits

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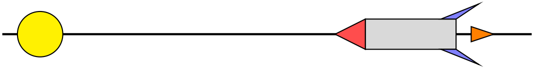

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What does *regularization* mean

When approaching a binary collision in the n-body problem (here $n = 2$), the gravitational acceleration and the velocity of the colliding bodies tend to infinity.

The Levi-Civita regularization (1904) transforms this singular motion into a smooth motion where all the variables remain finite, and which continues after the collision. This transformation couples a change of coordinates with a change of a time, as explained in the next proposition.

SUR LA RÉSOLUTION QUALITATIVE DU PROBLÈME RESTREINT DES TROIS CORPS

PAR

T. LEVI-CIVITA
à PADOUE.

Dans le problème des trois corps (points matériels, qui s'attirent suivant la loi de NEWTON) les forces et par conséquent les équations différentielles du mouvement se comportent d'une façon analytique régulière tant que les positions des trois points restent distinctes.

D'après cela il est presque évident qu'il ne peut y avoir autre raison de singularité pour le mouvement en dehors de la circonstance que deux des trois corps (ou tous les trois) se rapprochent indéfiniment.

Plus précisément M. PAINLEVÉ[1] a démontré qu'à partir de conditions initiales données des singularités peuvent se présenter alors seulement qu'une au moins des distances mutuelles tend vers zéro pour t convergent vers une valeur finie t_1.

Quoi qu'il en soit, les résultats récents de M. MITTAG-LEFFLER sur les représentations des branches monogènes des fonctions analytiques permettent d'affirmer que:

Dans le problème des trois corps les coordonnées sont exprimables *en tout cas et pendant toute la durée du mouvement* par des séries jouissant des propriétés fondamentales des séries de TAYLOR.

Soit en effet x une quelconque de ces coordonnées. D'après la conclusion de M. PAINLEVÉ, rappelée tout à l'heure, la fonction $x(t)$ reste

[1] Voir ses »Leçons etc., professées à Stockholm», chez A. Hermann, Paris 1897, p. 583.

Acta mathematica. 30. Imprimé le 17 février 1906. 39

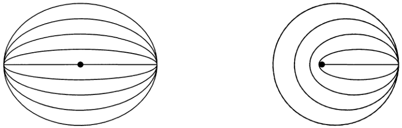
The map $z \to z^2$

Proposition

Suppose the motion of a point in the complex plane is given by $z(\tau)$ and satisfies Hooke's law $z'' = -Cz$. Then a point following the trajectory $\omega(t(\tau)) = [z(t)]^2$, where $dt = |z|^2 d\tau$, moves according to Newton's law

$$\frac{d^2\omega}{dt^2} = -\tilde{C}\frac{\omega}{|\omega|^3},$$

where $\tilde{C} = 2\left(|z'(0)|^2 + C|z(0)|^2\right)$



Effect of the transformation $z \to z^2$ on the Hooke ellipsis

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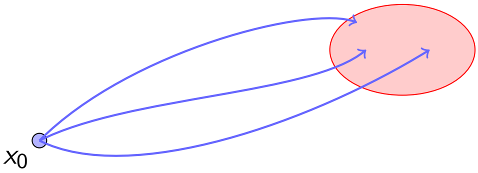
Setting

A controlled dynamical system is a smooth family of vector fields $f : M \times U \to TM$ where M is a smooth $n-$dimensional manifold, and $U \subset \mathbb{R}^m$ is the set of admissible controls.
First question: is some final state x_f accessible from some initial state x_0, i.e. does the system

$$\dot{x}(t) = f\left(x(t), u(t)\right), \qquad u(t) \in U$$
$$x(0) = x_0, \qquad x(t_f) = x_f$$

have a solution for some admissible control? The system is said to be *controllable* if the answer is positive for all possible initial and final states $x_0, x_f \in M$.



Controllability

Theorem

Let

$$\dot{x} = f_0(x) + \sum_{i=1}^{m} u_i f_i(x)$$

a controlled-affine system on a connected manifold, with $u = (u_1, \dots, u_m) \in U$. If

- f_0 is periodic
- The convex enveloppe of U is a neighbourhood of the origin
- $Lie_x \{f_0, \dots, f_m\} = T_x M$, $x \in M$

then the system is controllable.

Pontryagin maximum Principle (PMP)

Let $\dot{x} = f(x,u)$, $\quad \int_0^{t_f} L(x,u) \to \min$, defined on $M = \mathbb{R}^n$.

