Proofs for Stochastic Exam

This is a list of possible proofs that can be asked in the exam. It's still under construction. If anyone thinks of a possible proof (either the professor said it the class, or something you found in an old exam, etc...) please add it to this list!

1. Parceval's Theorem (exam 2011)
   $$\int_{-\infty}^{+\infty} [x(t)]^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$
2. Fourier Series coefficients (exam 2012)
   $$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t)\, dt,\ a_k = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos(k\omega_0 t)\, dt,\ b_k = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin(k\omega_0 t)\, dt$$
3. Convolution in time means multiplication in frequency domain and vice-versa (exam 2012)
4. Fourier transform of basic functions:

Signal $\tilde{x}(t)$	Fourier transform $X(\omega)$	Fourier transform $X(f)$
1	$2\pi\delta(\omega)$	$\delta(f)$
$\delta(t)$	1	1
$\sum_{n=-\infty}^{+\infty} \delta(t-nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(\omega - k\frac{2\pi}{T})$	$\frac{1}{T} \sum_{k=-\infty}^{+\infty} \delta(f - k\frac{1}{T})$
$\cos(k\omega_0 t)$	$\pi[\delta(\omega - k\omega_0) + \delta(\omega + k\omega_0)]$	$\frac{1}{2}[\delta(f - kf_0) + \delta(\omega + kf_0)]$
$\sin(k\omega_0 t)$	$\frac{\pi}{j}[\delta(\omega - k\omega_0) + \delta(\omega + k\omega_0)]$	$\frac{1}{2j}[\delta(f - kf_0) + \delta(\omega + kf_0)]$
$\tilde{x}(t) = 1,\ \lvert t\rvert < \frac{1}{2}T;$ $0,\ \lvert t\rvert > \frac{1}{2}T$	T sinc $(\frac{\omega T}{2\pi}) = T\frac{sin(\omega T/2)}{\omega T/2}$	T sinc $(fT) = T\frac{\sin(\pi fT)}{\pi fT}$
$\frac{W}{2\pi}$ sinc $(\frac{Wt}{2\pi})$	$B(\omega) = 1,\ \lvert\omega\rvert < \frac{1}{2}W;$ $0,\ \lvert\omega\rvert > \frac{1}{2}W$	$B(f) = 1,\ \lvert f\rvert < \frac{1}{2}F;$ $0,\ \lvert f\rvert > \frac{1}{2}F$

5. Fourier transform of properties:

Property	Signal	Fourier Transform
Linearity	$a\,x(t) + b\,y(t)$	$a\,X(\omega) + b\,Y(\omega)$
Time delay	$x(t - t_0)$	$\left\{e^{-j\omega t_0}\right\} X(\omega)$
Multiplication	$x(t)\,y(t)$	$\frac{1}{2\pi} X(\omega) * Y(\omega)$
Convolution	$x(t) * y(t)$	$X(\omega)\,Y(\omega)$
Time derivative	$\dot{x}(t)$	$j\omega\,X(\omega)$
Integral of Parseval	$\int\limits_{-\infty}^{+\infty} x^2(t)\,\mathrm{d}t$	$\frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} \lvert X(\omega)\rvert^2\,\mathrm{d}\omega$

Table 3.1: Properties of the Fourier transform.

6. This (2012 exam):

$$R_{\bar{x}\bar{y}}(\tau) = R_{\bar{y}\bar{x}}(-\tau)$$

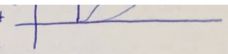
$$C_{\bar{x}\bar{y}}(\tau) = C_{\bar{y}\bar{x}}(-\tau)$$

$$K_{\bar{x}\bar{x}}(\tau) = K_{\bar{x}\bar{x}}(-\tau)$$

$$K_{\bar{x}\bar{x}}(0) = 1$$

See next page

that we shift $\tilde{y}$ (the slow system) to the left). For $C_{\tilde{y}\tilde{x}}(\tau)$ we can see that the coherence increases for negative τ (meaning that we shift $\tilde{x}$ to the right). This is in coherence with the figure over here →.



