The Costs of Economic Growth

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Abstract

The benefits of economic growth are widely touted in the literature. But what about the costs? Pollution, nuclear accidents, global warming, the rapid global transmission of disease, and bioengineered viruses are just some of the dangers created by technological change. How should these be weighed against the benefits, and in particular, how does the recognition of these costs affect the theory of economic growth? This paper shows that taking these costs into account has first-order consequences for economic growth. The rising value of life associated with standard utility functions generates a conservative bias to technological change, significantly slowing the optimal rate of economic growth.

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> Certain events quite within the realm of possibility, such as a major asteroid collision, global bioterrorism, abrupt global warming — even certain lab accidents— could have unimaginably terrible consequences up to and including the extinction of the human race... I am not a Green, an alarmist, an apocalyptic visionary, a catastrophist, a Chicken Little, a Luddite, an anticapitalist, or even a pessimist. But... I have come to believe that what I shall be calling the "catastrophic risks" are real and growing...
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> — Richard A. ?, p. v

1. Introduction

In October 1962, the Cuban missile crisis brought the world to the brink of a nuclear holocaust. President John F. Kennedy put the chance of nuclear war at "somewhere between one out of three and even." The historian Arthur Schlesinger, Jr., at the time an adviser of the President, later called this "the most dangerous moment in human history."[1] What if a substantial fraction of the world's population had been killed in a nuclear holocaust in the 1960s? In some sense, the overall cost of the technological innovations of the preceding 30 years would then seem to have outweighed the benefits.

While nuclear devastation represents a vivid example of the potential costs of technological change, it is by no means unique. The benefits from the internal combustion engine must be weighed against the costs associated with pollution and global warming. Biomedical advances have improved health substantially but made possible weaponized anthrax and lab-enhanced viruses. The potential benefits of nanotechnology stand beside the "grey goo" threat that a self-replicating machine could someday spin out of control. Experimental physics has brought us x-ray lithography techniques and superconductor technologies but also the remote possibility of devastating accidents as we smash particles together at ever higher energies.

[1]For these quotations, see (?, p. 26).

These and other technological dangers are detailed in a small but growing literature on so-called "existential risks"; ? is likely the most familiar of these references, but see also ?, ?, ?, and ?.

Technologies need not pose risks to the existence of humanity in order to have costs worth considering. New technologies come with risks as well as benefits. A new pesticide may turn out to be harmful to children. New drugs may have unforeseen side effects. Marie Curie's discovery of the new element radium led to many uses of the glow-in-the-dark material, including a medicinal additive to drinks and baths for supposed health benefits, wristwatches with luminous dials, and as makeup — at least until the dire health consequences of radioactivity were better understood. Other examples of new products that were intially thought to be safe or even healthy include thalidomide, lead paint, asbestos, and cigarettes.

The benefits of economic growth are truly amazing and have made enormous contributions to welfare. However, this does not mean there are not also costs. How does this recognition affect the theory of economic growth? This paper shows that taking these costs into account has first-order consequences. In particular, the rising value of life associated with standard utility functions generates a conservative bias to technological change, significantly slowing the optimal rate of economic growth.

2. The Model

At some level, this paper is about speed limits. You can drive you car slowly and safely, or fast and recklessly. Similarly, a key decision the economy must make is to set a safety threshold: researchers can introduce many new ideas without regard to safety, or they can select a very tight safety threshold and introduce fewer ideas each year, slowing growth to some extent.

The model below is a standard idea-based growth model, along the lines of ? and ?. Researchers introduce new varieties of intermediate goods, and the economy's productivity is increasing in the number of varieties. The key change relative to standard models is that each variety i also comes with a danger level, z^i. Some ideas are especially dangerous (nuclear weapons or lead paint) and have a high value of z^i, while other ideas are relatively harmless and have a low z^i. The mortality rate in the economy depends on the values of the z^i that are consumed as well as on the amount consumed.

In the equilibrium allocation we study, firms that sell dangerous products must pay a fee for each person they kill, and this fee is a price determined in equilibrium.

2.1. The Economic Environment

The economy features three types of goods: consumption goods (which come in a range of varieties), ideas, and people. People and ideas are the two key factors of production, combining to produce the consumption goods and new ideas.

At any point in time, a variety of consumption goods indexed by i on the interval $[0, A_t]$ are available for purchase. We could define utility directly over this variety of goods, but for the usual reasons, it is easier to handle the aggregation on the production side. Hence, we assume these varieties combine in a CES fashion to produce a single aggregate consumption good:

$$C_t = \left(\int_0^{A_t} X_{it}^{\theta} di \right)^{1/\theta}, \quad \theta > 1 \tag{1}$$

New varieties (ideas) are produced by researchers. If L_{at} units of labor are used in research with a current stock of knowledge A_t, then research leads to the discovery of $\alpha L_{at}^{\lambda} A_t^{\phi}$ new varieties. This technology for producing new ideas is similar to ?.

What's novel here is that each new variety i is also associated with a danger level, z^i. This danger level is drawn from a distribution with cdf $F(z)$ and is observed as soon as the variety is discovered. Researchers decide whether or not to complete the development of a new variety after observing its danger level. Given that varieties are otherwise symmetric, this leads to a cutoff level z_t: varieties with a danger level below z_t get implemented, whereas more dangerous varieties do not. z_t is a key endogenous variable determined within the model. The fraction $F(z_t) = \Pr\left[z^i \leq z_t\right]$ of candidate varieties get implemented, so the additional number of new varieties introduced at any point in time is

$$\dot{A}_t = \alpha F(z_t) L_{at}^{\lambda} A_t^{\phi}, \qquad A_0 \text{ given.} \tag{2}$$

One unit of labor can produce one unit of any existing variety, and labor used for different purposes cannot exceed the total amount available in the economy, N_t:

$$\int_0^{A_t} X_{it} di + L_{at} \leq N_t. \tag{3}$$

This total population is assumed to grow over time according to

$$\dot{N}_t = (\bar{n} - \delta_t) N_t, \qquad N_0 \text{ given.} \tag{4}$$

The parameter $\bar{n}$ captures exogenous fertility net of mortality unrelated to technological change.

Mortality from technological danger is denoted δ_t. In principle, it should depend on the amount of each variety consumed and the danger associated with each variety, and it could even be stochastic (nuclear weapons are a problem only if they are used). There could also be timing issues: the use of fossil fuels today creates global warming that may be a problem in the future.

These issues are interesting and could be considered in future work. To keep the present model tractable, however, we make some simplifying assumptions in determining mortality. In particular, all of the deaths associated with any new technology occur immediately when that technology is implemented, and the death rate depends on average consumption across all varieties. That is,

$$\delta_t = \bar{\delta} \dot{A}_t x_t \Gamma(z_t), \tag{5}$$

where $x_t \equiv \int_0^{A_t} X_{it}/N_t di/A_t$ is the average amount consumed of each variety and $\Gamma(z_t) \equiv E[z^i | z^i \leq z_t] = \int_0^{z_t} z dF(z)/F(z_t)$ is the average mortality rate associated with those new varieties. The mortality rate δ_t, then, is the product of the per

Figure 1: Flow Utility $u(c)$ for $\gamma > 1$

Note: Flow utility is bounded for $\gamma > 1$. If $\bar{u} = 0$, then flow utility is negative and dying is preferred to living.

capita quantity of new varieties consumed, $\dot{A}_t x_t$, and their average mortality rate.