Theorem (Pontryagin et al.)

iIf u is an optimal control on $[0, t_f]$, t_f not fixed, with response x, then there exists an absolutely continuous covector function p, valued in $(\mathbb{R}^n)^* \setminus \{0\}$, and $p^0 \leq 0$ such that, with

$$H(x, p, p^0, u) = \langle p, f(x,u) \rangle + p^0 L(x,u),$$

almost everywhere in $[0, t_f]$,

$$\dot{x} = \frac{\partial H}{\partial p}(x, p, p^0, u), \quad \dot{p} = -\frac{\partial H}{\partial x}(x, p, p^0, u)$$
$$H(x, p, p^0, u) = \max_{v \in U} H(x, p, p^0, v).$$

Morever, one has that $H = 0$. a.e. in $[0, t_f]$.

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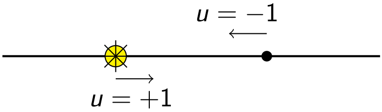


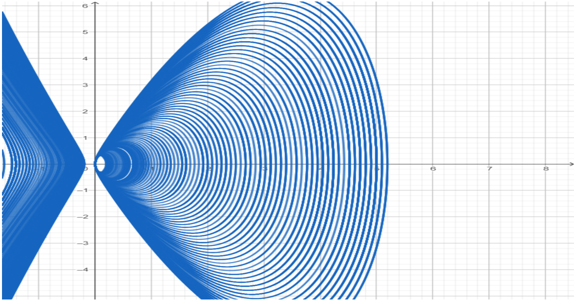
The equations for the state variable $x > 0$ of the *PMP* on the real line write as

$$a)\ \ddot{x} = -\frac{1}{x^2} - 1, \qquad b)\ \ddot{x} = -\frac{1}{x^2} + 1$$

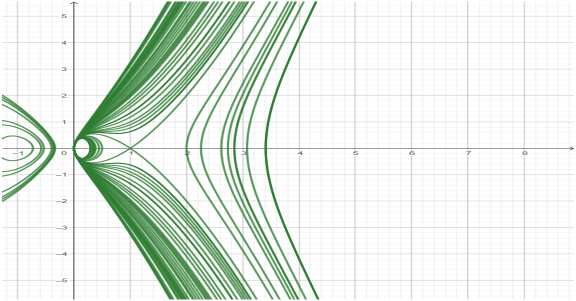
depending on the sign of the covector p_x. The equations can be integrated by means of the -respectively- conserved quantity

$$a)\ C_- = \frac{1}{2}\dot{x}^2 - \frac{1}{x} + x, \quad b)\ C_+ = \frac{1}{2}\dot{x}^2 - \frac{1}{x} - x$$

and the Sundman change of time $dt = x d\tau$, $x > 0$ in terms of the Weierstrass elliptic function $\wp$.



(x, x') phase space of: Above, system with $u = -1$; below, system with $u = +1$



To better understand the behavior of the control system, we need to regularize the whole system - and not just the extremals arising from the *PMP*.

It turns out that the Levi-Civita regularization written in the real plane as $(q_1, q_2) = \left(\xi_1^2 - \xi_2^2,\, 2\xi_1\xi_2\right)$ together with the change of time can be extended to the controlled system

$$\ddot{q} = -\frac{q}{|q|^3} + u$$

$q \in \mathbb{R}^2, |u| \leq 1$ by giving the affine-controlled system

$$x' = f_0(x) + u_1 f_1(x) + u_2 f_2(x)$$

where $x = (\xi_1, \xi_2, \eta_1, \eta_2, h) \in \mathbb{R}^5$, and f_0, f_1, f_2 real-analytical vector fields defined on the 4$-$dimensional manifold

$$M = \{2|\eta|^2 + (-h)|\xi|^2 = 1\} \subset \mathbb{R}^5$$

h be the energy of the non-controlled system.

Thanks to this change of - dependent and independent - variables, we can apply the thorem for the controllability to conclude that

Theorem

The planar controlled Kepler problem is everywhere controllable, whenever $h < 0$, where $h(t) = \frac{1}{2}|\dot{q}|^2 - \frac{1}{|q|}$ is the varying energy of the non-controlled system.

Proof: By calculation, it turns out that $\mathrm{Lie}_x \{f_0, f_1, f_2\} = T_x M$, $\forall x \in M$ and the vector field f_0 is periodic whenever $h < 0$.

