COSC-373: HOMEWORK 1

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Question 1. (a) Suppose the 3 students are $\{A, B, C\}$ and the 3 internships are $\{a, b, c\}$. The preferences for the students are:

- A: a, c, b
- B: a, c, b
- C: c, a, b

The preferences for the internships are:

- a: C, A, B
- b: B, C, A
- c: A, B, C

The 5 rounds proceed as follows:

(1) A applies to a, B applies to a, and C applies to c. a defers A, a rejects B, and c defers C.
(2) B applies to c. c defers B and c rejects C.
(3) C applies to a. a defers C and a rejects A.
(4) A applies to c. c defers A and c rejects B.
(5) B applies to b. b defers B.

The stable matchings found by the algorithm are thus $\{(A, c), (B, b), (C, a)\}$.

Question 2. (a) The stable matching found by the Gale-Shapley algorithm for such instances are $\{(s_1, u_1), (s_2, u_2), \ldots, (s_n, u_n)\}$. n rounds are required until the algorithm terminates.

(b) Since running the Gale-Shapley algorithm with students applying to internships gives the same stable matching as running the Gale-Shapley algorithm with internships applying to students, the stable matching is unique.

Question 3. (a) The centralized greedy algorithm executes as follows.

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M ← ∅
V_M ← ∅ // set of vertices that are matched in M
foreach v ∈ V do
    foreach u ∈ neighbors(v) do
        if v, u ∉ V_M then
            M ← M ∪ {(v, u)}
            V_M ← V_M ∪ {v, u}
return M
```

(b) Since we are iterating over the neighbors for each vertex, we are thus iterating over the set of edges in G. Assuming M and V_M are hash sets, where lookup and add operations take $O(1)$ time, the runtime of our procedure is therefore $O(m)$.

(c) Let $G = (V, E)$ where $V = \{a, b, c, d\}$ and $E = \{(a, b), (b, c), (c, d)\}$. Two maximal matchings are $M_1 = \{(a, b), (c, d)\}$ and $M_2 = \{(b, c)\}$. Observe that $|M_1| = 2|M_2|$.

Question 4. Suppose for the sake of contradiction that $\text{dist}(u, w) > \text{dist}(u, v) + \text{dist}(v, w)$. That is, the length of the shortest path from u to v plus the length of the shortest path from v to w is less than the length of the shortest path from u to w. The assumption thus implies that there exists a path from u to w that is shorter than the shortest path from u to w. Clearly, this is a contradiction. Therefore, dist satisfies the triangle inequality.