Individuals care about expected utility, where the expectation is taken with respect to mortality. Let S_t denote the probability a person survives until date t conditional on being alive at date 0. Expected utility is given by

$$U = \int_0^\infty e^{-\rho t} u(c_t) S_t dt, \tag{6}$$

where

$$\dot{S}_t = -\delta_t S_t, \quad S_0 = 1. \tag{7}$$

Finally, we assume that flow utility $u(c)$ is

$$u(c_t) = \bar{u} + \frac{c_t^{1-\gamma}}{1-\gamma}, \qquad c_t \equiv C_t/N_t, \quad \bar{u} > 0. \tag{8}$$

The key properties of this utility function are discussed next.

2.2. Flow Utility and the Value of Life

The cost of dying is the loss of future periods of utility. The flow lost in year t depends directly on $u(c_t)$. Figure 1 shows this flow utility for the CRRA formulation used in this paper, equation (8), for the special case in which $\gamma > 1$. This case turns out to be especially interesting in what follows. There are two points to notice in this graph. First, if $\bar{u} = 0$, then flow utility is negative. Since we have (implicitly) normalized the utility of death at zero in writing lifetime utility, utility would be maximized by never living in this case. Hence $\bar{u} > 0$ is required for this model to make sense.

Second, flow utility is bounded for $\gamma > 1$. Marginal utility goes to zero very quickly for these preferences. Eating more sushi on a given day when one is already eating sushi for breakfast, lunch, and dinner has very low returns. Instead, preserving extra days of life on which to eat sushi is the best way to increase utility.

This point can also be made with the algebra. Valued in units of consumption instead of utils and expressed as a ratio to current consumption, the flow value of a year of life is

$$\frac{u(c_t)}{u'(c_t)c_t} = \bar{u}c_t^{\gamma-1} - \frac{1}{1-\gamma} \tag{9}$$

For $\gamma > 1$, the value of life rises faster than consumption; this is the essential mechanism that leads the economy to tilt its allocation away from consumption growth and toward preserving life in the model. This point is more general than the particular utility function assumed here. For example, any bounded utility function will deliver this result, as will log utility.[2]

2.3. A Rule of Thumb Allocation

Given the symmetry of X_{it}, there are two nontrivial allocative decisions that have to be made in this economy at each date. First is the allocation of labor between L_{at} and X_{it}. Second is the key tradeoff underlying this paper, the choice of the safety threshold z_t. A high cutoff for z_t implies that more new ideas are introduced in each period but it also means a higher mortality rate. This is the model's analog to driving fast and recklessly instead of slowly and safely.

In the next main section, we will let markets allocate resources and study an equilibrium allocation. To get a sense for how the model works, however, it is convenient to begin first with a simple rule of thumb allocation. For this example, we assume the economy puts a constant fraction $\bar{s}$ of its labor in research and allocates the remainder symmetrically to the production of the consumption goods. In addition, we assume the safety cutoff is constant over time at $\bar{z}$.

Let g_x denote the growth rate of some variable x along a balanced growth path. Then, we have the following result (proofs for this and other propositions are given

[2]For log preferences, $u(c) = \bar{u} + \log c$. Because $u'(c)c = 1$, the value of a year of life in consumption units is just $u(c)$ itself, which increases without bound in consumption.

Figure 2: Growth under the Rule of Thumb Allocation

Note: There is a medium-run tradeoff between growth and technological danger, but no long-run tradeoff. In the long run, safer choices lead to faster net population growth and therefore faster consumption growth.

in Appendix A):

Proposition 1 (BGP under the Rule of Thumb Allocation): Under the rule of thumb allocation, there exists a balanced growth path such that $g_c = \sigma g_A$ and

$$\delta^* = \bar{\delta} g_A (1 - \bar{s}) \Gamma(\bar{z}) \tag{10}$$

$$g_N = \bar{n} - \delta^* \tag{11}$$

$$g_A = \frac{\lambda(\bar{n} - \delta^*)}{1 - \phi} = \frac{\lambda \bar{n}}{1 - \phi + \lambda \bar{\delta}(1 - \bar{s})\Gamma(\bar{z})}. \tag{12}$$

Along the balanced growth path, the mortality rate is constant and depends on (a) how fast the economy grows, (b) the intensity of consumption, and (c) the danger threshold. As in ?, the steady-state growth rate is proportional to the rate of population growth. However, the population growth rate is now an endogenous variable because of endenous mortality. For example, an increase in research intensity $\bar{s}$ will reduce the steady-state mortality rate (a lower consumption intensity) and therefore increase the long-run growth rate.

The effect of changing the danger threshold $\bar{z}$ is more subtle and is shown graphically in Figure 2. As emphasized earlier, there is indeed a basic tradeoff in this model between growth and safety. Over the first 300+ years in the example, the safer choice of $\bar{z}$ leads to slower growth as researchers introduce fewer new varieties. However, this tradeoff disappears in the long run because the growth rate itself depends on population growth. A safer technology choice reduces the mortality rate, raises the population growth rate, and therefore raises consumption growth in the long run.

3. A Competitive Equilibrium with Patent Buyouts

The rule of thumb allocation suggests that this model will deliver a balanced growth path with an interesting distinction between the medium-run and long-run tradeoffs between growth and safety. Moreover, the model features endogenous growth in the strong sense that changes in policy can affect the long-run growth rate. Somewhat surprisingly, neither of these results will hold in the competitive equilibrium, and our rule of thumb allocation turns out not to be a particularly good guide to the equilibrium dynamics of the competitive equilibrium.

3.1. An Overview of the Equilibrium

A perfectly competitive equilibrium will not exist in this model because of the nonrivalry of ideas (?). Instead of following Romer and introducing imperfect competition, we use a mechanism advocated by ?. That is, we consider an equilibrium in which research is funded entirely by "patent buyouts": the government in our model purchases new ideas at a price P_{at} and makes the designs publicly available at no charge. The motivation for this approach is largely technical: it simplifies the model so it is easier to understand. However, there is probably some interest in studying this institution in its own right.

The other novel feature of this equilibrium is that we introduce a competitive market for mortality: idea producers pay a price v_t for every person they kill, and households "sell" their mortality as if survival were a consumer durable. This market bears some resemblence to one that emerges in practice through the legal system of torts and liabilities.

In equilibrium, these two institutions determine the key allocations. Patent buyouts pin down the equilibrium amount of research and the mortality market pins down the danger cutoff.

3.2. Optimization Problems

The equilibrium introduces three prices: a wage w_t, the price of mortality v_t, and the price of new ideas P_{at}. The equilibrium then depends on three optimization problems.

First, a representative household supplies a unit of labor, chooses how much of her life to sell in the mortality market, pays a lump-sum tax τ_t, and eats the proceeds. Our timing assumption is that mortality is realized at the end of the period, after consumption occurs.

HH Problem: Given $\{w_t, v_t, \tau_t\}$, the representative household solves

$$\max_{\{\delta_t^h\}} \int_0^\infty e^{-\rho t} u(c_t) S_t dt$$

$$\text{s.t. } c_t = w_t + v_t \delta_t^h - \tau_t \text{ and } \dot{S}_t = -\delta_t^h S_t.$$

Next, a representative firm in the perfectly competitive market for the final good (FG) solves the following profit maximization problem:

FG Problem: At each date t, given w_t and A_t,

$$\max_{\{X_{it}\}} \left(\int_0^{A_t} X_{it}^\theta di \right)^{1/\theta} - w_t \int_0^{A_t} X_{it} di.$$

Finally, a representative research firm produces ideas in the perfectly competitive idea sector and chooses a cutoff danger level z_t based on the price of mortality. The research firm sees constant returns to idea production at productivity α_t, so the effects associated with $\lambda < 1$ and $\phi \neq 0$ are external:

R&D Problem: At each date t, given $P_{at}, w_t, v_t, x_t, \alpha_t$,

$$\max_{L_{at}, z_t} P_{at} \alpha_t F(z_t) L_{at} - w_t L_{at} - v_t \delta_t N_t \;\; \text{s.t} \;\; \delta_t = \bar{\delta} x_t \alpha_t F(z_t) L_{at} \Gamma(z_t).$$

3.3. Defining the Competitive Equilibrium

The competitive equilibrium in this economy solves the optimization problems given in the previous section and the relevant markets clear. The only remaining issue to discuss is the government purchase of ideas. We've already assumed the government pays a price P_{at} for any idea and releases the design into the public domain. We assume this is the only option for researchers — there is no way to keep ideas secret and earn a temporary monopoly profit. As discussed above, the reason for this is to keep the model simple; nothing would change qualitatively if we introduced monopolistic competition, either through secrecy or patents.

