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HORIZONTAL CHORD THEOREMS

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For any real function f, defined on a bounded or unbounded interval, the set

$$H(f) = \{h \in [0, \infty): f(x) = f(x+h) \text{ for some } x\}$$

is called the **chord set** of f. The purpose of this note is to present some generalizations of known theorems concerning these sets, and to test their generality by means of counter examples.

1. Functions having every chord. It is well known, and very easy to prove [**1**, p. 78], that if f is periodic and continuous on the real line R, then $H(f) = [0, \infty)$; briefly, a continuous periodic function has every chord. This result was generalized by Diaz and Metcalf [**3**], who showed that it is sufficient to assume that f is periodic, continuous at some point, and that for each $h > 0$, the function $f(x+h) - f(x)$ has a connected range. In particular, a periodic derivative has every chord. It is not sufficient to assume that f itself has a connected range, or even to assume that f is a Darboux function of Baire class 1. (A function f is a Darboux function if its domain is an interval and if it has the intermediate value property, that is, maps each subinterval onto a connected set. It is of Baire class 1 if it can be represented as the limit of a convergent sequence of continuous functions.) For example,

$$f(x) = \cos\left(\frac{1}{\sin x}\right) + \tfrac{1}{2}(-1)^{[x/\pi]}, \quad \cos\left(\frac{1}{0}\right) = 0, \tag{1}$$

where $[x]$ denotes the largest integer less than or equal to x, is a Darboux function of Baire class 1 with period 2π, but it has no chord of length π.

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The hypothesis that f be continuous at some point cannot be omitted. Let B be a Hamel basis (a maximal set of real numbers linearly independent over the rationals) that includes $b_1 = 1$. Define $f(\sum_1^n x_i b_i) = \sum_2^n x_i b_i$ whenever $b_1, \cdots, b_n$ are distinct members of B and $x_1, \cdots, x_n$ are rational numbers. Then $f(x + h) - f(x) = f(h)$ for all x and h. This function is periodic (every rational number is a period), and each of the functions $f(x + h) - f(x)$ is continuous (in fact, constant), but f has no chord of irrational length.

Generalizing in another direction, Tews [7] showed that a continuous *almost* periodic function has every chord. Actually, a much simpler and more general theorem holds, as we shall now show.

Let us say that a function f, defined on an interval, is **positively recurrent at** x_0 if for every $\varepsilon > 0$ the set $\{x: |f(x) - f(x_0)| < \varepsilon\}$ is unbounded above, and that f is **positively recurrent** if it is positively recurrent at each point of its domain. Note that the domain of such a function must be of the form (a, ∞), $[a, \infty)$, or $(-\infty, \infty)$. Replacing "above" by "below" gives the corresponding definitions for **negatively recurrent**. These definitions are consistent with the notion of recurrence used in topological dynamics [**4**]. Since an almost periodic function is obviously recurrent (both positively and negatively), Tews's result is a corollary of the following theorem.

THEOREM 1. *If f is continuous and either positively or negatively recurrent on an interval, then f has every chord.*

Proof. Suppose some positive number h does not belong to $H(f)$. Then $f(x + h) - f(x)$ never changes sign. Since $f(x)$, $f(-x)$, $-f(x)$, and $f(x - c)$ all have the same chord set, we may assume that f is positively recurrent, that $f(x + h) > f(x)$ for all x, and that the domain of f includes $[0, \infty)$. Let f attain its minimum value on $[0, h]$ at x_0, and its minimum on $[h, 2h]$ at x_1. If $x > h$, then $x - nh \in (h, 2h]$ for some integer $n \geqq 0$, and

$$f(x) \geqq f(x - nh) \geqq f(x_1) > f(x_1 - h) \geqq f(x_0).$$

Thus, when $\varepsilon = f(x_1) - f(x_0)$, the set

$$\{x: |f(x) - f(x_0)| < \varepsilon\}$$

is bounded above by h. This contradicts the hypothesis that f is positively recurrent.

