For this reason, we assume that the birth rate (predator) is $k_3x_1(t)x_2(t)$. The death rate of the predator will be taken as simply proportional to the number of predators alive at the time; that is, death rate (predator) $= k_4x_2(t)$.

Since $x_1'(t)$ and $x_2'(t)$ represent the change in the prey and predator populations, respectively, with respect to time, the problem is expressed by the system of nonlinear differential equations

$$x_1'(t) = k_1x_1(t) - k_2x_1(t)x_2(t) \quad \text{and} \quad x_2'(t) = k_3x_1(t)x_2(t) - k_4x_2(t).$$

Use Runge-Kutta of order 4 for systems to solve this system for $0 \le t \le 4$, assuming that the initial population of the prey is 1000 and of the predators is 500 and that the constants are $k_1 = 3$, $k_2 = 0.002$, $k_3 = 0.0006$, and $k_4 = 0.5$. Is there a stable solution to this population model? If so, for what values x_1 and x_2 is the solution stable?

6. In Exercise 5 we considered the problem of predicting the population in a predator-prey model. Another problem of this type is concerned with two species competing for the same food supply. If the numbers of species alive at time t are denoted by $x_1(t)$ and $x_2(t)$, it is often assumed that, although the birth rate of each of the species is simply proportional to the number of species alive at that time, the death rate of each species depends on the population of both species. We will assume that the population of a particular pair of species is described by the equations

$$\frac{dx_1(t)}{dt} = x_1(t)[4 - 0.0003x_1(t) - 0.0004x_2(t)]$$

and

$$\frac{dx_2(t)}{dt} = x_2(t)[2 - 0.0002x_1(t) - 0.0001x_2(t)].$$

If it is known that the initial population of each species is 10,000, find the solution to this system for $0 \le t \le 4$. Is there a stable solution to this population model? If so, for what values of x_1 and x_2 is the solution stable?

5.8 Stiff Differential Equations

All the methods for approximating the solution to initial-value problems have error terms that involve a higher derivative of the solution of the equation. If the derivative can be reasonably bounded, then the method will have a predictable error bound that can be used to estimate the accuracy of the approximation. Even if the derivative grows as the step sizes increase, the error can be kept in relative control, provided that the solution also grows in magnitude. Problems frequently arise, however, where the magnitude of the derivative increases, but the solution does not. In this situation, the error can grow so large that it dominates the calculations. Initial-value problems for which this is likely to occur are called **stiff equations** and are quite common, particularly in the study of vibrations, chemical reactions, and electrical circuits.

Stiff systems derive their name from the motion of spring and mass systems that have large spring constants.

Stiff differential equations have an exact solution with a term of the form e^{-ct}, where c is a large positive constant. This is usually only a part of the solution, called the *transient* solution; the more important portion of the solution is called the *steady-state* solution. A transient portion of a stiff equation will rapidly decay to zero as t increases, but since the nth derivative of this term has magnitude c^ne^{-ct}, the derivative does not decay as quickly, and for large values of c it can grow very large. In addition, the derivative in the error term is evaluated not at t, but at a number between zero and t, so the derivative terms may increase as t increases—and very rapidly indeed. Fortunately, stiff equations can generally be predicted from the physical problem from which the equation is derived, and with care the error can be kept under control. The manner in which this is done is considered in this section.

Illustration The system of initial-value problems

$$u_1' = 9u_1 + 24u_2 + 5\cos t - \frac{1}{3}\sin t, \quad u_1(0) = \frac{4}{3}$$
$$u_2' = -24u_1 - 51u_2 - 9\cos t + \frac{1}{3}\sin t, \quad u_2(0) = \frac{2}{3}$$

has the unique solution

$$u_1(t) = 2e^{-3t} - e^{-39t} + \frac{1}{3}\cos t, \quad u_2(t) = -e^{-3t} + 2e^{-39t} - \frac{1}{3}\cos t.$$

The transient term e^{-39t} in the solution causes this system to be stiff. Applying the Runge-Kutta Fourth-Order Method for Systems program RKO4SY57 gives results listed in Table 5.17. When $h = 0.05$, stability results and the approximations are accurate. Increasing the step size to $h = 0.1$, however, leads to the disastrous results shown in the table. □