In addition, we assume the idea purchases are financed with lump-sum taxes on households and that the government's budget balances in each period. Moreover, we assume the government sets the price at which ideas are purchased so that total purchases of ideas are a constant proportion β of aggregate consumption; we will relax this assumption later.

The formal definition of the equilibrium allocation follows:

Definition A CE with public support for R&D consists of quantities $\{c_t, \delta_t^h, X_{it}, A_t, L_{at}, N_t, \tau_t, \delta_t, z_t, \alpha_t, x_t\}$ and prices $\{w_t, P_{at}, v_t\}$ such that

1. $\{c_t, \delta_t^h\}$ solve the HH Problem.
2. X_{it} solve the FG Problem.
3. $L_{at}, z_t, \delta_t, A_t$ solve the R&D Problem.
4. w_t clears the labor market: $\int_0^{A_t} X_{it} di + L_{at} = N_t$.
5. v_t clears the mortality market: $\delta_t^h = \delta_t$.
6. The government buys ideas: $P_{at}\dot{A}_t = \beta c_t N_t$.
7. Lump sum taxes τ_t balance the budget: $\tau_t = P_{at}\dot{A}_t / N_t$.
8. Other conditions: $\dot{N}_t = (\bar{n} - \delta_t)N_t$, $\alpha_t = \alpha L_{at}^{\lambda-1} A_t^{\phi}$, and $x_t \equiv \frac{1}{A_t}\int_0^{A_t} X_{it} di / N_t$.

3.4. The Benchmark Case

The equilibrium behavior of the economy depends in important ways on a few parameters. We specify a benchmark case that will be studied in detail, and then in subsequent sections consider the effect of deviating from this benchmark. In specifying the benchmark, it is helpful to note that in equilbrium $c_t = A_t^\sigma(1-s_t)$, where $\sigma \equiv \frac{1-\theta}{\theta}$ is the elasticity of consumption with respect to the stock of ideas. The benchmark case is then given by

Assumption A. (Benchmark Case) Let $\eta \equiv \lim_{z\to 0} F'(z)z/F(z)$. Assume

A1. Finite elasticity of $F(z)$ as $z \to 0$: $\eta \in (0,\infty)$

A2. Rapidly declining marginal utility of consumption: $\gamma > 1$

A3. Knowledge spillovers are not too strong: $\phi < 1 + \eta\sigma(\gamma - 1)$.

We will discuss the nature and role of each of these assumptions in more detail as we develop the results. The least familiar assumption is A1, but note that both the exponential and the Weibull distributions have this property. In contrast, the lognormal and Fréchet distributions have an infinite elasticity in the limit as z goes to zero. This has interesting implications that we will explore in detail.

3.5. The Equilibrium Balanced Growth Path

We are now ready to solve for the equilibrium in this growth model with dangerous technologies. In particular, we can characterize a balanced growth path:

Proposition 2 (Equilibrium Balanced Growth): Under Assumption A, the competitive equilibrium exhibits an asymptotic balanced growth path as $t \to \infty$ such that

$$s_t \to \frac{\beta}{1+\beta+\eta} \tag{13}$$

$$z_t \to 0 \text{ (and therefore } \delta_t \to 0) \tag{14}$$

$$\dot{z}_t/z_t \to g_z = -(\gamma-1)g_c \tag{15}$$

$$\dot{c}_t/c_t \to g_c = \sigma g_A \equiv \frac{\lambda\sigma\bar{n}}{1-\phi+\eta\sigma(\gamma-1)} \tag{16}$$

$$u'(c_t)v_t \to \frac{\bar{u}}{\rho}. \tag{17}$$

The somewhat surprising result that emerges from the equilibrium under Assumption A is that mortality and the danger threshold, rather than being constant in steady state, decline at a constant exponential rate. Technological change becomes increasingly conservative over time, as an increasing fraction of possible new ideas are rejected because they are too dangerous.

The consequence of this conservative bias in technological change is no less surprising: it leads to a slowdown in steady-state growth. There are several senses in which this is true, and these will be explored as the paper goes on. But two are evident now. First, the negative growth rate of z_t introduces a negative trend in TFP growth for the idea sector, other things equal. Recall that the idea production function is $\dot{A}_t = \alpha F(z_t) L_{at}^{\lambda} A_t^{\phi}$. z_t declines at the rate $(\gamma-1)g_c$, ultimately getting arbitrarily close to zero. By Assumption A1, the elasticity of the distribution $F(z)$ at zero is finite and given by η, so $F(z_t)$ declines at rate $\eta(\gamma-1)g_c = \eta(\gamma-1)\sigma g_A$.

The second way to see how this bias slows down growth is to focus directly on consumption growth itself. The steady-state rate of consumption growth is

$$g_c = \frac{\lambda\sigma\bar{n}}{1-\phi+\eta\sigma(\gamma-1)}. \tag{18}$$

The last term in the denominator directly reflects the negative TFP growth in the idea production function resulting from the tightening of the danger threshold.

That this slows growth can be seen by considering the following thought experiment. A feasible allocation in this economy is to follow the equilibrium path until z_t is arbitrarily small and then keep it constant at this value. This results in a mortality rate that is arbitrarily close to zero, and the growth rate in this case will be arbitrarily close to $\lambda\sigma\bar{n}/(1-\phi)$, which is clearly greater than the equilibrium growth rate. Rather than keep z constant at a small level, the equilibrium continues to reduce the danger cutoff, slowing growth. Some numerical examples at the end of this paper suggest that this slowdown can be substantial.

Of course, this raises a natural question: Why does the equilbrium allocation lead the danger threshold to fall exponentially to zero? To see the answer, first consider the economic interpretation of the mortality price v_t. This is the price at which firms must compensate households per unit of mortality that their inventions inflict. In the terminology of the health and risk literatures, it is therefore equal to the value of a statistical life (VSL).[3]

Along the balanced growth path, the value of life satisfies equation (17):

$$u'(c_t)v_t \to \frac{\bar{u}}{\rho}. \tag{19}$$

This equation simply says that the value of life measured in utils is asymptotically equal to the present discounted value of utility: as consumption goes to infinity, flow utility converges to $\bar{u}$, so lifetime utility is just $\bar{u}/\rho$.

Viewed in another way, this equation implies that the value of life grows faster than consumption. Given our functional form assumption for preferences, $u'(c_t) = c_t^{-\gamma}$. So $c_t^{-\gamma} v_t$ converges to a constant, which means that $g_v \to \gamma g_c$. Because γ is larger than one, marginal utility falls rapidly, and the value of life rises faster than consumption.