Proofs Slide 37:

$R_{\tilde{x}\tilde{x}}(\tau) = R_{\tilde{x}\tilde{x}}(-\tau)$:

$$R_{\tilde{x}\tilde{x}}(\tau) = E\{\tilde{x}(t)\tilde{x}(t+\tau)\} \quad \text{with } \sigma = t+\tau$$
$$= E\{\tilde{x}(\sigma-\tau)\tilde{x}(\sigma)\} \quad t = \sigma-\tau$$
$$= E\{\tilde{x}(\sigma)\tilde{x}(\sigma-\tau)\} \rightarrow \text{stationary process so}$$
$$= E\{\tilde{x}(t)\tilde{x}(t-\tau)\} \quad \text{holds } \forall \sigma \text{ including } \sigma = t.$$
$$\hookrightarrow = R_{\tilde{x}\tilde{x}}(-\tau) \text{ so even.}$$

$R_{\tilde{x}\tilde{y}}(\tau) = E\{\tilde{x}(t)\tilde{y}(t+\tau)\}$ with $\sigma = t+\tau$

$= E\{\tilde{x}(\sigma-\tau)\tilde{y}(\sigma)\}$ $\quad t = \sigma-\tau$

$= E\{\tilde{y}(t)\tilde{x}(t-\tau)\} = R_{\tilde{y}\tilde{x}}(-\tau)$

This also implies that it's not $R_{\tilde{x}\tilde{y}}(-\tau)$

$C_{\tilde{x}\tilde{x}}(\tau) = C_{\tilde{x}\tilde{x}}(-\tau)$:

Follows from $R_{\tilde{x}\tilde{x}}(\tau) = R_{\tilde{x}\tilde{x}}(-\tau)$

$C_{\tilde{x}\tilde{y}}(\tau) = C_{\tilde{y}\tilde{x}}(-\tau) \neq C_{\tilde{x}\tilde{y}}(-\tau) \rightarrow$ Follows from the previous one.

$K_{\tilde{x}\tilde{x}}(\tau) = K_{\tilde{x}\tilde{x}}(-\tau)$:

Follows from $R_{\tilde{x}\tilde{x}}(\tau) = C_{\tilde{x}\tilde{x}}(-\tau)$

$K_{\tilde{x}\tilde{y}}(\tau) = K_{\tilde{y}\tilde{x}}(-\tau) \neq K_{\tilde{x}\tilde{y}}(-\tau) \rightarrow$ Follows from Previous one.

$R_{\tilde{x}\tilde{x}}(0) = E\{\tilde{x}(t)\tilde{x}(t)\} = E\{\tilde{x}(t)^2\}$

This equals $m_2 = \sigma_{\tilde{x}}^2 + \mu_{\tilde{x}}^2$

$C_{\tilde{x}\tilde{x}}(0) = E\{(\tilde{x}-\mu_{\tilde{x}})(\tilde{x}-\mu_{\tilde{x}})\} = E\{(\tilde{x}-\mu_{\tilde{x}})^2\}$

This equals $m_2' = \sigma_{\tilde{x}}^2$

$K_{\tilde{x}\tilde{x}} = \frac{C_{\tilde{x}\tilde{x}}}{\sigma_x^2} = 1$

$R_{\tilde{x}\tilde{y}}(0) = E\{\tilde{x}(t)\tilde{y}(t)\} = R_{\tilde{x}\tilde{y}} \rightarrow$ stationary process so we do not care about t.

$C_{\tilde{x}\tilde{y}} = E\{(\tilde{x}-\mu_x)(\tilde{y}-\mu_y)\}$

$= E\{\tilde{x}\tilde{y} - \tilde{x}\mu_y - \tilde{y}\mu_x + \mu_x\mu_y\}$

$= R_{\tilde{x}\tilde{y}} - \mu_{\tilde{x}}\mu_y$

So, $R_{\tilde{x}\tilde{y}} = C_{\tilde{x}\tilde{y}} + \mu_{\tilde{x}}\mu_y$

The other two follow from the first. All due to the assumption of stationary process.

7. This (2015 exam):

3. FOURIER TRANSFORM (5 points)

Prove the modulation theorem of the Fourier Transform.

I.e. when: $\qquad y(t) = x(t) \cdot \cos(\omega_1 t),$

and with $X(\omega) = \mathcal{F}\{x(t)\}$, $Y(\omega) = \mathcal{F}\{y(t)\}$

then: $\qquad Y(\omega) = \frac{1}{2}X(\omega - \omega_1) + \frac{1}{2}X(\omega + \omega_1).$