In Theorem 1 it is not sufficient to assume that f is positively or negatively recurrent at each point. Any horizontal line that meets the graph of

$$f(x) = 2 \sin 2\pi x + \tanh x \tag{2}$$

meets it in an unbounded set. (When $|y| < 1$ the set $f^{-1}(y)$ is even relatively dense.) But f has no chord of length 1. More surprising is the fact that a recurrent derivative need not have every chord. For each x in R, let m be the largest integer less than x and define

(3) $$f(x) = 2^m\left(1 + x - m + \sin\frac{\pi}{x - m}\right).$$

This function is continuous when x is not an integer. When n is an integer, f is continuous on the left but assumes all values between 0 and 2^{n+1} in every right neighborhood of n. It follows that the set $\{x: f(x) = f(x_0)\}$ is unbounded above, for each x_0. Hence f is positively recurrent. To verify that f is the derivative of the function

$$F(x) = \int_{-\infty}^{x} f(t)\,dt,$$

note that when n is an integer and $0 < h < 1$ we have

$$F(n + h) - F(n) - hf(n) = 2^{n-1}h^2 + 2^n \int_0^h \sin\frac{\pi}{t}\,dt.$$

Replacing $\sin \pi/t$ by

$$\frac{d}{dt}\left(\frac{t^2}{\pi}\cos\frac{\pi}{t}\right) - \frac{2t}{\pi}\cos\frac{\pi}{t},$$

the right member takes the form

$$2^{n-1}h^2 + \frac{2^n h^2}{\pi}\cos\frac{\pi}{h} - \frac{2^{n+1}}{\pi}\int_0^h t\cos\frac{\pi}{t}\,dt,$$

which is numerically less than $2^{n+1}h^2$. Hence F has a right derivative at n equal to $f(n)$. The fundamental theorem then completes the proof that $F'(x) = f(x)$ everywhere. Nevertheless, f has no chord of length 1, since $f(x + 1) - f(x) = f(x) > 0$ for all x.

2. The universal chord theorem. For continuous functions, the universal chord theorem of P. Lévy [**6**] [see also **1**, p. 79] asserts:

(i) *If $h \in H(f)$, then $h/n \in H(f)$ for every positive integer n.*

(ii) *If a and h are positive numbers and a is not a submultiple of h, then there exists a continuous function f with $h \in H(f)$ and $a \notin H(f)$.*

Lévy's example for (ii) was

(4) $$f(x) = \sin^2\left(\frac{\pi x}{a}\right) - \frac{x}{h}\sin^2\left(\frac{\pi h}{a}\right).$$

The proof of (i) depends only on the fact that each of the functions $f[x + (h/n)] - f(x)$ has the intermediate value property. Hence the theorem holds for derivatives, as Boas [**1**, p. 81] has remarked, and also for approximate derivatives [**2**, p. 31]. However, the function defined by equation (1) shows that the theorem can fail for a Darboux function of Baire class 1, despite the fact that any such function is topologically equivalent to a derivative, by Maximoff's theorem [**2**, p. 49].

3. A stronger form of Hopf's theorem. H. Hopf [**5**] obtained a complete characterization of the chord sets of functions continuous on a bounded closed interval and of plane continua. (If K is a subset of the plane, then

$$\{h \in [0,\infty) : (x+h, y) \in K \text{ for some } (x,y) \in K\}$$

is called the **chord set** of K.) A subset of $(0,\infty)$ is called **additive** if it contains the sum of any two of its members. Let us call a set $H \subset [0,\infty)$ **co-additive** if $0 \in H$ and the set $H^* = (0,\infty) - H$ is additive. Hopf's theorem reads as follows:

(i) *The chord set of any non-empty compact connected subset of the plane is compact and co-additive.*

(ii) *Any compact co-additive subset of* $[0,\infty)$ *is the chord set of some plane continuum; more particularly, it is the chord set of some continuous function on a bounded closed interval.*

The function that Hopf used to prove (ii) was not differentiable. Nevertheless, the following theorem is true.

THEOREM 2. *For any compact co-additive set* $H \subset [0,\infty)$ *there exists a function* F *of class* C^∞ *on* R *such that* H *is the chord set of* F *and also of the restriction of* F *to* $[0,B]$, *where* $B = \sup H$.

Proof. We shall obtain this result by smoothing Hopf's function. Accordingly, we begin by repeating his proof of (ii), with a few minor changes. Let $\dot{H}$ denote the boundary of H, and define

$$f(x) = \begin{cases} d(x,\dot{H}) & \text{for } x \in H \cup (-\infty,0) \\ -d(x,\dot{H}) & \text{for } x \in H^*, \end{cases} \tag{5}$$

where d denotes ordinary distance in R.

