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CMPS 242 Homework 1

Question 1

a) There are two ways to think about this problem. Suppose we use ***c*** as the notation of a cat, and ***d*** as the notation of a dog.

i) Here is the first solution. Since Alice at least has a cat, all the cases of the remaining two pets that Alice has are (***c***, ***c***), (***d***, ***d***), (***d***, ***c***), (***c***, ***d***). Thus, the probability that at least one of Alice's pets is a dog is 3/4.

ii) Here is the second solution. There is only one case that Alice doesn't have a dog, and the probability of such case is $1/2 \times 1/2 = 1/4$. Therefore, the probability that at least one of Alice's pets is a dog is $1 - 1/4 = 3/4$.

b) This question can be translated into the following version:
We randomly choose one neighbor from the three, and randomly choose one pet which belongs to the chosen neighbor. We want to know which neighbor is most likely the owner of the pet if the chosen pet is a dog.
In other words, we want to compare $p(O = Alice \mid P = dog)$, $p(O = Bob \mid P = dog)$ and $p(O = Cathy \mid P = dog)$.

According to the product rule,

$$p\left(O = Alice \mid P = dog\right) = \frac{p\left(O = Alice, P = dog\right)}{p\left(P = dog\right)},$$

and

$$p\left(O = Alice, P = dog\right) = p\left(P = dog \mid O = Alice\right) p\left(O = Alice\right).$$

Suppose

$$p\left(O = Alice\right) = p\left(O = Bob\right) = p\left(O = Cathy\right) = \frac{1}{3},$$

we know $p(P = dog \mid O = Alice) = 2/3$, thus $p(O = Alice,\ P = dog) = 2/9$. On the other hand,

$$\begin{aligned} p\left(P = dog\right) = {} & p\left(P = dog \mid O = Alice\right) p\left(O = Alice\right) \\ & + p\left(P = dog \mid O = Bob\right) p\left(O = Bob\right) \\ & + p\left(P = dog \mid O = Cathy\right) p\left(O = Cathy\right) = \frac{2}{3} \end{aligned}$$

Thus,

$$p\left(O = Alice \mid P = dog\right) = \frac{\frac{2}{9}}{\frac{2}{3}} = \frac{1}{3}.$$

Similarly,

$$p\left(O = Bob \mid P = dog\right) = \frac{p\left(O = Bob, P = dog\right)}{p\left(P = dog\right)} = \frac{1}{6},$$

$$p\left(O = Cathy \mid P = dog\right) = \frac{p\left(O = Cathy, P = dog\right)}{p\left(P = dog\right)} = \frac{1}{2}.$$

Therefore, Cathy is most likely the owner of the dog and I should first knock her door.

Question 2

a) The 1-d Gaussian distribution is defined by

$$\mathcal{N}\left(x \mid \mu, \sigma^2\right) = \frac{1}{\left(2\pi\sigma^2\right)^{1/2}} exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\}$$

Claim. $\int_{-\infty}^{\infty} \mathcal{N}\left(x \mid \mu, \sigma^2\right) dx = 1$

Pf.

Since

$$\int_{-\infty}^{\infty} \frac{1}{\left(2\pi\sigma^2\right)^{1/2}} exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\} dx = \int_{-\infty}^{\infty} \frac{1}{\left(2\pi\sigma^2\right)^{1/2}} exp\left\{-\frac{1}{2\sigma^2}\left(y-\mu\right)^2\right\} dy,$$

thus,

$$\left[\int_{-\infty}^{\infty} \frac{1}{\left(2\pi\sigma^2\right)^{1/2}} exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\} dx\right]^2$$

$$= \int_{-\infty}^{\infty} \frac{1}{\left(2\pi\sigma^2\right)^{1/2}} exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\} dx \int_{-\infty}^{\infty} \frac{1}{\left(2\pi\sigma^2\right)^{1/2}} exp\left\{-\frac{1}{2\sigma^2}\left(y-\mu\right)^2\right\} dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} exp\left\{-\frac{\left(x-\mu\right)^2+\left(y-\mu\right)^2}{2\sigma^2}\right\} dxdy.$$