8. Chapter 3, Table 6

$H(\omega)$	I
$\frac{1}{1+j\omega\tau}$	$\frac{1}{2\tau}$
$\frac{1}{1+2\zeta\frac{j\omega}{\omega_0}+\left(\frac{j\omega}{\omega_0}\right)^2}$	$\frac{\omega_0}{4\zeta}$
$\frac{1}{(1+j\omega\tau_1)(1+j\omega\tau_2)}$	$\frac{1}{2(\tau_1+\tau_2)}$
$\frac{1+j\omega\tau}{1+2\zeta\frac{j\omega}{\omega_0}+\left(\frac{j\omega}{\omega_0}\right)^2}$	$\frac{\omega_0}{4\zeta}(1+\omega_0^2\tau^2)$
$\frac{1+j\omega\tau_1}{(1+j\omega\tau_2)(1+j\omega\tau_3)}$	$\frac{1}{2(\tau_2+\tau_3)}\left(1+\frac{\tau_1^2}{\tau_2\tau_3}\right)$
$\frac{1}{(1+j\omega\tau)\left\{1+2\zeta\frac{j\omega}{\omega_0}+\left(\frac{j\omega}{\omega_0}\right)^2\right\}}$	$\frac{1}{2}\frac{\frac{\omega_0}{2\zeta}+\omega_0^2\tau}{1+2\zeta\omega_0\tau+\omega_0^2\tau^2}$

Table 3.6: Standard integrals for the calculation of the variance.

You have to be able to use the Lyapunov equations to find I in table 3.6, so that's what we're going to do right now.

Number 1:

We get $H(\omega) = \frac{K}{1+j\omega\tau}$ prove using Lyapunov that if we drive this system with white noise with intensity W as an input we get output: $\sigma_y^2 = \frac{K^2}{2\tau}W$

$$s=j\omega \rightarrow H(s) = \frac{Y(s)}{U(s)} = \frac{K}{1+s\tau} \quad \text{so we get that: } y(s) + \tau s y(s) = k u(s)$$
$$= y(t) + \tau\dot{y}(t) = k u(t)$$

Now we define $x(t) = y(t)$ and $C=1 \wedge D=0$: $\dot{x}(t) = -\frac{1}{\tau}x(t) + \frac{k}{\tau}u(t) \longrightarrow$ First order system.

$$\underbrace{-\tfrac{1}{\tau}}_{A} \qquad \underbrace{\tfrac{k}{\tau}}_{B}$$

Now we take the Lyapunov equation: $0 = A C_{\bar{x}\bar{x}ss} + C_{\bar{x}\bar{x}ss}A^T + BWB^T$

$$0 = -\frac{1}{\tau}C_{\bar{x}\bar{x}ss} + C_{\bar{x}\bar{x}ss}\left(-\frac{1}{\tau}\right) + \frac{k}{\tau}W\frac{k}{\tau}$$

$$C_{\bar{x}\bar{x}ss}\left(\frac{2}{\tau}\right) = \frac{k^2}{\tau^2}W \rightarrow C_{\bar{x}\bar{x}ss} = \frac{k^2}{2\tau}W \quad \text{now } x=y \text{ and } C_{\bar{x}\bar{x}ss} = \sigma_x^2$$

we get that $\sigma_y^2 = \frac{k^2}{2\tau}W$

Number 2:

$$H(\omega) = \frac{1}{1 + 2\zeta\frac{j\omega}{\omega_0} + \left(\frac{j\omega}{\omega_0}\right)^2} \quad \text{with } s=j\omega \rightarrow H(s) = \frac{1}{1 + 2\zeta\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2} = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

$$H(s) = \frac{Y(s)}{U(s)} = s^2y(s) + 2\zeta\omega_0 s y(s) + \omega_0^2 y(s) = \omega_0^2 u(s)$$

$$\ddot{y}(t) + 2\zeta\omega_0\dot{y}(t) + \omega_0^2 y(t) = \omega_0^2 u(t) \longrightarrow \ddot{y}(t) = -2\zeta\omega_0\dot{y}(t) - \omega_0^2 y(t) + \omega_0^2 u(t)$$

$$\begin{bmatrix}\dot{y}(t)\\ \ddot{y}(t)\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1\\ -\omega_0^2 & -2\zeta\omega_0\end{bmatrix}}_{A}\begin{bmatrix}y(t)\\ \dot{y}(t)\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ \omega_0^2\end{bmatrix}}_{B}u(t)$$