Note that 0 and B belong to $\dot{H}$, and that H^* is open. If we regard H as a subset of the x-axis, the graph of f has a right-angled peak, with endpoints in $\dot{H}$, above each component of $H - \dot{H}$, and a right-angled trough, with endpoints in $\dot{H}$, below each component of $(0,B) - H$. It also includes the points of $\dot{H}$ itself and the rays with slope -1 to the left of the origin and to the right of B.

If $B = 0$, then $f(x) = -x$ for all x, and the theorem is true in this case. Hence we may assume $B > 0$. Clearly f is continuous and satisfies a Lipschitz condition. It is not differentiable, but $f'(x) = \pm 1$ for any x that is not in $\dot{H}$ and is not the midpoint of one of the components of $(0,B) - \dot{H}$. The proof that H is the chord set of f and of its restriction f_B to $[0,B]$ rests on two lemmas:

1° *If* H *contains an interval of length* λ, *then* $(0,\lambda) \subset H$.

2° *If* $a \in \dot{H} \cup H^*$ *and* $a \leqq a + h \in H$, *then* $h \in H$.

Statement 1° follows from the fact that if $0 < a < \lambda$, then any interval of length λ contains a multiple of a. Consequently, no element of H^* can belong to $(0, \lambda)$. To prove 2°, let the hypotheses be satisfied and suppose $h \notin H$. Then h belongs to H^*, and therefore $(h-\varepsilon, h+\varepsilon) \subset H^*$ for some $\varepsilon > 0$. Moreover, a cannot be 0. Hence a belongs to $\dot{H} - \{0\}$ or to H^*. In either case, a is a cluster point of H^*. Therefore $a + x \in H^*$ for some $|x| < \varepsilon$, and $h - x \in H^*$. By additivity, $(a + x) + (h - x) = a + h$ belongs to H^*, contrary to hypothesis.

If $h \in \dot{H}$, then $f(0) = f(h) = 0$. Since $0 \leqq h \leqq B$, it follows that $h \in H(f_B)$. If $h \in H - \dot{H}$, let $h + 2a$ be the first point of $\dot{H}$ to the right of h. (Such a point exists, since $h < B \in \dot{H}$.) Then $h + 2a$ is a point of $\dot{H}$ nearest to $a + h$, and $f(a+h) = a > 0$. Since $(h, h + 2a) \subset H$, 1° implies that $(0, 2a) \subset H$. Consequently, 0 is a point of $\dot{H}$ nearest to a, and therefore $f(a) = a = f(a + h)$. Since $0 < a < a + h < B$, it follows that $h \in H(f_B)$. Thus $H \subset H(f_B)$.

If $h \in H(f)$, then $h \geqq 0$ and $f(x) = f(x + h)$ for some $x \in R$. To show that $h \in H$ we distinguish three cases. If $f(x) = f(x + h) = 0$, then both x and $x + h$ belong to $\dot{H}$, and the conclusion follows from 2°, with $a = x$. If $f(x) = f(x + h) > 0$, let a be a point of $\dot{H}$ nearest to x. Since $f(x + h) = |x - a|$, the interval with endpoints $x + h \pm |x - a|$ is contained in H. Its endpoints also belong to H; in particular, $a + h \in H$. Then 2° implies that $h \in H$. Lastly, if $f(x) = f(x + h) < 0$, let $a + h$ be a point of $\dot{H}$ nearest to $x + h$. Then $f(x) = -|x - a|$. All points of the open interval with endpoints $x \pm |x - a|$ belong to H^*. One of these endpoints is a. Hence a belongs either to H^* or to $\dot{H}$, and 2° implies that $h \in H$. This completes the proof that $H = H(f) = H(f_B)$.

(The foregoing argument becomes intuitively clear if one observes that when a chord of f lying above the x-axis is slid downward, keeping its left endpoint on the graph of f, its right endpoint ends up in a point of H; and when a chord lying below the x-axis is slid upward, keeping its right endpoint on the graph of f, its left endpoint ends up in a point of H^* or in a point of $\dot{H}$.)

To obtain Theorem 2 from this result, observe that if ϕ is any 1-1 mapping of R into R, then the composite function $F = \phi \circ f$ has the same chord set as f. By suitable choice of ϕ, we shall show that F can be made to be of class C^∞.