Let

$$\frac{x-\mu}{\sigma} = r\cos\theta, \quad \frac{y-\mu}{\sigma} = r\sin\theta,$$

then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} exp\left\{-\frac{\left(x-\mu\right)^2+\left(y-\mu\right)^2}{2\sigma^2}\right\} dxdy$$

$$= \int_{0}^{2\pi}\int_{0}^{\infty} \frac{1}{2\pi\sigma^2} exp\left\{-\frac{r^2\cos^2\theta + r^2\sin^2\theta}{2}\right\} |J|\, drd\theta,$$

where

$$J = \frac{\partial\left(x,y\right)}{\partial\left(r,\theta\right)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \sigma\cos\theta & -\sigma r\sin\theta \\ \sigma\sin\theta & \sigma r\cos\theta \end{vmatrix} = r\sigma^2\cos^2\theta + r\sigma^2\sin^2\theta$$

$$= r\theta^2.$$

Therefore,

$$\int_{0}^{2\pi}\int_{0}^{\infty} \frac{1}{2\pi\sigma^2} exp\left\{-\frac{r^2\cos^2\theta + r^2\sin^2\theta}{2}\right\} |J|\, drd\theta = \frac{1}{2\pi}\int_{0}^{2\pi} d\theta \int_{0}^{\infty} exp\left\{-\frac{r^2}{2}\right\} rdr$$

$$= \frac{1}{2\pi}\int_{0}^{2\pi} d\theta \int_{0}^{\infty} exp\left\{-\frac{r^2}{2}\right\} d\left(\frac{r^2}{2}\right) = exp\left\{-\frac{r^2}{2}\right\}\Bigg|_{-\infty}^{0} = 1.$$

Since we know $\int_{-\infty}^{\infty} \mathcal{N}\left(x \mid \mu, \sigma^2\right) dx > 0$,

so $\int_{-\infty}^{\infty} \mathcal{N}\left(x \mid \mu, \sigma^2\right) dx = \sqrt{1} = 1$, which means Gaussian distribution is well normalized.

b)

i) **Claim.** $\mathbb{E}\left[x\right]=\mu$

Pf.

Approach 1 - By Symmetry Argument

This approach involves defining the integration variable such that it is symmetric about the origin. We know

$$\mathbb{E}\left[x\right]=\int_{-\infty}^{\infty}\frac{1}{\left(2\pi\sigma^2\right)^{1/2}}exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\}xdx$$.

Let $y=x-\mu$, then $dx=dy$,

$$\int_{-\infty}^{\infty}\frac{1}{\left(2\pi\sigma^2\right)^{1/2}}exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\}xdx$$

$$=\int_{-\infty}^{\infty}\frac{1}{\left(2\pi\sigma^2\right)^{1/2}}exp\left\{-\frac{1}{2\sigma^2}y^2\right\}\left(y+\mu\right)dy$$

$$=\int_{-\infty}^{0}\frac{1}{\left(2\pi\sigma^2\right)^{1/2}}exp\left\{-\frac{1}{2\sigma^2}y^2\right\}ydy+\int_{0}^{\infty}\frac{1}{\left(2\pi\sigma^2\right)^{1/2}}exp\left\{-\frac{1}{2\sigma^2}y^2\right\}ydy$$

$$+\mu\int_{-\infty}^{\infty}\frac{1}{\left(2\pi\sigma^2\right)^{1/2}}exp\left\{-\frac{1}{2\sigma^2}y^2\right\}dy$$

We notice that the sum of the first term and the second term in the above cancel out since the domain of their integrations are symmetric about the origin. Furthermore, because

$$\frac{1}{\left(2\pi\sigma^2\right)^{1/2}}exp\left\{-\frac{1}{2\sigma^2}y^2\right\}=\mathcal{N}\left(y\mid 0,\sigma^2\right)$$

and

$$\int_{-\infty}^{\infty}\mathcal{N}\left(y\mid 0,\sigma^2\right)dy=1$$,

we now can conclude

$$\mathbb{E}\left[x\right]=\mu\int_{-\infty}^{\infty}\frac{1}{\left(2\pi\sigma^2\right)^{1/2}}exp\left\{-\frac{1}{2\sigma^2}y^2\right\}dy=\mu$$

Approach 2 - Differentiating Under the Integral Sign

This approach involves differentiating under the integral with respect to parameter μ.

We consider the Gaussian integral

$$\int_{-\infty}^{\infty}exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\}dx=\left(2\pi\sigma^2\right)^{1/2}$$

We now differentiate both sides with respect to parameter μ, which leads to

$$\int_{-\infty}^{\infty}\frac{\partial}{\partial\mu}exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\}dx=0$$.

