Proposition

Let X_i be a collection of size n of random variables with means $\gamma_i\mu$, where γ_i are some arbitrary known constants, and $i \in 1, 2, \ldots, n$. Also let $\widehat{\mu}$ be some other estimator for the mean, such that

$$\widehat{\mu} = \frac{1}{k}\sum_i^n c_i X_i,$$

where c_i, k are some other arbitrary constants. Then $\widehat{\mu}$ is an **unbiased estimator** of μ if and only if

$$k = \sum_i^n c_i \gamma_i.$$

Proof

Let X_i be a collection of size n of random variables with means $\gamma_i\mu$, where γ_i are some arbitrary known constants, and $i \in 1, 2, \ldots, n$. Also let $\widehat{\mu}$ be some other estimator for the mean, such that

$$\widehat{\mu} = \frac{1}{k}\sum_i^n c_i X_i,$$

where c_i, k are some other arbitrary constants.

It is known that if $\widehat{\mu}$ is an unbiased estimator of μ, then

$$E(\widehat{\mu}) = \mu.$$

Therefore it must be that

$$\begin{aligned}
\mu &= E\left\{\frac{1}{k}\sum_i^n c_i X_i\right\} \\
&= \frac{1}{k}E\left\{\sum_i^n c_i X_i\right\} \\
k\mu &= E\{c_1 X_1 + \cdots + c_n X_n\} \\
&= E(c_1 X_1) + \cdots + E(c_n X_n) \\
&= c_1 E(X_1) + \cdots + c_n E(X_n) \\
&= c_1\gamma_1\mu + \cdots + c_n\gamma_n\mu \\
k\mu &= \mu(c_1\gamma_1 + \cdots c_n\gamma_n) \\
k &= \sum_i^n c_i\gamma_i.
\end{aligned}$$

Examples

(1)

> Each of the independent random variables X_1, X_2 and X_3 has mean μ.
>
> Consider the estimator
>
> $$\widehat{\mu} = \frac{1}{8}(2X_1 + 3X_2 + 4X_3),$$
>
> Is $\widehat{\mu}$ an unbiased estimator of μ?
>
> (Practice quiz 7, q2)

For $\widehat{\mu} = \frac{1}{k}\sum_i^n c_i X_i$ to be an unbiased estimator of μ, it must be that

$$k = \sum_i^n c_i \gamma_i.$$

Here, let $c_i = 2, 3, 4$, $\gamma_i = 1$, and $k = 8$. Then

$$2(1) + 3(1) + 4(1) = 9 \neq 8.$$

Therefore $\widehat{\mu}$ is not an unbiased estimator, as $k \neq \sum c_i \gamma_i$.

(2)

> Suppose that X_1, X_2 and X_3 independent random variables such that $X_1 \sim N(\theta, \sigma^2)$, $X_2 \sim N(2\theta, \sigma^2)$ and $X_3 \sim N(3\theta, \sigma^2)$.
>
> Consider the estimator
>
> $$\hat{\theta} = \frac{1}{3}(X_1 + X_2 + X_3),$$
>
> Is $\hat{\theta}$ an unbiased estimator of θ?
>
> (Practice quiz 7, q6)

For $\hat{\theta} = \frac{1}{k} \sum_i^n c_i X_i$ to be an unbiased estimator of θ, it must be that

$$k = \sum_i^n c_i \gamma_i.$$

Here, let $c_i = 1$, $\gamma_i = 1, 2, 3$, and $k = 3$. Then

$$1(1) + 1(2) + 1(3) = 6 \neq 3.$$

Therefore $\hat{\theta}$ is not an unbiased estimator, as $k \neq \sum c_i \gamma_i$.

(3)

> Three independent random variables, X, Y and Z, are such that the mean of X is 4θ, the mean of Y is 2θ, and the mean of Z is θ.
>
> Consider the estimator
>
> $$\hat{\theta} = \frac{1}{8}(X + Y + 2Z),$$
>
> Is $\hat{\theta}$ an unbiased estimator of θ?
>
> (Practice quiz 7, q7)

For $\hat{\theta} = \frac{1}{k} \sum_i^n c_i X_i$ to be an unbiased estimator of θ, it must be that

$$k = \sum_i^n c_i \gamma_i.$$

Here, let $c_i = 1, 1, 2$, $\gamma_i = 4, 2, 1$, and $k = 8$. Then

$$1(4) + 1(2) + 2(1) = 8 = 8.$$

Therefore $\hat{\theta}$ is an unbiased estimator, as $k = \sum c_i \gamma_i$.