The open set $E = (0, B) - \dot{H}$ has only a finite number of components whose length exceeds any given positive number. Let $y_0 > y_1 > y_2 > \cdots$ be a strictly decreasing sequence of positive numbers, tending to 0, such that the half-length of each component of E is a term of the sequence. Before defining ϕ, we shall show that if ϕ is of class C^∞ on R, and if

$$\phi^{(n)}(0) = \phi^{(n)}(y_i) = \phi^{(n)}(-y_i) = 0 \tag{6}$$

for $n \geqq 1$ and $i \geqq 0$, then $F = \phi \circ f$ is of class C^∞ on R.

Let I be any component of E, and denote its midpoint by x_1. Then $f(x_1) = \pm y_i$ for some $i \geqq 0$. On one half of I we have $f' = 1$; on the other half, f' is equal to -1. On both of these intervals, $|F^{(n)}| = |\phi^{(n)} \circ f|$ for $n \geqq 1$, and (6) implies that

$F^{(n)}(x) \to 0$ as $x \to x_1$. Since F is continuous, it follows by induction and l'Hospital's Rule that $F^{(n)}(x_1) = 0$ for $n \geqq 1$. Thus $|F^{(n)}| = |\phi^{(n)} \circ f|$ on each component of E, and also on $(-\infty, 0)$ and (B, ∞), for $n \geqq 0$.

Let $x_0 \in \dot{H}$. If $x \notin \dot{H}$, the component of $R - \dot{H}$ to which x belongs has an endpoint a in $[x_0, x)$ or in $(x, x_0]$, and $a \in \dot{H}$. By the mean value theorem,

$$|F(x) - F(x_0)| = |F(x) - F(a)| = |(x-a)F'(\xi)| \leqq |x - x_0| \cdot |\phi'[f(\xi)]|$$

for some ξ between a and x. It follows from (6) and the constancy of F on $\dot{H}$ that $F'(x_0) = 0$. Assuming $F^{(n)} = 0$ on $\dot{H}$, similar reasoning shows that $F^{(n+1)} = 0$ on $\dot{H}$. Thus, by induction, F is of class C^∞ on R, and all its derivatives vanish on $\dot{H}$ as well as at the midpoints of the components of E.

It only remains to define a function ϕ having all the properties we have assumed. Recall that the function

$$\omega(x) = \exp\left(-\frac{1}{x^2}\right), \quad \omega(0) = 0, \tag{7}$$

is of class C^∞ on R and that $\omega^{(n)}(0) = 0$ for $n \geqq 0$. For each positive integer i, let b_i be an upper bound of the function $\omega(x - y_i) \cdot \omega(x - y_{i-1})$ and of the absolute values of its first $i - 1$ derivatives on the interval $[y_i, y_{i-1}]$. Put $a_i = 1/ib_i$ and define

$$\psi(x) = \begin{cases} \omega(x - y_0) & \text{on } (y_0, \infty) \\ a_i\omega(x - y_i) \cdot \omega(x - y_{i-1}) & \text{on } (y_i, y_{i-1}] \end{cases} \tag{8}$$

for $i \geqq 1$. Then define $\psi(0) = 0$ and $\psi(x) = \psi(-x)$ when $x < 0$. It is clear that ψ and each of its derivatives tends to 0 at each of the points y_i, and also as $x \to 0$, since

$$|\psi^{(n)}(x)| \leqq 1/i \text{ on } [y_i, y_{i-1}] \quad \text{for } i > n \geqq 0.$$

Therefore, ψ is of class C^∞ on R. Moreover, $\psi(x) > 0$ except at the points $0, \pm y_0, \pm y_1, \cdots$, where it vanishes together with its derivatives of all orders. Consequently, the function

$$\phi(x) = \int_0^x \psi(t)\, dt \tag{9}$$

is strictly increasing, of class C^∞ on R, and satisfies conditions (6). This completes the proof of Theorem 2.

4. Chord sets of analytic functions. Is it possible to find an *analytic* function having any prescribed chord set? (Recall that Lévy used an elementary function (4) to satisfy the much less stringent requirements of part (ii) of his theorem.) To see that no such improvement of Theorem 2 is possible, let C be a nowhere dense perfect subset of $[1,2]$, and take $H = [0,1] \cup C$. Then H is compact, $0 \in H$, and $H^* = (1, \infty) - C$ is additive. Since $\dot{H} = \{0,1\} \cup C$ is uncountable, the following theorem shows that no analytic function can have H for its chord set.