When differentiated, the following is obtained

$$\int_{-\infty}^{\infty}(x-\mu)exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\}dx=0$$, which can be reexpressed as

$$\int_{-\infty}^{\infty}xexp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\}dx=\mu\int_{-\infty}^{\infty}exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\}dx$$

It is easily seen that the integral to the right side of the above equality is the non-normalized Gaussian and hence equals $\left(2\pi\sigma^2\right)^{1/2}$. After some simple algebra, we easily obtain

$$\frac{1}{\left(2\pi\sigma^2\right)^{1/2}}\int_{-\infty}^{\infty} xexp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\}dx = \mu = \mathbb{E}\left[x\right]$$

Approach 3 - Integration by Parts

This is the most straightforward (and most explicit) approach. We begin with the expected value of the Gaussian,

$$\mathbb{E}\left[x\right] = \frac{1}{\left(2\pi\sigma^2\right)^{1/2}}\int_{-\infty}^{\infty} xexp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\}dx.$$

Let $z = \frac{x-\mu}{\sqrt{2}\sigma}$ which implies that $dz = \frac{1}{\sqrt{2}\sigma}dx$ and $x = \sqrt{2}\sigma z + \mu$. Plugging these relations into the stated equation for expected value yields

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}(\sqrt{2}\sigma z+\mu)exp\{-z^2\}dz = \sqrt{\frac{2}{\pi}}\sigma\int_{-\infty}^{\infty} zexp\{-z^2\}dz+\frac{\mu}{\sqrt{\pi}}\int_{-\infty}^{\infty} exp\{-z^2\}dz.$$

We know that the second integral equals $\sqrt{\pi}$, thereby cancelling the denominator and simply leaving μ. We integrate $\int_{-\infty}^{\infty} zexp\{-z^2\}dz$ by parts using the well known formula $uv - \int vdu$. We pick the variables as follows: $u = z \Rightarrow du = dz$ and $dv = exp\{-z^2\}dz \Rightarrow v = \int_{-\infty}^{\infty} exp\{-z^2\}dz$. We know that, as aforementioned, $v = \sqrt{\pi}$, and a doing a little algebra reveals that $uv - \int vdu = 0$. Therefore, we are simply left with μ, which is the expected value of the Gaussian.

ii) **Claim.** $\mathrm{var}\left[x\right] = \sigma^2$

Pf.

Here, we proceed by differentiating under the integral with respect to σ^2. Integration by parts is possible, but is cumbersome and is hence not presented.

We already know

$$\int_{-\infty}^{\infty}\frac{1}{\left(2\pi\sigma^2\right)^{1/2}}exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\}dx = 1.$$

Let's move $\frac{1}{\left(2\pi\sigma^2\right)^{1/2}}$ to the right of this equation, and let $t = \sigma^2$, we get

$$\int_{-\infty}^{\infty} exp\left\{-\frac{1}{2t}\left(x-\mu\right)^2\right\}dx = \left(2\pi t\right)^{1/2}.$$

Then differentiate both sides of the equation with respect to t,

$$\int_{-\infty}^{\infty} exp\left\{-\frac{\left(x-\mu\right)^2}{2t}\right\}\frac{\left(x-\mu\right)^2}{2t^2}dx = \frac{1}{2}\left(2\pi\right)^{1/2}t^{-1/2}.$$

Then rearrange

$$\int_{-\infty}^{\infty}\frac{1}{\left(2\pi t\right)^{1/2}}exp\left\{-\frac{\left(x-\mu\right)^2}{2t}\right\}\left(x-\mu\right)^2dx = t,$$

which is the same as

$$\int_{-\infty}^{\infty} \frac{1}{(2\pi\sigma^2)^{1/2}} exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}(x-\mu)^2\,dx = \sigma^2$$
.

Based on the definition of $\mathbb{E}[x]$, $\text{var}[x] = \mathbb{E}\left[(x-\mathbb{E}[x])^2\right]$ and $\mathbb{E}[x] = \mu$, we can conclude $\mathbb{E}\left[(x-\mu)^2\right] = \text{var}[x] = \sigma^2$

c)

i) **Claim.** $\mu_{ML} = \frac{1}{N}\sum_{n=1}^{N} x_n$

Pf.