With this key piece of information, we can turn to the first-order condition for the choice of z_t in the R&D Problem. That first-order condition is

$$z_t = \frac{P_{at}}{v_t N_t \bar{\delta} x_t} = \frac{\beta c_t}{v_t \bar{\delta} g_{At}(1 - s_t)}. \tag{20}$$

The first part of this equation says that the danger threshold z_t equals the ratio of two terms. The numerator is related to the marginal benefit of allowing more dangerous technologies to be used, which is proportional the price at which the additional ideas could be sold. The denominator is related to the marginal cost, which depends on the value of the additional lives that would be lost.

[3]Suppose the mortality rate is $\delta_t = .001$ and $v_t = \$1$ million. In this example, each person receives \$1000 ($= v_t\delta_t$) for the mortality risk they face. For every thousand people in the economy, one person will die, and the total compensation paid out for this death will equal \$1 million.

The second equation in (20) uses the fact that $P_{at}\dot{A}_t = \beta C_t$ to eliminate the price of ideas. This last expression illustrates the key role played by the value of life. In particular, we saw above that v_t/c_t grows over time since $\gamma > 1$; the value of life rises faster than consumption. Because both g_{At} and $1 - s_t$ are constant along the balanced growth path, the rapidly rising value of life leads to the exponential decline in z_t. More exactly, v_t/c_t grows at rate $(\gamma - 1)g_c$, so this is the rate at which z_t declines, as seen in equation (15).

What is the economic intuition? Because γ exceeds 1, flow utility $u(c)$ is bounded and the marginal utility of additional consumption falls very rapidly. This leads the value of life to rise faster than consumption. The benefit of using more dangerous technologies is that the economy gets more consumption. The cost is that more people die. Because the marginal utility of consumption falls so quickly, the costs of people dying exceed the benefit of increasing consumption and the equilibrium delivers a declining threshold for technological danger. Safety trumps economic growth.

3.6. Growth Consequences

From the standpoint of growth theory, there are some interesting implications of this model. First, as we saw in the context of the rule-of-thumb allocation, this model is potentially a fully endogenous growth model. Growth is proportional to population growth, but because the mortality rate is endogenous, policy changes can affect the mortality rate and therefore affect long-run growth, at least potentially.

Interestingly, however, that is not the case in the equilibrium allocation. Instead, the mortality rate trends to zero and is unaffected by policy changes in the long run. Hence, the equilibrium allocation features semi-endogenous growth, where policy changes have long-run level effects but not growth effects. In particular, notice that the patent buyout parameter β, which influences the long-run share of labor going to research, is not a determinant of the long-run growth rate in (18). Moreover, the invariance of long-run growth to policy is true even though a key preference parameter, γ, influences the long-run growth rate.

Finally, it is interesting to consider the special case of $\phi = 1$, so the idea production function resembles that in ?. In this case,

$$\frac{\dot{A}_t}{A_t} = \alpha F(z_t) L_{at}^{\lambda}.$$

Here, growth does not explode even in the presence of population growth with $\phi = 1$, as can be seen in equation (18). Instead, the negative trend in z_t and the fact that the mortality rate depends on the growth rate conspire to keep growth finite.

4. Extensions

The crucial assumptions driving the results have been collected together and labeled as Assumption A. In this section, we illustrate how things change when these assumptions are relaxed. Briefly, there are two main findings. First, when we consider distributions with an infinite elasticity at $z = 0$, the concern for safety exhibits an even more extreme technological bias: equilibrium growth slows all the way to zero asymptotically. Second, we highlight the role played by $\gamma > 1$: if instead $\gamma < 1$, then the equilibrium allocation looks like the rule of thumb allocation, selecting a constant danger threshold in steady state.

4.1. Relaxing A1: Letting $\eta = \infty$

Recall that η is the elasticity of the danger distribution $F(z)$ in the limit as $z \to 0$. Intuitively, this parameter plays an important role in the model because $F(z)$ is the fraction of new ideas that are used in the economy, and z is trending exponentially to zero. The term $\eta g_z = -\eta(\gamma - 1)g_c$ therefore plays a key role in determining the growth rate of ideas along the balanced growth path:

$$g_A = \frac{\lambda \bar{n}}{1 - \phi + \eta\sigma(\gamma - 1)}. \tag{21}$$

Assumption A1 says that η is finite. This is true for a number of distributions, including the exponential ($\eta = 1$), the Weibull, and the gamma distributions. However, it is not true for a number of other distributions. Both the lognormal and the Fréchet distributions have an infinite elasticity at zero, for example. Given that we have no prior over which of these distributions is most relevant to our problem, it is essential to consider carefully the case of $\eta = \infty$.

In fact, it is easy to get a sense for what will happen by considering equation (21). As η rises in this equation, the steady-state growth rate of the economy declines. Intuitively, a 1% reduction in z has a larger and larger effect on $F(z)$: an increasing fraction of ideas that were previously viewed as safe are now rejected as too dangerous. This reasoning suggests that as η gets large, the steady-state growth rate falls to zero, and this intuition is confirmed in the following proposition:

Proposition 3 (Equilibrium Growth with $\eta = \infty$): Let Assumptions A2 and A3 hold, but instead of A1, assume $\eta = \infty$. In the competitive equilibrium, as $t \to \infty$

1. The growth rate of consumption falls to zero: $\dot{c}_t/c_t \to 0$
2. The level of consumption goes to infinity: $c_t \to \infty$
3. The technology cutoff, the mortality rate, and the share of labor devoted to research all go to zero: $z_t \to 0$, $\delta_t \to 0$, $s_t \to 0$.

When $\eta = \infty$, the increasingly conservative bias of technological change slows the exponential growth rate all the way to zero. However, this does not mean that growth ceases entirely. Instead, the level of consumption still rises to infinity, albeit at a slower and slower rate.

4.2. Relaxing A2: Assume $\gamma < 1$

The most important assumption driving the results in this paper is that marginal utility diminishes quickly, in the sense that $\gamma > 1$. For example, the value of a year of life in year t as a ratio to consumption is

$$\frac{u(c_t)}{u'(c_t)c_t} = \bar{u}c_t^{\gamma-1} + \frac{1}{1-\gamma}.$$

For $\gamma > 1$, this rises to infinity as consumption grows. But for $\gamma < 1$, it converges to $1/(1-\gamma)$: the value of life is proportional to consumption. In this case, the elasticity of utility with respect to consumption remains positive rather than falling to zero, which keeps the value of life and consumption on equal footing. The result is that the conservative bias of technological change disappears: the economy features exponential growth in consumption with a constant danger cutoff and a constant, positive mortality rate:

Proposition 4 (Equilibrium Growth with $\gamma < 1$): Let Assumptions A1 and A3 hold, but instead of A2, assume $\gamma < 1$. The competitive equilibrium exhibits an asymptotic balanced growth path as $t \to \infty$ such that

$$z_t \to z^* \in (0,\infty), \quad \delta_t \to \delta^* \in (0,\infty)$$

$$s_t \to \frac{\beta(1-\frac{\Gamma(z^*)}{z^*})}{1+\beta(1-\frac{\Gamma(z^*)}{z^*})}$$

$$\frac{\dot{c}_t}{c_t} \to g_c = \frac{\lambda(\bar{n}-\delta^*)}{1-\phi} = \frac{\lambda\sigma\bar{n}}{1-\phi+\lambda\bar{\delta}(1-s^*)\Gamma(z^*)}$$

$$\frac{v_t}{c_t} \to \frac{1}{1-\gamma}\cdot\frac{1}{\rho+\delta^*-(1-\gamma)g_c}$$