THEOREM 3. *If $f:[a,b] \to R$ is continuous on $[a,b]$ and analytic on (a,b), then the boundary $\dot{H}$ of $H = H(f)$ is countable.*

Proof. If f is constant, then $\dot{H} = \{0, b-a\}$ has only two elements. We may therefore assume that f is not constant on (a,b). Then the set

$$D = \{x \in (a,b): f'(x) = 0\} \cup \{a,b\}$$

is countable, since the zeros of f' are isolated. The set

$$E_1 = \{h \in [0, b-a]: f(x) = f(x \pm h) \text{ for some } x \in D\}$$

is also countable, since f can assume any value at most countably many times. Let E_2 denote the set of endpoints of components of the open set $H - \dot{H}$. Evidently E_2 is countable. We shall show that $\dot{H} - (E_1 \cup E_2)$ is finite.

Let $h \in \dot{H} - (E_1 \cup E_2)$. Then $h \in H$ and $f(x_0) = f(x_0 + h)$ for some $x_0 \in [a, b-h]$. Since $h \notin E_1$, neither x_0 nor $x_0 + h$ can belong to D, hence

$$a < x_0 < x_0 + h < b, \; f'(x_0) \neq 0, \text{ and } f'(x_0 + h) \neq 0.$$

It follows that f is locally invertible at x_0 and at $x_0 + h$; there exist continuous functions ϕ and ψ, defined on an open interval I containing $y_0 = f(x_0)$, such that $\phi(y_0) = x_0$, $\psi(y_0) = x_0 + h$, and

$$f[\phi(y)] = f[\psi(y)] = y \text{ for all } y \in I. \tag{10}$$

Since $\psi(y_0) - \phi(y_0) = h > 0$, we may assume that $\psi(y) - \phi(y) > 0$ on I. Then (10) implies that $\psi(y) - \phi(y) \in H$ for all $y \in I$. Since $\psi - \phi$ is continuous, it maps I onto a connected subset of H that contains h. Since $h \in \dot{H} - E_2$, no connected subset of H can contain h and a number different from h. Therefore $\psi(y) - \phi(y) = h$ for all $y \in I$. Putting $x = \phi(y)$, it follows that $f(x) = f(x+h)$ on the subinterval $\phi(I)$ of (a,b). By analytic continuation, this equation holds for all $a < x < b + h$. Hence h is a member of the set

$$E_3 = \{h \in (0, b-a): f(x) = f(x+h) \text{ on } (a, b-h)\}.$$

Thus $\dot{H} - (E_1 \cup E_2) \subset E_3$.

If h_1 and h_2 are in E_3 and $h_1 < h_2$, then

$$a < a + h_1 < b - h_2 + h_1 < b.$$

If x is in the interval $J = (a + h_1, b - h_2 + h_1)$, then $x - h_1$ is in both $(a, b - h_2)$ and $(a, b - h_1)$. From the definition of E_3 it follows that $f(x - h_1) = f(x - h_1 + h_2)$ and $f(x - h_1) = f(x - h_1 + h_1) = f(x)$. Therefore $f(x) = f(x + h_2 - h_1)$ on J.

By analytic continuation, this equation holds on $(a, b - h_2 + h_1)$, hence $h_2 = h_1 \in E_3$. Thus E_3 contains positive differences of its members. If E_3 were infinite it would follow that E_3 has arbitrarily small members, and then f would be constant. Therefore E_3 must be finite. Consequently, $\dot{H}$ is countable.

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WOMEN IN MATHEMATICS

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As I looked out over the audience at the Monday afternoon session of the 1971 Summer MAA meeting, I observed that over twenty percent were women, far more than at any other MAA session in my memory. This influx of female mathematicians was due, of course, to the subject of the panel: Women in Mathematics. Indeed, many in the audience told me later that they had come to the Penn State meeting solely or primarily because of the panel.

Under the direction of its moderator, Christine Ayoub of Penn State, the panel decided to focus on two questions: 1. Is there discrimination against women in mathematics? 2. What can, or should, be done to improve the status of women in the field?

The positioning of the members came to be symbolic, with the conservatives — the moderator and panelist Mary Ellen Rudin of Wisconsin on the right and the more militant Gloria Hewitt of Montana and myself on the left. There was an omission — there probably should have been representation of graduate students and/or assistant professors on the panel rather than only those who have more or less "made it."