From the textbook we know the log likelihood function can be written in this form

$$\ln p\left(\mathbf{x}|\mu,\sigma^2\right) = -\frac{1}{2\sigma^2}\sum_{n=1}^{N}(x_n-\mu)^2 - \frac{N}{2}\ln\sigma^2 - \frac{N}{2}\ln(2\pi)$$
.

Differentiate the above function with respect to μ and let the result to be equals to 0, we get

$$-\frac{1}{\sigma^2}\sum_{n=1}^{N}(x_n-\mu) = 0$$
.

After rearranging, we obtain

$$\mu = \frac{1}{N}\sum_{n=1}^{N} x_n$$

ii) **Claim.** $\sigma^2_{ML} = \frac{1}{N}\sum_{n=1}^{N}(x_n-\mu_{ML})^2$

Pf.

We differentiate the log likelihood function with respect to σ^2 and let the result equal 0, to get

$$\frac{1}{2\sigma^4}\sum_{n=1}^{N}(x_n-\mu)^2 - \frac{N}{2\sigma^2} = 0$$

After rearranging and substituting μ for μ_{ML}, we obtain

$$\sigma^2_{ML} = \frac{1}{N}\sum_{n=1}^{N}(x_n-\mu_{ML})^2$$

d)

i) **Claim.** $\mathbb{E}[\mu_{ML}] = \mu$

Pf.

<u>Approach 1</u>

According to

$$\mu_{ML} = \frac{1}{N}\sum_{n=1}^{N} x_n$$
,

we know

$$\mathbb{E}[\mu_{ML}] = \mathbb{E}\left[\frac{1}{N}\sum_{n=1}^{N} x_n\right]$$

$$= \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}\left[x_n\right]$$
.

Since $\mathbb{E}\left[x\right] = \mu$, we get

$$\mathbb{E}\left[\mu_{ML}\right] = \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}\left[x_n\right] = \frac{1}{N}\sum_{n=1}^{N}\mu = \mu$$

Approach 2

Another approach may also be employed, using direct integration.

$$\mathbb{E}\left[\mu_{ML}\right] = \frac{1}{N}\sum_{n=1}^{N}\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} x_n exp\left\{-\frac{1}{2\sigma^2}\left(x_n-\mu\right)^2\right\} dx_n = \frac{1}{N}(\mu N)$$

Clearly, the above expression evaluates to μ.

ii) **Claim.** $\mathbb{E}\left[\sigma^2_{ML}\right] = \left(\frac{N-1}{N}\right)\sigma^2$

Pf.

According to

$$\sigma^2_{ML} = \frac{1}{N}\sum_{n=1}^{N}\left(x_n-\mu_{ML}\right)^2$$
,

we know

$$\mathbb{E}\left[\sigma^2_{ML}\right] = \mathbb{E}\left[\frac{1}{N}\sum_{n=1}^{N}\left(x_n-\mu_{ML}\right)^2\right] = \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}\left[\left(x_n^2 - 2x_n\mu_{ML} + \mu^2_{ML}\right)\right]$$
.

According to

$$\mu_{ML} = \frac{1}{N}\sum_{n=1}^{N}x_n$$
, we have

$$\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}\left[\left(x_n^2 - 2x_n\mu_{ML} + \mu^2_{ML}\right)\right] = \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}\left[\left(x_n^2 - \frac{2x_n}{N}\sum_{i=1}^{N}x_i + \frac{1}{N^2}\sum_{j=1}^{N}\sum_{k=1}^{N}x_jx_k\right)\right]$$

Since we already know

$$\mathbb{E}\left[x^2\right] = \mu^2 + \sigma^2$$
,

and when $i \neq j$,

$$\mathbb{E}\left[x_ix_j\right] = \mathbb{E}\left[x_i\right]\mathbb{E}\left[x_i\right] = \mu^2$$
.