Now we take Lyapunov:
$$0 = \begin{bmatrix}0 & 1\\ -\omega_0^2 & -2\zeta\omega_0\end{bmatrix}C_{\bar{x}\bar{x}ss} + C_{\bar{x}\bar{x}ss}\begin{bmatrix}0 & -\omega_0^2\\ 1 & -2\zeta\omega_0\end{bmatrix} + \begin{bmatrix}0\\ \omega_0^2\end{bmatrix}\underbrace{W}_{\text{we set } W=1}\begin{bmatrix}0 & \omega_0^2\end{bmatrix}$$

$$\begin{bmatrix}0 & 1\\ -\omega_0^2 & -2\zeta\omega_0\end{bmatrix}\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix} = \begin{bmatrix}C_{21} & C_{22}\\ -\omega_0^2C_{11} - 2\zeta\omega_0C_{21} & -\omega_0^2C_{12} - 2\zeta\omega_0C_{22}\end{bmatrix}$$

$$\begin{bmatrix}C_{11} & C_{12}\\ C_{21} & C_{22}\end{bmatrix}\begin{bmatrix}0 & -\omega_0^2\\ 1 & -2\zeta\omega_0\end{bmatrix} = \begin{bmatrix}C_{12} & -\omega_0^2C_{11} - 2\zeta\omega_0C_{12}\\ C_{22} & -\omega_0^2C_{21} - 2\zeta\omega_0C_{22}\end{bmatrix}$$

so,
$$\begin{bmatrix}C_{21} + C_{12} & C_{22} - \omega_0^2C_{11} - 2\zeta\omega_0C_{12}\\ C_{22} - \omega_0^2C_{11} - 2\zeta\omega_0C_{21} & -\omega_0^2(C_{12}+C_{21}) - 4\zeta\omega_0C_{22}\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & \omega_0^4\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

SO, $C_{21} + C_{12} = 0$

$$\left.\begin{aligned} C_{22} - \omega_0^2 C_{11} - 2\zeta\omega_0 C_{21} = 0 \\ C_{22} - \omega_0^2 C_{11} - 2\zeta\omega_0 C_{12} = 0 \end{aligned}\right\} \longrightarrow 2C_{22} - 2\omega_0^2 C_{11} = 0 \quad \text{so } C_{11} = \frac{C_{22}}{\omega_0^2} = \frac{\omega_0}{4\zeta}$$

$$\omega_0^4 - \omega_0^2 (C_{12} + C_{21}) - 4\zeta\omega_0 C_{22} = 0 \longrightarrow C_{22} = \frac{\omega_0^3}{4\zeta} \quad \text{SO } C_{21} = C_{12} = 0$$

$$C_{\xi\xi_{ss}} = \begin{bmatrix} \frac{\omega_0}{4\zeta} & 0 \\ 0 & \frac{\omega_0^3}{4\zeta} \end{bmatrix} \quad \text{so from } C_{\xi\xi} = \begin{bmatrix} \sigma_{\bar{x}_1}^2 & C_{\bar{x}_1\bar{x}_2} & \\ & \ddots & \\ & & \sigma_{\bar{x}_n}^2 \end{bmatrix} \quad \text{we find that } \sigma_{\xi_{m}}^2 = \frac{\omega_0}{4\zeta}$$

Number 3:

$$H(s) = \frac{1}{(1+s\tau_1)(1+s\tau_2)} = \frac{1}{s^2\tau_1\tau_2 + (\tau_1+\tau_2)s + 1} = \frac{y(s)}{u(s)}$$

SO, $\tau_1\tau_2 s^2 y(s) + (\tau_1+\tau_2) s y(s) + y(s) = u(s)$

$$\tau_1\tau_2 \ddot{y}(t) + (\tau_1+\tau_2)\dot{y}(t) + y(t) = u(t)$$

$$\ddot{y}(t) = -\frac{(\tau_1+\tau_2)}{\tau_1\tau_2}\dot{y}(t) - \frac{1}{\tau_1\tau_2} y(t) + \frac{1}{\tau_1\tau_2} u(t)$$

SO,

$$\begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{1}{\tau_1\tau_2} & \frac{-(\tau_1+\tau_2)}{\tau_1\tau_2} \end{bmatrix}}_{A} \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{\tau_1\tau_2} \end{bmatrix}}_{B} u(t)$$

now, $0 = A C_{\xi\xi_{ss}} + C_{\xi\xi_{ss}} A^T + B \overset{=1}{W} B^T$

$$\begin{bmatrix} 0 & 1 \\ -\frac{1}{\tau_1\tau_2} & \frac{-(\tau_1+\tau_2)}{\tau_1\tau_2} \end{bmatrix}\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} C_{21} & C_{22} \\ \frac{-C_{11}}{\tau_1\tau_2} - \frac{(\tau_1+\tau_2)}{\tau_1\tau_2} C_{21} & \frac{-C_{12}}{\tau_1\tau_2} - \frac{(\tau_1+\tau_2)}{\tau_1\tau_2} C_{22} \end{bmatrix}$$