Table 5.17

t	Exact $u_1(t)$	$w_1(t)$ $h = 0.05$	$w_1(t)$ $h = 0.1$	Exact $u_2(t)$	$w_2(t)$ $h = 0.05$	$w_2(t)$ $h = 0.1$
0.1	1.793061	1.712219	−2.645169	−1.032001	−0.8703152	7.844527
0.2	1.423901	1.414070	−18.45158	−0.8746809	−0.8550148	38.87631
0.3	1.131575	1.130523	−87.47221	−0.7249984	−0.7228910	176.4828
0.4	0.9094086	0.9092763	−934.0722	−0.6082141	−0.6079475	789.3540
0.5	0.7387877	9.7387506	−1760.016	−0.5156575	−0.5155810	3520.00
0.6	0.6057094	0.6056833	−7848.550	−0.4404108	−0.4403558	15697.84
0.7	0.4998603	0.4998361	−34989.63	−0.3774038	−0.3773540	69979.87
0.8	0.4136714	0.4136490	−155979.4	−0.3229535	−0.3229078	311959.5
0.9	0.3416143	0.3415939	−695332.0	−0.2744088	−0.2743673	1390664.
1.0	0.2796748	0.2796568	−3099671.	−0.2298877	−0.2298511	6199352.

Although stiffness is usually associated with systems of differential equations, the approximation characteristics of a particular numerical method applied to a stiff system can be predicted by examining the error produced when the method is applied to a simple *test equation*,

$$y' = \lambda y, \quad \text{with } y(0) = \alpha,$$

where λ is a negative real number. The solution to this equation contains the transient solution $e^{\lambda t}$ and the steady-state solution is zero, so the approximation characteristics of a method are easy to determine. (A more complete discussion of the round-off error associated with stiff systems requires an examination of the test equation when λ is a complex number with negative real part.)

One-Step Methods

Suppose that we apply Euler's method to the test equation. Letting $h = (b - a)/N$ and $t_j = jh$, for $j = 0, 1, 2, \ldots, N$, implies that

$$w_0 = \alpha$$

and, for $j = 0, 1, \ldots, N - 1$,

$$w_{j+1} = w_j + h(\lambda w_j) = (1 + h\lambda)w_j,$$

so

$$w_{j+1} = (1 + h\lambda)^{j+1} w_0 = (1 + h\lambda)^{j+1}\alpha, \quad \text{for } j = 0, 1, \ldots, N-1. \tag{5.6}$$

Since the exact solution is $y(t) = \alpha e^{\lambda t}$, the absolute error is

$$|y(t_j) - w_j| = |e^{jh\lambda} - (1 + h\lambda)^j|\,|\alpha| = |(e^{h\lambda})^j - (1 + h\lambda)^j|\,|\alpha|,$$

and the accuracy depends on how well the term $1 + h\lambda$ approximates $e^{h\lambda}$. When $\lambda < 0$, the exact solution, $(e^{h\lambda})^j$, decays to zero as j increases, but, by Eq. (5.6), the approximation will have this property only if $|1 + h\lambda| < 1$. This effectively restricts the step size h for Euler's method to satisfy $|1 + h\lambda| < 1$, which in turn implies that $h < 2/|\lambda|$.

Suppose now that a round-off error δ_0 is introduced in the initial condition for Euler's method,

$$w_0 = \alpha + \delta_0.$$

At the jth step the round-off error is

$$\delta_j = (1 + h\lambda)^j \delta_0.$$

If $\lambda < 0$, the condition for the control of the growth of round-off error, $|1 + h\lambda| < 1$, is the same as the condition for controlling the absolute error, so we need to have $h < 2/|\lambda|$.

Illustration The test differential equation

$$y' = -30y, \quad 0 \le t \le 1.5, \quad y(0) = \frac{1}{3}$$

has exact solution $y = \frac{1}{3}e^{-30t}$. Using $h = 0.1$ for Euler's method, the Runge-Kutta method of order 4, and the Adams Predictor-Corrector method, gives the results at $t = 1.5$ in Table 5.18. □

Table 5.18

Exact solution	9.54173×10^{-21}
Euler's method	-1.09225×10^{4}
Runge-Kutta method	3.95730×10^{1}
Predictor-Corrector method	8.03840×10^{5}

The situation is similar for other one-step methods. In general, a function Q exists with the property that the difference method, when applied to the test equation, gives

$$w_{j+1} = Q(h\lambda) w_j.$$

The accuracy of the method depends upon how well $Q(h\lambda)$ approximates $e^{h\lambda}$, and the error will grow without bound if $|Q(h\lambda)| > 1$.