Then we get

$$\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}\left[\left(x_n^2 - \frac{2x_n}{N}\sum_{i=1}^{N}x_i + \frac{1}{N^2}\sum_{j=1}^{N}\sum_{k=1}^{N}x_jx_k\right)\right]$$

$$= \mu^2+\sigma^2-\frac{2}{N^2}\left\{\sum_{n=1}^{N}\left[\left(\sigma^2+\mu^2\right)+\left(N-1\right)\mu^2\right]\right\}+\frac{1}{N^3}\sum_{n=1}^{N}\left\{N\left[\left(\sigma^2+\mu^2\right)+\left(N-1\right)\mu^2\right]\right\}$$

$$= \mu^2 + \sigma^2 - \frac{2}{N}\left(N\mu^2+\sigma^2\right) + \frac{1}{N}\left(N\mu^2+\sigma^2\right)$$

$$= \mu^2 + \sigma^2 - \mu^2 - \frac{\sigma^2}{N} = \left(\frac{N-1}{N}\right)\sigma^2.$$

Therefore, we can conclude

$$\mathbb{E}\left[\sigma_{ML}^2\right] = \left(\frac{N-1}{N}\right)\sigma^2.$$

Question 3

a)

i) **Claim.** $p(X,Y) = p(X)\cdot p(Y)$

Pf.

By using the product rule, we know $p(X,Y) = p\left(Y|X\right)p(X)$.

Since X and Y are independent random variables, we have $p\left(Y|X\right) = p(Y)$.

Therefore, $p(X,Y) = p(X)\cdot p(Y)$ holds for both continuous random variables and discrete random variables.

ii) **Claim.** $\mathbb{E}[XY] = \mathbb{E}[X]\cdot\mathbb{E}[Y]$

Pf.

Let's assume that the independent random variables are discrete. The definition of expectation shows

$$\mathbb{E}[XY] = \mathbb{E}[f(x)\,g(y)] = \sum_x\sum_y f(x)\,g(y)\,p_{\mathbf{XY}}(x,y),$$

where $p_{\mathbf{XY}}$ is the joint probability density of X and Y.

Since X and Y are independent, $p_{\mathbf{XY}}$ can actually be rewrite to

$p_{\mathbf{XY}} = p_{\mathbf{X}}(x)\,p_{\mathbf{Y}}(y)$.

Thus,

$$\sum_x\sum_y f(x)\,g(y)\,p_{\mathbf{XY}}(x,y) = \sum_x\sum_y f(x)\,p_{\mathbf{X}}(x)\,g(y)\,p_{\mathbf{Y}}(y)$$

$$= \sum_x f(x)\,p_{\mathbf{X}}(x)\sum_y g(y)\,p_{\mathbf{Y}}(y) = \mathbb{E}[f(x)]\,\mathbb{E}[g(y)]$$

Therefore, from

$\mathbb{E}[f(x)\,g(y)] = \mathbb{E}[f(x)]\,\mathbb{E}[g(y)]$,

we can conclude

$\mathbb{E}[XY] = \mathbb{E}[X]\cdot\mathbb{E}[Y]$.

For continuous random variables, we can just replace the summations with integrals in the previous formula derivation.

b) **Summary:** We call *random.randint(1, 2)* twice to generate two random integers at a time, and loop this procedure 10000 times to simulate the coin tosses with two independent coins. For a specific $x = a$ and $y = b$, $p(x = a,\ y = b) = (\textit{number of } x = a \textit{ and } y = b) / 10000$, $p(x = a) = (\textit{number of } x = a) / 10000$ and $p(y = b) = (\textit{number of } y = b) / 10000$. We notice that:

- $p(x = a)$ and $p(y = b)$ are very closed to 0.5;
- $p(x = a,\ y = b)$ and the results of $p(x = a)\cdot p(y = b)$ are very close to each other. The absolute distance between them is smaller than 10^{-2}.

On the other hand, $E(XY) = \sum_{all\ results\ of\ XY} (XY)p(XY) = 2.25$. $E(X) = E(Y) = 1.5$ because they both obey discrete uniform distribution. We notice that $E(XY)$ is exactly equals to $E(X) \cdot E(Y)$.

c) **Summary:** We call *random.sample(range(1, 3), 2)* to generate two dependent toss results for 10000 times. The dependent relation derives from this python function because it restricts the second toss result cannot be equals to the previous one. We use the same code as described in the previous experiment to calculate $p(x = a)$, $p(y = b)$ and $p(x = a,\ y = b)$. We notice that:

- $p(x = a)$ and $p(y = b)$ are both still very closed to 0.5;
- $p(x = a,\ y = b)$ is closed to 0.5 iff $a \neq b$, and equals to 0 iff $a = b$;
- The absolute distance between $p(x = a,\ y = b)$ and $p(x = a) \cdot p(y = b)$ is closed to 0.25.