For $\gamma < 1$, the economy looks very similar to the rule of thumb allocation; for example, compare the growth rate to that in Proposition 1. The economy features a constant danger cutoff as well as endogenous growth: an increase in idea purchases by the government (a higher β) will shift more labor into research, lower the mortality rate, and increase the long-run growth rate.[4]

4.3. Optimality

In this section, we study the allocation of resources that maximizes a social welfare function. There are three reasons for this. First, it is important to verify that the

[4]The intermediate case of log utility ($\gamma = 1$) requires separate consideration. In this case, the technology cutoff z_t still declines to zero, but this decline is slower than exponential. The long-run growth rate is then precisely back to the semi-endogenous growth case: $g_A = \lambda\bar{n}/(1-\phi)$.

declining danger threshold we uncovered in the equilibrium allocation is not a perverse feature of our equilibrium. Second, in the equilibrium allocation, individuals put no weight on the welfare of future generations; it is a purely selfish equilibrium. It is interesting to study the effect of deviations from this benchmark. Finally, our equilibrium allocation employed a particular institution for funding research: patent buyouts where spending on new ideas is in constant proportion to consumption. This institution is surely special (and not generally optimal), so it is important to confirm that it is not driving the results. The bottom line of this extension to an optimal allocation is that all of the previous results hold up well.

In this environment with multiple generations, there is no indisputable social welfare function. However, a reasonably natural choice that serves our purposes is to treat flows of utility from different people symmetrically and to discount flows across time at rate ρ. This leads to the following definition of an optimal allocation:

> Definition An optimal allocation in this economy is a time path for $\{s_t, z_t\}$ that solves

$$\max_{\{s_t,z_t\}} \int_0^\infty e^{-\rho t} N_t u(c_t) dt \quad \text{subject to}$$

$$c_t = A_t^\sigma (1 - s_t)$$

$$\dot{A}_t = \alpha F(z_t) s_t^\lambda N_t^\lambda A_t^\phi$$

$$\dot{N}_t = (\bar{n} - \delta_t) N_t$$

$$\delta_t = \bar{\delta} \alpha s_t^\lambda N_t^\lambda A_t^{\phi-1} (1 - s_t) \int_0^{z_t} z f(z) dz.$$

The optimal allocation can then be characterized as follows.

Proposition 5 (Optimal Balanced Growth): Under Assumption A and the optimal allocation, the economy exhibits an asymptotic balanced growth path as $t \to \infty$ such that

$$\frac{s_t}{1 - s_t} \to \frac{\lambda \sigma g_A}{(1 + \eta)(\rho - \bar{n} + (\gamma - 1) g_c) + (1 - \phi) g_A - \eta \sigma g_A}$$

$$z_t \to 0 \text{ (and therefore } \delta_t \to 0)$$

$$\dot{z}_t/z_t \to -(\gamma - 1)g_c$$

$$\dot{c}_t/c_t \to g_c,$$

where g_c and g_A are the same as in the competitive equilibrium.

The key properties of the competitive equilibrium carry over into the optimal allocation. In particular, the danger threshold declines exponentially to zero at the rate $(\gamma - 1)g_c$, and this technological bias slows the growth rate of the economy. The long-run growth rate is the same as in the equilibrium allocation.

5. Numerical Examples

We now report a couple of numerical examples to illustrate how this economy behaves along the transition path. We make some attempt to choose plausible parameter values and to produce simulations that have a "realistic" look to them. However, the model abstracts from a number of important forces shaping economic growth and mortality, so the examples should not be taken too literally. Mainly, they will illustrate the extent to which growth can be slowed by concerns about the dangers of certain technologies.

The first example features sustained exponential growth ($\eta < \infty$). The second assumes a distribution $F(z)$ with an elasticity that rises to infinity as z falls to zero. According to Proposition 3, this example exhibits a growth rate that declines to zero, even though consumption itself rises indefinitely.

5.1. Benchmark Example

The basic parameterization of the benchmark case is described in Table 1. For the curvature of marginal utility, we choose $\gamma = 1.5$; large literatures on intertemporal choice (?), asset pricing (?), and labor supply (?) suggest that this is a reasonable value. For $F(z)$, we assume an exponential distribution so that $\eta = 1$; we also

Table 1: Benchmark Parameter Values

Parameter Value	Comment
$\gamma = 1.5$	Slightly more curvature than log utility
$\eta = 1$	$F(z)$ is an exponential distribution
$\beta = 0.02$	Government spends 2% of consumption on ideas
$\lambda = 1$, $\phi = 1/2$	Idea production function: $\dot{A}_t = \alpha F(z_t) A_t^{1/2} L_{at}$
$\bar{n} = .01$	Long-run population growth rate
$\sigma = 2$	Elasticity of consumption wrt ideas
$\rho = .05$	Rate of time preference
$\bar{\delta} = 50$	Mortality rate intercept

Note: These are the baseline parameter values for the numerical examples.

Figure 3: Equilibrium Dynamics: Benchmark Case

Note: Simulation results for the competitive equilibrium using the parameter values from Table 1. Consumption growth settles down to a constant positive rate, substantially lower than what is feasible. The danger threshold and mortality rate converge to zero.

assume this distribution has a mean of one. We set $\beta = .02$: government spending on ideas equals 2% of aggregate consumption. For the idea production function, we choose $\lambda = 1$ and $\phi = 1/2$, implying that in the absence of declines in z_t, the idea production function itself exhibits productivity growth. Finally, we assume a constant population growth rate of 1% per year. The other parameter values are relatively unimportant and are shown in the table. Other reasonable choices for parameter values will yield similar results qualitatively. The model is solved using a reverse shooting technique, discussed in more detail in Appendix B.

Figure 3 shows an example of the equilibrium dynamics that occur in this economy for the benchmark case. The economy features a steady-state growth rate of per capita consumption of 1.33%. This constant growth occurs while the danger threshold and the mortality rate decline exponentially to zero; both z_t and δ_t grow at -0.67%.

Several other features of the growth dynamics are worth noting. First, the particular initial conditions we've chosen have the growth rate of consumption declining along the transition path; a different choice could generate a rising growth rate, although declining growth appears to be more consistent with the value of life in the model (more on this below).

Second, consider total factor productivity for the idea production function. With $\lambda = 1$, TFP is $\alpha F(z_t)A_t^{\phi}$. Because we've assumed $\phi = 1/2$, this production function has the potential to exhibit positive TFP growth as knowledge spillovers rise over time. However, a declining danger threshold can offset this. In steady state, TFP growth for the idea production function is $\phi g_A + \eta g_z = -0.33\%$. That is, even though a given number of researchers are generating more and more candidate ideas over time, the number that get implemented is actually declining because of safety considerations.

Finally, the steady-state growth rate of 1.33% can be compared to an alternative path. It is feasible in this economy to let the technology-induced mortality rate fall to some arbitrarily low level — such as 1 death per billion people — and then to keep it constant at that rate forever by maintaining a constant technology cutoff $\bar{z}$. As this constant cutoff gets arbitrarily small, the steady state growth rate of the economy converges to $\lambda\sigma\bar{n}/(1-\phi)$ — that is, to the rule-of-thumb growth rate from Proposition 1. For our choice of parameter values, this feasible steady-state growth rate is 4.0% per year. That is, concerns for safety make it optimal in this environment to slow growth considerably relative to what is possible in the steady state.

The reason for this, of course, is the rising value of life, shown for this example in Figure 4. The value of life begins in period 0 at about 100 times annual consumption; if we think of per capita consumption as $30,000 per year, this corresponds to a value of life of $3 million, very much in the range considered in the literature (???). Over time, the value of life relative to consumption rises exponentially at a rate that

Figure 4: The Value of Life: Benchmark Case

Note: Simulation results for the competitive equilibrium using the parameter values from Table 1. The value of life rises faster than consumption.