Multistep Methods

The problem of determining when a method is stable is more complicated in the case of multistep methods, due to the interplay of previous approximations at each step. Explicit multistep methods tend to have stability problems, as do predictor-corrector methods, because they involve explicit techniques. In practice, the techniques used for stiff systems are implicit multistep methods. Generally, w_{i+1} is obtained by iteratively solving a nonlinear equation or nonlinear system, often by Newton's method. To illustrate the procedure, consider the following implicit technique.

Implicit Trapezoidal Method

$$w_0 = \alpha$$
$$w_{j+1} = w_j + \frac{h}{2}[f(t_j, w_j) + f(t_{j+1}, w_{j+1})]$$

where $j = 0, 1, \ldots, N-1$.

To determine w_1 using this technique, we apply Newton's method to find the root of the equation

$$0 = F(w) = w - w_0 - \frac{h}{2}[f(t_0, w_0) + f(t_1, w)] = w - \alpha - \frac{h}{2}[f(a, \alpha) + f(t_1, w)].$$

To approximate this solution, select $w_1^{(0)}$ (usually as w_0) and generate $w_1^{(k)}$ by applying Newton's method to obtain

$$w_1^{(k)} = w_1^{(k-1)} - \frac{F\left(w_1^{(k-1)}\right)}{F'\left(w_1^{(k-1)}\right)} = w_1^{(k-1)} - \frac{w_1^{(k-1)} - \alpha - \frac{h}{2}\left[f(a, \alpha) + f\left(t_1, w_1^{(k-1)}\right)\right]}{1 - \frac{h}{2} f_y\left(t_1, w_1^{(k-1)}\right)}$$

The program TRAPNT58 implements the Implicit Trapezoidal method.

until $|w_1^{(k)} - w_1^{(k-1)}|$ is sufficiently small. Normally only three or four iterations are required. Once a satisfactory approximation for w_1 has been determined, the method is repeated to find w_2 and so on.

Alternatively, the Secant method can be used in the Implicit Trapezoidal method in place of Newton's method, but then two distinct initial approximations to w_{j+1} are required. To determine these, the usual practice is to let $w_{j+1}^{(0)} = w_j$ and obtain $w_{j+1}^{(1)}$ from some explicit multistep method. When a system of stiff equations is involved, a generalization is required for either Newton's or the Secant method. These topics are considered in Chapter 10.

Illustration The stiff initial-value problem

$$y' = 5e^{5t}(y - t)^2 + 1, \quad 0 \le t \le 1, \quad y(0) = -1$$

has solution $y(t) = t - e^{-5t}$. To show the effects of stiffness, the Implicit Trapezoidal method and the Runge-Kutta method of order 4 are applied both with $N = 4$, giving $h = 0.25$, and with $N = 5$, giving $h = 0.20$.

The Trapezoidal method performs well in both cases using a maximum of 10 iterations per step and $TOL = 10^{-6}$, as does Runge-Kutta with $h = 0.2$. However, $h = 0.25$ is outside the region of absolute stability of the Runge-Kutta method, which is evident from the results in Table 5.19. □

Table 5.19

	Runge–Kutta Method		Trapezoidal Method	
	$h = 0.2$		$h = 0.2$	
t_i	w_i	$\lvert y(t_i) - w_i \rvert$	w_i	$\lvert y(t_i) - w_i \rvert$
0.0	-1.0000000	0	-1.0000000	0
0.2	-0.1488521	1.9027×10^{-2}	-0.1414969	2.6383×10^{-2}
0.4	0.2684884	3.8237×10^{-3}	0.2748614	1.0197×10^{-2}
0.6	0.5519927	1.7798×10^{-3}	0.5539828	3.7700×10^{-3}
0.8	0.7822857	6.0131×10^{-4}	0.7830720	1.3876×10^{-3}
1.0	0.9934905	2.2845×10^{-4}	0.9937726	5.1050×10^{-4}
	$h = 0.25$		$h = 0.25$	
t_i	w_i	$\lvert y(t_i) - w_i \rvert$	w_i	$\lvert y(t_i) - w_i \rvert$
0.0	-1.0000000	0	-1.0000000	0
0.25	0.4014315	4.37936×10^{-1}	0.0054557	4.1961×10^{-2}
0.5	3.4374753	3.01956×10^{0}	0.4267572	8.8422×10^{-3}
0.75	1.44639×10^{23}	1.44639×10^{23}	0.7291528	2.6706×10^{-3}
1.0	Overflow		0.9940199	7.5790×10^{-4}