Existence of optimal solutions

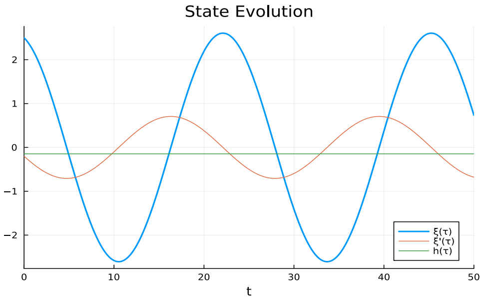
The existence of an optimal couple trajectory - control (x^*, u^*) which solves the regularized problem

$$x' = f_0(x) + u_1 f_1(x) + u_2 f_2(x), \qquad \int_0^{\tau_f} |\xi|^2 d\tau \to \min$$

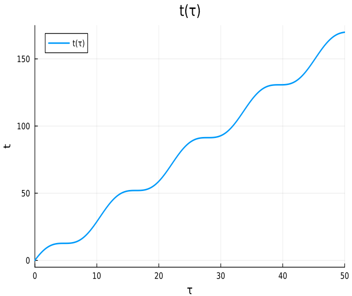
is guaranteed if a blow-up in finite time does not occour. But also by restricting to the one dimensional case, it turns out that the Weierstrass $\wp-$function has poles!

Proposition

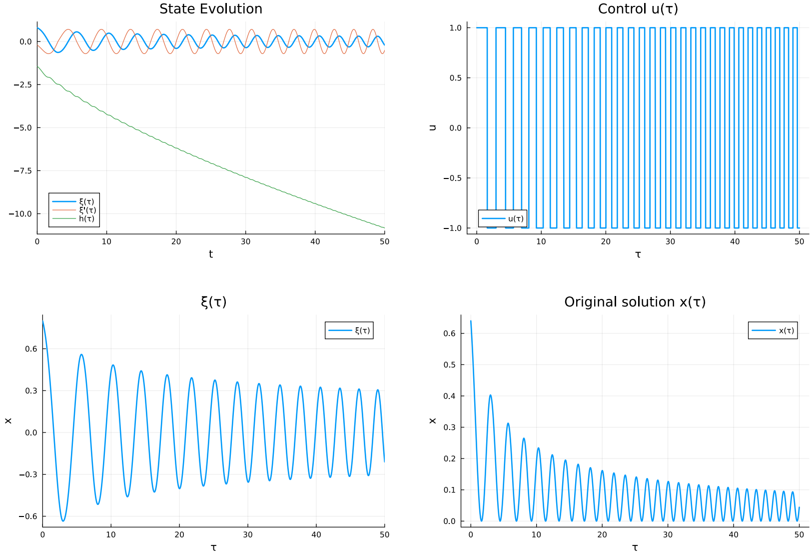
The existence of an optimal solution for the regularized Kepler problem is guaranteed only on the subset of M defined by $h < 0$



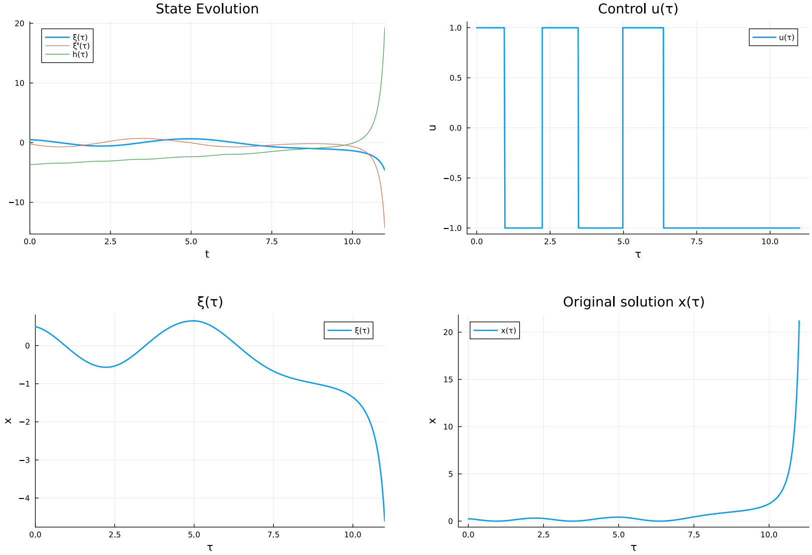
Above, reparametrized system with $u = 0$. It is an harmonic oscillator; below, original and new time τ



What happens if we force the control so that $h'(\tau) \to \min$, i.e. $u = -\frac{\xi\xi'}{|\xi\xi'|}$



What happens if we force the control so that $h'(\tau) \to$ max, i.e. $u = \frac{\xi\xi'}{|\xi\xi'|}$... blow-up in finite time!



What we are working on

- Bound the system in the region $h < 0$, so that we have existence of optimal solutions
- Solve the equations of the *PMP* in the regular setting
- Extend the results to the three-dimensional case

thank you!