On the other hand, we get $E(XY) = 2$, and both of $E(X)$ and $E(Y)$ are equals to 1.5. The absolute distance between $E(XY)$ and the result of $E(X) \cdot E(Y)$ is 0.25.

Appendix

A. Python code for Question 3(b): X and Y are independent random variables.

```
from __future__ import division
import random

# Generate tosses results
NUM_OF_TOSSES = 10000
tosses = []
for _ in range(NUM_OF_TOSSES):
    toss = []
    toss.append(random.randint(1, 2))
    toss.append(random.randint(1, 2))
    tosses.append(toss)

# Verify p(X,Y) = p(X) * p(Y)
x = 2
y = 1

count_x_y = 0
count_x = 0
count_y = 0
for toss in tosses:
    if toss[0] == x and toss[1] == y:
        count_x_y += 1
    if toss[0] == x:
        count_x += 1
    if toss[1] == y:
        count_y += 1
p_x_y = count_x_y / NUM_OF_TOSSES
p_x = count_x / NUM_OF_TOSSES
p_y = count_y / NUM_OF_TOSSES

## Show results summary
print "p_x =", p_x
print "p_y =", p_y
print "p_x * p_y =", (p_x * p_y)
print "p_x_y =", p_x_y
print "distance between p_x_y and p_x * p_y : ", abs(p_x_y - p_x * p_y), "\n"


# Verify E(XY) = E(X) * E(Y)
rg = range(1, 3)

## Calculate E(XY)
x_y_values=[]
x_y_count=[]
for i in rg:
    for j in rg:
        m = i * j
        if m not in x_y_values:
            x_y_values.append(m)
            x_y_count.append(1)
        else:
            x_y_count[x_y_values.index(m)] += 1

cases = sum(x_y_count)
x_y_expt = 0
for i in range(len(x_y_values)):
    x_y_expt += x_y_values[i] * x_y_count[i] / cases

print "x_y_expt =", x_y_expt

## Calculate E(X) or E(Y)
x_expt = 0
```

```
for i in rg:
    x_expt += i * 1 / len(rg)

y_expt = x_expt

## Show results summary
print "x_expt =", x_expt
print "y_expt =", y_expt

print "distance between x_y_expt and x_expt * y_expt : ", abs(x_y_expt - x_expt * y_expt)
```

B. Python code for Question 3(c): X and Y are NOT independent random variables.

```
from __future__ import division
import random

# Generate tosses results
NUM_OF_TOSSES = 10000
tosses = []
for _ in range(NUM_OF_TOSSES):
    # this ensures X and Y cannot be repeated, such that
    # X and Y is not independent
    tosses.append(random.sample(range(1, 3), 2))

# Verify p(X,Y) = p(X) * p(Y)
x = 1
y = 2

count_x_y = 0
count_x = 0
count_y = 0
for toss in tosses:
    if toss[0] == x and toss[1] == y:
        count_x_y += 1
    if toss[0] == x:
        count_x += 1
    if toss[1] == y:
        count_y += 1
p_x_y = count_x_y / NUM_OF_TOSSES
p_x = count_x / NUM_OF_TOSSES
p_y = count_y / NUM_OF_TOSSES

## Show results summary
print "p_x =", p_x
print "p_y =", p_y
print "p_x * p_y =", (p_x * p_y)
print "p_x_y =", p_x_y
print "distance between p_x_y and p_x * p_y : ", abs(p_x_y - p_x * p_y), "\n"


# Verify E(XY) = E(X) * E(Y)
rg = range(1, 3)

## Calculate E(XY)
x_y_values=[]
x_y_count=[]
for i in rg:
    for j in rg:
        if i == j: continue
        m = i * j
        if m not in x_y_values:
            x_y_values.append(m)
            x_y_count.append(1)
        else:
            x_y_count[x_y_values.index(m)] += 1
```

```
cases = sum(x_y_count)
x_y_expt = 0
for i in range(len(x_y_values)):
    x_y_expt += x_y_values[i] * x_y_count[i] / cases

print "x_y_expt =", x_y_expt

## Calculate E(X) or E(Y)
x_expt = 0
for i in rg:
    x_expt += i * 1 / len(rg)

y_expt = x_expt

## Show results summary
print "x_expt =", x_expt
print "y_expt =", y_expt

print "distance between x_y_expt and x_expt * y_expt : ", abs(x_y_expt - x_expt * y_expt)
```