MATLAB has specific routines to solve stiff systems. To use the variable order multistep method `ode15s` to solve the stiff system in the Illustration at the beginning of the section:

$$u_1' = 9u_1 + 24u_2 + 5\cos t - \frac{1}{3}\sin t, \quad u_1(0) = \frac{4}{3},$$
$$u_2' = -24u_1 - 51u_2 - 9\cos t + \frac{1}{3}\sin t, \quad u_2(0) = \frac{2}{3},$$

we first define the right-hand side of the system by creating an `M-file` called `F1.m`.

```
function dy = F1(t,y)
dy = zeros(2,1);
dy(1) = 9*y(1)+24*y(2)+5*cos(t)-sin(t)/3;
dy(2) = -24*y(1)-51*y(2)-9*cos(t)+sin(t)/3;
```

and make `F1.m` known to MATLAB with

```
F3 = @F1
```

The initial conditions $u_1(0) = 4/3$ and $u_2(0) = 2/3$ are defined by

```
y10 = 4/3, y20 = 2/3
```

and the t values where we want the approximations by

```
tspan = [0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0]
```

The MATLAB response to

```
[T,YY]=ode15s(F3,tspan,[y10 y20])
```

gives the t values in the array `T` and in the array `YY` are the approximations to $u_1(t)$ in the first column and to $u_2(t)$ in the second.

$$T = \begin{bmatrix} 0 \\ 0.100000000000000 \\ 0.200000000000000 \\ 0.300000000000000 \\ 0.400000000000000 \\ 0.500000000000000 \\ 0.600000000000000 \\ 0.700000000000000 \\ 0.800000000000000 \\ 0.900000000000000 \\ 1.000000000000000 \end{bmatrix} \text{ and } YY = \begin{bmatrix} 1.333333333333333 & 0.666666666666667 \\ 1.792822173096780 & -1.031510588260698 \\ 1.423841705229490 & -0.874543162822479 \\ 1.131606544984823 & -0.725026524390037 \\ 0.909737405881405 & -0.608356645400698 \\ 0.739488715596804 & -0.515984454184929 \\ 0.606600245598731 & -0.440862764215503 \\ 0.500693300085545 & -0.377829713594847 \\ 0.414374560936528 & -0.323304488389232 \\ 0.342214086384561 & -0.274706988032098 \\ 0.280123638387467 & -0.230112384428887 \end{bmatrix}$$

EXERCISE SET 5.8

1. Solve the following stiff initial-value problems using Euler's method and compare the results with the actual solution.
   a. $y' = -9y$, for $0 \le t \le 1$, with $y(0) = e$ and $h = 0.1$; actual solution $y(t) = e^{1-9t}$.
   b. $y' = -20(y - t^2) + 2t$, for $0 \le t \le 1$, with $y(0) = \frac{1}{3}$ and $h = 0.1$; actual solution $y(t) = t^2 + \frac{1}{3}e^{-20t}$.
   c. $y' = -20y + 20\sin t + \cos t$, for $0 \le t \le 2$, with $y(0) = 1$ and $h = 0.25$; actual solution $y(t) = \sin t + e^{-20t}$.
   d. $y' = \dfrac{50}{y} - 50y$, for $0 \le t \le 1$, with $y(0) = \sqrt{2}$ and $h = 0.1$; actual solution $y(t) = (1 + e^{-100t})^{1/2}$.
2. Repeat Exercise 1 using the Runge-Kutta Fourth-Order method.
3. Repeat Exercise 1 using the Adams Fourth-Order Predictor-Corrector method.
4. Repeat Exercise 1 using the Trapezoidal method with a tolerance of 10^{-5}.
5. The Backward Euler One-Step method is defined by

$$w_{i+1} = w_i + hf(t_{i+1}, w_{i+1}) \quad \text{for } i = 0, 1, \ldots, N-1.$$

   Repeat Exercise 1 using the Backward Euler method incorporating Newton's method to solve for w_{i+1}.
6. The differential equation

$$\frac{dp(t)}{dt} = rb(1 - p(t))$$

   can be used as a model for studying the proportion $p(t)$ of nonconformists in a society whose birth rate is b, and where r represents the rate at which an offspring becomes nonconformist when at least one of the parents is a conformist. Use the Trapezoidal method to find an approximation for $p(50)$ when $p(0) = 0.01$, $b = 0.02$, $r = 0.1$, and t takes on the integral values from 1 to 50.

5.9 Survey of Methods and Software

In this chapter we have considered methods to approximate the solutions to initial-value problems for ordinary differential equations. We began with a discussion of the most elementary numerical technique, Euler's method. This procedure is not sufficiently accurate to be of use in applications, but it illustrates the general behavior of the more powerful techniques, without the accompanying algebraic difficulties. The Taylor methods were then considered