Figure 5: Equilibrium Dynamics: Fréchet Case

Note: Dynamics when $F(z)$ is Fréchet, so $\eta = \infty$: growth slows to zero asymptotically. See notes to Figure 3.

converges to 0.67%, the same rate at which mortality declines.

5.2. Numerical Example When $\eta = \infty$

One element of the model that is especially hard to calibrate is the distribution from which technological danger is drawn, $F(z)$. The previous example assumed an exponential distribution so that $\eta = 1$; in particular, the elasticity of the distribution as z approaches zero is finite. However, this need not be the case. Both the Fréchet and the lognormal distributions feature an infinite elasticity. In Proposition 3, we showed that this leads the growth rate of consumption to converge to zero asymptotically. For this example, we consider the Fréchet distribution to illustrate this result: $F(z) = e^{-z^{-\psi}}$ and we set $\psi = 1.1$.[5] Other parameter values are unchanged from the benchmark case shown in Table 1, except we now set $\bar{\delta} = 1$, which is needed to put the value of life in the right ballpark.

Figure 5 shows the dynamics of the economy for this example. The growth rate of consumption now converges to zero as $\eta(z)$ gets larger and larger, meaning that a given decline in the danger threshold eliminates more and more potential ideas. Interestingly, this rising elasticity means that the danger threshold itself declines much more gradually in this example.

[5] We require $\psi > 1$ so that the mean (and hence conditional expectation) exist. The elasticity of this cdf is $\eta(z) = \psi z^{-\psi}$, so a small value of ψ leads the elasticity to rise to infinity relatively slowly.

Figure 6: Consumption and the Value of Life: Fréchet Case

Note: Dynamics when $F(z)$ is Fréchet, so $\eta = \infty$: consumption still rises to infinity. See notes to Figure 4.

Figure 6 — with its logarithmic scale — suggests that this declining consumption growth rate occurs as consumption gets arbitrarily high. The value of life still rises faster than consumption, but the increase is no longer exponential.

6. Discussion and Evidence

The key mechanism at work in this paper is that the marginal utility of consumption falls quickly, leading the value of life to rise faster than consumption. This tilts the allocation in the economy away from consumption growth and toward preserving lives. Exactly this same mechanism is at work in ?, which studies health spending. In that paper, $\gamma > 1$ leads to an income effect: as the economy gets richer over time (exogenously), it is optimal to spend an increasing fraction of income on health care in an effort to reduce mortality. The same force is at work here in a very different context. Economic growth combines with sharply diminishing marginal utility to make the preservation of life a luxury good. The novel finding is that this force has first-order effects on the determination of economic growth itself.

6.1. Empirical Evidence on the Value of Life

Direct evidence on how the value of life has changed over time is surprisingly difficult to come by. Most of the empirical work in this literature is cross-sectional in nature; see ? and ?, for example. Two studies that do estimate the value of life over time are ? and ?. These studies find that the value of life rises roughly twice as fast as income, supporting the basic mechanism in this paper.

Less direct evidence may be obtained by considering our changing concerns regarding safety. It is a common observation that parents today are much more

Figure 7: Mortality Rate from Accidental Drowning

Note: Taken from various issues of the National Center for Health Statistics, Vital Statistics Data. Breaks in the data imply different sources and possibly differences in methodology.

careful about the safety of their children than parents a generation ago. Perhaps that is because the world is a more dangerous place, but perhaps it is in part our sensitivity to that danger which has changed.

I am searching for formal data on how safety standards have changed over time and how they compare across countries. One source of information comes from looking at accidental deaths from drowning. Perhaps to a greater extent than for other sources of mortality, it does not seem implausible that advances in health technologies may have had a small effect on drowning mortality: if one is underwater for more than several minutes, there is not much that can be done. Nevertheless, there have been large reductions in the mortality rate from accidental drowning in the United States, as shown in Figure 7. At least in part, these are arguably due to safety improvements.

Safety standards also appear to differ significantly across countries, in a way that is naturally explained by the model. While more formal data is clearly desirable, different standards of safety in China and the United States have been vividly highlighted by recent events in the news. Eighty-one deaths in the United States have been linked to the contamination of the drug heparin in Chinese factories (?). In the summer of 2007, 1.5 million toys manufactured for Mattel by a Chinese supplier were recalled because they were believed to contain lead paint (?). And in an article on the tragic health consequences for workers producing toxic cadmium batteries in China, the Wall Street Journal reports

> As the U.S. and other Western nations tightened their regulation of cadmium, production of nickel-cadmium batteries moved to less-developed countries, most of it eventually winding up in China. "Everything was

> transferred to China because no one wanted to deal with the waste from cadmium," says Josef Daniel-Ivad, vice president for research and development at Pure Energy Visions, an Ontario battery company. (?)

6.2. The Environmental Kuznets Curve

Another interesting application of the ideas in this paper is to the environmental Kuznets curve. As documented by ? and ?, pollution exhibits a hump-shaped relationship with income: it initially gets worse as the economy develops but then gets better. To the extent that one of the significant costs of pollution is higher mortality — as the Chinese cadmium factory reminds us — the declines in pollution at the upper end of the environmental Kuznets curve are consistent with the mechanism in this paper. As the economy gets richer, the value of life rises substantially and the economy features an increased demand for safety.

In fact, the consequences for economic growth are also potentially consistent with the environmental Kuznets curve. One of the ways in which pollution has been mitigated in the United States is through the development of new, cleaner technologies. Examples include scrubbers that remove harmful particulates from industrial exhaust and catalytic converters that reduce automobile emissions. Researchers can spend their time making existing technologies safer or inventing new technologies. Rising concerns for safety lead them to divert effort away from new inventions, which reduces the output of new varieties and slows growth.

7. Conclusion

Safety is a luxury good. For a large class of standard preferences used in applied economics, the value of life rises faster than consumption. The marginal utility associated with more consumption on a given day runs into sharp diminishing returns, and adding additional days of life on which to consume is a natural, welfare-enhancing response. Economic growth therefore leads to a disproportionate concern for safety.

This force is so strong, in fact, that concerns for safety eventually outweigh a society's demand for economic growth. In the economy studied here, safety considerations lead to a conservative bias in technological change that slows growth considerably relative to what could otherwise be achieved. Depending on exactly how the model is specified, this can take the form of an overall reduction in exponential growth to a lower but still positive rate. Alternatively, the exponential growth rate itself may be slowed to zero.

From the standpoint of the growth literature, this is a somewhat surprising result. Some literatures focus on the importance of finding policies to increase the long-run growth rate; others emphasize the invariance of long-run growth to policies. Hence, the result that concerns for safety lead to a substantial reduction in the optimal growth rate is noteworthy.

The finding can also be viewed from another direction, however. A literature on sustainability questions the wisdom of economic growth; for example, see ?, ?, and ?. The model studied here permits very strong concerns for safety and human life. And while the consequences are slower rates of economic growth — even rates that slow to zero asymptotically — it is worth noting that the key driving force in the model is an income effect that operates only as consumption goes to infinity. That is, even the most aggressive slowing of growth found in this paper features unbounded growth in individual consumption; it is never the case that all growth should cease entirely.

This paper suggests a number of different directions for future research on the economics of safety. It would clearly be desirable to have precise estimates of the value of life and how this has changed over time; in particular, does it indeed rise faster than income and consumption? More empirical work on how safety standards have changed over time — and estimates of their impacts on economic growth — would also be valuable. Finally, the basic mechanism at work in this paper over time also applies across countries. Countries at different levels of income may have very different values of life and therefore different safety standards. This may have interesting implications for international trade, standards for pollution and global warming, and international relations more generally.