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}\begin{bmatrix} 0 & \frac{-1}{\tau_1\tau_2} \\ 1 & \frac{-(\tau_1+\tau_2)}{\tau_1\tau_2} \end{bmatrix} = \begin{bmatrix} C_{12} & \frac{-1}{\tau_1\tau_2} C_{11} - \frac{(\tau_1+\tau_2)}{\tau_1\tau_2} C_{12} \\ C_{22} & \frac{-1}{\tau_1\tau_2} C_{21} - \frac{(\tau_1+\tau_2)}{\tau_1\tau_2} C_{22} \end{bmatrix}$$

SO,

$$\begin{bmatrix} C_{21} + C_{12} & C_{22} - \frac{C_{11}}{\tau_1\tau_2} - \frac{(\tau_1+\tau_2)}{\tau_1\tau_2} C_{12} \\ C_{22} - \frac{C_{11}}{\tau_1\tau_2} - \frac{(\tau_1+\tau_2)}{\tau_1\tau_2} C_{21} & \frac{-1}{\tau_1\tau_2}(C_{12} + C_{21}) - \frac{2(\tau_1+\tau_2)}{\tau_1\tau_2} C_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{(\tau_1\tau_2)^2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

SO, $C_{21} + C_{12} = 0$

$$\left.\begin{aligned} C_{22} - \frac{C_{11}}{\tau_1\tau_2} - \frac{(\tau_1+\tau_2)}{\tau_1\tau_2} C_{21} = 0 \\ C_{22} - \frac{C_{11}}{\tau_1\tau_2} - \frac{(\tau_1+\tau_2)}{\tau_1\tau_2} C_{12} = 0 \end{aligned}\right\} \longrightarrow 2C_{22} - \frac{2C_{11}}{\tau_1\tau_2} = 0 \rightarrow C_{11} = C_{22}(\tau_1\tau_2) = \frac{1}{2(\tau_1+\tau_2)}$$

$$\frac{1}{(\tau_1\tau_2)^2} - \frac{1}{\tau_1\tau_2}(C_{12} + C_{21}) - 2\frac{(\tau_1+\tau_2)}{\tau_1\tau_2} C_{22} = 0 \rightarrow \frac{1}{\tau_1\tau_2} = 2(\tau_1+\tau_2) C_{22}$$

$$C_{22} = \frac{1}{2(\tau_1+\tau_2)\tau_1\tau_2}$$

$$C_{11} = \sigma_y^2 = \frac{1}{2(\tau_1+\tau_2)}$$

9. Prove that Auto-PSD is a real function and that Cross-PSD is a complex function.
   Auto-PSd = E{X(w) X(-w)} = E{|X(w)^2|} which is the expectation of a real function.

10. Prove that the fourier transform of a pulse function is Tsinc(omega T/ (2*pi))

PROOF FOURIER TRANSFORMS OF BLOCK FUNCTIONS

$$B(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$
$$= \int_{-T/2}^{T/2} e^{-j\omega t}\, dt$$

using
$$\begin{cases} \cos(\omega t) = \frac{1}{2} e^{j\omega t} + \frac{1}{2} e^{-j\omega t} \\ \sin(\omega t) = \frac{1}{2j} e^{j\omega t} - \frac{1}{2} e^{-j\omega t} \end{cases}$$

$$\cos(\omega t) - j\sin(\omega t) = \cancel{\frac{1}{2} e^{j\omega t}} + \frac{1}{2} e^{-j\omega t} - \cancel{\frac{1}{2} e^{j\omega t}} + \frac{1}{2} e^{-j\omega t}$$
$$= e^{-j\omega t}$$

$$= \int_{-T/2}^{T/2} (\cos(\omega t) - j\sin(\omega t))\, dt$$
$$= \int_{-T/2}^{T/2} \cos(\omega t)\, dt - \cancel{j \int_{-T/2}^{T/2} \sin(\omega t)\, dt}^{\,0}$$

Since Sin is an odd function

$$= 2 \int_{0}^{T/2} \cos(\omega t)\, dt$$
$$= 2 \int_{0}^{T/2} d \frac{1}{\omega} \sin(\omega t)\, dt$$
$$= 2 \left[\frac{1}{\omega} \sin \omega t \right]_{0}^{T/2}$$
$$= 2 \frac{1}{\omega} \sin\left(\omega \frac{T}{2}\right)$$
$$= 2 \frac{T/2}{T/2} \frac{1}{\omega} \sin\left(\omega \frac{T}{2}\right)$$
$$= T \frac{\sin(\omega T/2)}{\omega T/2}$$

QED