

## COSC-373: HOMEWORK 1

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**Question 1.** (a) Suppose the 3 students are  $\{A, B, C\}$  and the 3 internships are  $\{a, b, c\}$ . The preferences for the students are:

- $A$ :  $a, c, b$
- $B$ :  $a, c, b$
- $C$ :  $c, a, b$

The preferences for the internships are:

- $a$ :  $C, A, B$
- $b$ :  $B, C, A$
- $c$ :  $A, B, C$

The 5 rounds proceed as follows:

- (1)  $A$  applies to  $a$ ,  $B$  applies to  $a$ , and  $C$  applies to  $c$ .  $a$  defers  $A$ ,  $a$  rejects  $B$ , and  $c$  defers  $C$ .
- (2)  $B$  applies to  $c$ .  $c$  defers  $B$  and  $c$  rejects  $C$ .
- (3)  $C$  applies to  $a$ .  $a$  defers  $C$  and  $a$  rejects  $A$ .
- (4)  $A$  applies to  $c$ .  $c$  defers  $A$  and  $c$  rejects  $B$ .
- (5)  $B$  applies to  $b$ .  $b$  defers  $B$ .

The stable matchings found by the algorithm are thus  $\{(A, c), (B, b), (C, a)\}$ .

**Question 2.** (a) The stable matching found by the Gale-Shapley algorithm for such instances are  $\{(s_1, u_1), (s_2, u_2), \dots, (s_n, u_n)\}$ .  $n$  rounds are required until the algorithm terminates.

- (b) Since running the Gale-Shapley algorithm with students applying to internships gives the same stable matching as running the Gale-Shapley algorithm with internships applying to students, the stable matching is unique.

**Question 3.** (a) The centralized greedy algorithm executes as follows.

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1  $M \leftarrow \emptyset$ 
2  $V_M \leftarrow \emptyset$  // set of vertices that are matched in  $M$ 
3 foreach  $v \in V$  do
4   foreach  $u \in \text{neighbors}(v)$  do
5     if  $v, u \notin V_M$  then
6        $M \leftarrow M \cup \{(v, u)\}$ 
7        $V_M \leftarrow V_M \cup \{v, u\}$ 
8 return  $M$ 
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- (b) Since we are iterating over the neighbors for each vertex, we are thus iterating over the set of edges in  $G$ . Therefore, the runtime of our procedure is  $O(m)$ .

- (c) Let  $G = (V, E)$  where  $V = \{a, b, c, d\}$  and  $E = \{(a, b), (b, c), (c, d)\}$ . Two maximal matchings are  $M_1 = \{(a, b), (c, d)\}$  and  $M_2 = \{(b, c)\}$ . Observe that  $|M_1| = 2|M_2|$ .

**Question 4.** Suppose for the sake of contradiction that  $\text{dist}(u, w) > \text{dist}(u, v) + \text{dist}(v, w)$ . That is, the length of the shortest path from  $u$  to  $v$  plus the length of the shortest path from  $v$  to  $w$  is less than the length of the shortest path from  $u$  to  $w$ . The assumption thus implies that there exists a path from  $u$  to  $w$  that is shorter than the shortest path from  $u$  to  $w$ . Clearly, this is a contradiction. Therefore,  $\text{dist}$  satisfies the triangle inequality.