A Appendix: Proofs of the Propositions

This appendix contains outlines of the proofs of the propositions reported in the paper.

Proof of Proposition 1. BGP under the Rule of Thumb Allocation

Equations (10) and (11) follow immediately from the setup. The growth rate of ideas then comes from taking logs and derivatives of both sides of the following equation, evaluated along a balanced growth path, and using the fact that $\delta^* = \bar{\delta} g_A (1-\bar{s})\Gamma(\bar{z})$:

$$\frac{\dot{A}_t}{A_t} = \alpha F(z_t) \frac{\bar{s}^\lambda N_t^\lambda}{A_t^{1-\phi}}.$$

QED.

Proof of Proposition 2. Equilibrium Balanced Growth

Solving the optimization problems that help define the equilibrium and making some substitutions leads to the following seven key equations that pin down the equilibrium values of $\{c_t, s_t, v_t, z_t, A_t, \delta_t, N_t\}$:

$$c_t = A_t^\sigma (1-s_t) \tag{22}$$

$$\frac{s_t}{1-s_t} = \beta - \frac{v_t \delta_t}{c_t} = \beta(1 - \frac{\Gamma(z)}{z}) \tag{23}$$

$$v_t = \frac{u(c_t)/u'(c_t)}{\rho + \delta_t + \gamma g_{ct} - g_{vt}} \tag{24}$$

$$z_t = \frac{\beta c_t}{v_t \bar{\delta} g_{At}(1-s_t)} \tag{25}$$

$$\frac{\dot{A}_t}{A_t} = \alpha F(z_t) \frac{s_t^\lambda N_t^\lambda}{A_t^{1-\phi}} \tag{26}$$

$$\delta_t = \bar{\delta} g_{At} (1-s_t) \Gamma(z_t) \tag{27}$$

$$\frac{\dot{N}_t}{N_t} = \bar{n} - \delta_t \tag{28}$$

We begin by studying the value of life, in equation (24). With our specification of utility, $u(c_t)/u'(c_t) = \bar{u}c_t^{\gamma} + c_t/(1-\gamma)$. Since $\gamma > 1$, the growth rate of the value of life must be equal to γg_c along an (asymptotic) balanced growth path. Equation (24) then implies the last result in the proposition, namely that $u'(c_t)v_t \to \bar{u}/\rho$.

The fact that $g_v = \gamma g_c$ immediately implies from (25) that the danger cutoff z_t converges to zero along a balanced growth path, because v_t/c_t rises to infinity with $\gamma > 1$. Similarly, $g_z = -(\gamma-1)g_c$. And since $z_t \to 0$, equation (27) implies that $\delta_t \to 0$ as well.

To get the growth rate of the economy as a whole, recall that

$$\frac{\dot{A}_t}{A_t} = \alpha F(z_t) \frac{\bar{s}^{\lambda} N_t^{\lambda}}{A_t^{1-\phi}}.$$

Taking logs and derivatives of this equation along the balanced growth path, and using the fact that $\eta \equiv \lim_{z \to 0} F'(z)z/F(z)$ is finite from Assumption A, we have

$$(1-\phi)g_A = \eta g_z + \lambda \bar{n}.$$

The growth rate results of the proposition then follow quickly, after we note that $g_c = \sigma g_A$ and $g_z = -(\gamma-1)g_c$. For example

$$g_A = \frac{\lambda \bar{n}}{1 - \phi + \eta\sigma(\gamma - 1)}.$$

Finally, the share of labor devoted to research in steady state can be found from equation (23). By L'hopital's rule, $\lim \Gamma(z)/z = \Gamma'(0)$. Using the definition of the conditional expectation, one can calculate that

$$\Gamma'(z) = \left(1 - \frac{\Gamma(z)}{z}\right)\eta(z)$$

where $\eta(z) \equiv zF'(z)/F(z)$. Taking the limit as $z \to 0$ and noting that η is finite reveals that $\lim \Gamma(z)/z = \eta/(1+\eta)$. Substituting this into (23) yields the asmptotic value for s. QED.

Proof of Proposition 3. Equilibrium Growth with $\eta = \infty$

First, we show $c_t \to \infty$, by contradiction. Suppose not. That is, suppose $c_t \to c^* \in (0, \infty)$. The contradiction arises because the model has a strong force for idea growth and therefore consumption growth. By (24), $v_t \to v^*$ and (23) and (25) imply that $s_t \to s^* \in (0,1)$ and $z_t \to z^* > 0$. Studying the system of differential equations in (26), (27), and (28) reveals that

$$\frac{\dot{A}_t}{A_t} \to \frac{\lambda \bar{n}}{1 - \phi + \bar{\delta}(1 - s^*)\Gamma(z^*)} > 0.$$

But since $c_t = A_t^\sigma(1 - s_t)$, this means $c_t \to \infty$, which contradicts our original supposition that $c_t \to c^*$. Therefore this supposition was wrong, and c_t does in fact go to ∞.

Next, we show everything else, such as $g_{ct} \to 0$. With $c_t \to \infty$ and $\gamma > 1$, $v_t/c_t \to \infty$ from (24), as the value of life rises faster than consumption. Then (25) implies that

$$\frac{v_t}{c_t} = \frac{\beta}{\bar{\delta} z_t g_{At}(1 - s_t)} = \frac{\beta(1 + \beta(1 - \Gamma(z_t)/z_t)}{\bar{\delta} z_t g_{At}}$$

But then $v_t/c_t \to \infty$ if and only if $z_t g_{At} \to 0$.

First, we show that z_t has to go to zero. Why? Suppose not. That is, suppose $z_t \to z^*$ and $g_{At} \to 0$. From (23), $s_t \to s^* \in (0,1)$. And then from (27), it must be that $\delta \to 0$. But then population grows at rate $\bar{n}$ eventually and the law of motion for ideas (26) would lead to exponential growth in A_t, which is a contradiction. Therefore z_t has to go to zero.

Is it possible that g_{At} does not then also go to zero? No. Notice that $z_t \to 0$ as rapidly as consumption (and therefore A_t) go to infinity. But the fact that $\eta = \infty$ means that $F(z_t)$ goes to zero faster than N_t and A_t are rising.

The fact that $s_t \to 0$ comes from (23) because, as we show next, $\lim_{z \to 0} \Gamma(z)/z = 1$ when $\eta = \infty$. By L'hopital's rule, $\lim \Gamma(z)/z = \Gamma'(0)$. Using the definition of the conditional expectation, one can calculate that

$$\Gamma'(z) = \left(1 - \frac{\Gamma(z)}{z}\right)\eta(z)$$

where $\eta(z) \equiv zF'(z)/F(z)$. Since $\eta(z) \neq 0$, we can divide both sides by $\eta(z)$ and consider the limit as $z \to 0$:

$$\lim \frac{\Gamma'(z)}{\eta(z)} = 1 - \lim \frac{\Gamma(z)}{z}.$$

The left-hand size is zero since $\eta(z) \to \infty$ by assumption, which proves the result.

The fact that $z_t \to 0$, $g_{At} \to 0$, and $s_t \to 0$ imply that $\delta_t \to 0$ and $g_{ct} \to 0$. QED.

Proof of Proposition 4. Equilibrium Growth with $\gamma < 1$

The equilibrium with $\gamma < 1$ is characterized by the same seven equations listed above in the proof of Proposition 2, equations (22) through (28). The proof begins in the same way, by studying the value of life in equation (24). With our specification of utility, $u(c_t)/u'(c_t) = \bar{u}c_t^{\gamma} + c_t/(1-\gamma)$. Since $\gamma < 1$, the constant term disappears asymptotically and the growth rate of the value of life equals g_c along an (asymptotic) balanced growth path. Equation (24) then implies the last result in the proposition, giving the constant ratio of the value of life to consumption.

The fact that v_t/c_t converges to a constant means that z_t converges to a nonzero value, according to equation (25). Similarly, δ_t does as well, according to equation (27). The solution for the growth rate and the research share are found in ways similar to those in the proof of Proposition 2. QED.

Proof of Proposition 5. Optimal Balanced Growth

The Hamiltonian for the optimal growth problem is

$$H = N_t u(c_t) + \mu_{1t}\alpha F(z_t) s_t^{\lambda} N_t^{\lambda} A_t^{\phi} + \mu_{2t} N_t \left(\bar{n} - \bar{\delta}\alpha F(z_t) s_t^{\lambda} N_t^{\lambda} A_t^{\phi-1}(1-s_t)\int_0^{z_t} z f(z)dz\right).$$

Applying the Maximum Principle, the first order necessary conditions are

$H_s = 0$: $\quad N_t u'(c_t) A_t^\sigma = \mu_{1t} \lambda \frac{\dot{A}_t}{s_t} - \mu_{2t} N_t (\lambda \frac{\delta_t}{s_t} - \frac{\delta_t}{1-s_t})$

$H_z = 0$: $\quad \mu_{1t} = \mu_{2t} N_t \bar{\delta} z_t \cdot \frac{1-s_t}{A_t}$

Arbitrage(A_t): $\quad \rho = \frac{\dot{\mu}_{1t}}{\mu_{1t}} + \frac{1}{\mu_{1t}} \left(N_t u'(c_t) \sigma \frac{c_t}{A_t} + \mu_{1t} \phi \frac{\dot{A}_t}{A_t} - \mu_{2t} N_t (\phi - 1) \frac{\delta_t}{A_t} \right)$

Arbitrage(N_t): $\quad \rho = \frac{\dot{\mu}_{2t}}{\mu_{2t}} + \frac{1}{\mu_{2t}} \left(u(c_t) + \mu_{1t} \lambda \frac{\dot{A}_t}{N_t} + \mu_{2t} (\bar{n} - \delta_t) - \mu_{2t} N_t \lambda \frac{\delta_t}{N_t} \right)$

together with two transversality conditions: $\lim_{t\to\infty} \mu_{1t} A_t e^{-\rho t} = 0$ and $\lim_{t\to\infty} \mu_{2t} N_t e^{-\rho t} = 0$.

Combining these first order conditions (use the first and second to get an expression for μ_{2t} and substitute this into the arbitrage equation for N_t) and rearranging yields:

$$\frac{\rho - g_{\mu 2t} - (\bar{n} - \delta_t) + \lambda \delta_t - \lambda g_{At} \bar{\delta} (1 - s_t) z_t}{\lambda \bar{\delta} g_{At} z_t \frac{1-s_t}{s_t} - \frac{\delta_t}{1-s_t} (\lambda \frac{1-s_t}{s_t} - 1)} = \frac{u(c_t)}{u'(c_t) c_t} \cdot (1 - s_t). \tag{29}$$

This is the key equation for determining the asymptotic behavior of z_t. In particular, along a balanced growth path, the right-hand-side goes to infinity for $\gamma > 1$. This requires that $z_t \to 0$ so that the denominator of the left side goes to zero (since $\delta_t \to 0$ as well). Moreover, with some effort, one can show that the denominator on the left side grows at the same rate as z_t along the balanced growth path, which implies that $g_z = -(\gamma - 1) g_c$ from the usual value-of-life argument used earlier.

The result for the growth rate of A_t and c_t follows by the same arguments as in the proof of Proposition 2. Finally, one can combine the first order conditions to solve for the allocation of research. QED.

B Appendix: Solving the Model Numerically

The transition dynamics of the equilibrium allocation can be studied as a system of four differential equations in four "state-like" variables that converge to constant values: ℓ_t, m_t, δ_t, and w_t. These variables, their meaning, and their steady-state values are displayed in Table 2.

Table 2: Key "State-Like" Variables for Studying Transition Dynamics

Variable	Meaning	Steady-State Value
$\ell_t \equiv \frac{v_t \delta_t}{c_t}$	Value of life $\times$ mortality	$\ell^* = \beta \cdot \frac{\eta}{1+\eta}$
$m_t \equiv g_{At}$	Growth rate of A_t	$m^* = g_A$
δ_t	Mortality rate	$\delta^* = 0$
$w_t \equiv \frac{u(c_t)}{u'(c_t)c_t} \cdot \frac{c_t}{v_t}$	Value of a year of life relative to mortality price	$w^* = \rho$

Letting a "hat" denote a growth rate, the laws of motion for these state-like variables are

$$\hat{\ell}_t = \frac{\rho + \delta_t - w_t + (\gamma - 1)\sigma m_t + \lambda(\bar{n} - \delta_t) - (1-\phi)m_t}{1 + k_t \left(\frac{\lambda}{\beta - \ell_t} - \gamma\right) + \frac{\eta(z_t) + \theta(z_t)}{1 - \theta(z_t)}} \tag{30}$$

$$\hat{m}_t = -\left(\frac{\eta(z_t)}{1 - \theta(z_t)} + \frac{\lambda k_t}{\beta - \ell_t}\right)\hat{\ell}_t + \lambda(\bar{n} - \delta_t) - (1-\phi)m_t \tag{31}$$

$$\hat{\delta}_t = \hat{m}_t + \left(k_t - \frac{\theta(z_t)}{1 - \theta(z_t)}\right)\hat{\ell}_t \tag{32}$$

$$\hat{w}_t = \hat{\delta}_t - \hat{\ell}_t + \left(\gamma - 1 + \frac{\delta_t}{w_t \ell_t}\right)\left(\sigma m_t + k_t \hat{\ell}_t\right) \tag{33}$$

where $\theta(z) \equiv z\Gamma'(z)/\Gamma(z) = \eta(z)(z/\Gamma(z) - 1)$ is the elasticity of the conditional expectation function and $k_t \equiv \ell_t/(1 + \beta - \ell_t)$. The only other variable that must be obtained in order to solve these differential equations is z_t, and it can be gotten as follows. First, $s_t = (\beta - \ell_t)/(1 + \beta - \ell_t)$. With s_t in hand, $\Gamma(z_t)$ can be recovered from the state-like variables using the mortality rate: $\delta_t = \bar{\delta} m_t (1 - s_t)\Gamma(z_t)$. Finally, $z_t = \beta\Gamma(z_t)/\ell_t$.

We solve the system of differential equations using "reverse shooting"; see ?, p. 355. That is, we start from the steady state, consider a small departure, and then run time backwards. For the results using the exponential distribution, we set $T = 600$; for the results using the Fréchet distribution, we set $T = 12150$.

An interesting feature of the numerical results is that $\hat{\ell}_t \approx 0$ holds even far away from the steady state. The reason is that $\lim_{z \to 0} \theta(z) = 1$: if z changes by a small percent starting close to zero, the conditional expectation changes by this same percentage. But this means that $\hat{\ell}_t \approx 0$ since there is a $1/(1-\theta(z_t))$ term in the denominator. But since $z_t/\Gamma(z_t) = \beta/\ell_t$, if ℓ_t does not change by much, then z_t will not change by much either. QED.