Just draw a line tangent to the unit circle parallel to the boundary. The desired heading is then AB'.

For a space-varying wind field, we have the following known result (due to the authors) which appears as Problem 61-4, Flight in an Irrotational Wind Field, SIAM Review, April 1962, pp. 155–156.

If an aircraft travels at a constant speed relative to the wind and traverses any closed curve, the time taken is always less when there is no wind than when there is any irrotational wind field. (It would be of interest if this result would have some effect on track records. Apparently a track record is not official if the wind was blowing over four miles per hour. However, if one ran an integral number of revolutions around a track subject to any irrotational wind field, his record time should be counted all the more. Maybe in the future, as our approach to sports gets more and more scientific, the wind velocity will be recorded continuously at different positions around the track to see whether or not the wind actually helped.)

It is easy to show that if one is flying in an irrotational wind field, the minimum time path between two given points is not necessarily the segment joining the two given points. This suggests a new problem which will be considered in a subsequent paper, i.e.,

What is the most general wind field such that the minimum time path between any two given points is along the segment joining them?

Another problem to be considered in the subsequent paper is the minimum time path in three dimensions. Even in a gravitational field, if the air speed with respect to the wind were held constant (by continuously adjusting the throttle setting) the results would be the same as before. However, if we maintain a fixed throttle setting, the relative air speed will increase as one descends and we get (under certain simplifying assumptions), the classical brachistochrone problem but subject to a wind field.

For the latter two problems, it seemed necessary to use the calculus of variations.

HISTORY OF A FORMULA FOR PRIMES

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In 1947, Mills [7] published the following striking result:

THEOREM A. *There is a real number θ such that $[\theta^{3^n}]$ is a prime for all n, $n=1, 2, \cdots$.*

($[x]$ denotes the integral part of x.) This simple formula yielding prime values exclusively inspired several other papers; it is the purpose of this note to show the development of the idea used in the proof of Theorem A through these later contributions.

Proof of Theorem A. The proof depends on the following result of Ingham [5]: if p_n denotes the nth prime, then there is a constant k such that

$$p_{n+1} - p_n < kp_n^{5/8} \tag{1a}$$

for $n=1, 2, \cdots$. If $N>k^8$ and p_n is the largest prime $<N^3$, then from (1a),

$$p_n < N^3 < p_{n+1} < p_n + kp_n^{5/8} < N^3 + N^{1/8}N^{15/8} = N^3 + N^2 < (N+1)^3 - 1. \tag{2a}$$

Thus, given any integer $N>k^8$, we can find a prime between N^3 and $(N+1)^3-1$. This allows us to construct a sequence of primes $\{q_n\}$ as follows: let $q_1>k^8$ be a prime and let q_{n+1} be a prime such that

$$q_n^3 < q_{n+1} < (q_n + 1)^3 - 1; \tag{3a}$$

for definiteness, we could choose q_{n+1} to be the smallest prime in that range.

Let

$$u_n = q_n^{3^{-n}} \quad \text{and} \quad v_n = (q_n + 1)^{3^{-n}}, \tag{4a}$$

$n=1, 2, \cdots$. We see that $\{u_n\}$ is increasing and $\{v_n\}$ is decreasing:

$$u_{n+1} = q_{n+1}^{3^{-n-1}} > (q_n^3)^{3^{-n-1}} = q_n^{3^{-n}} = u_n,$$

$$v_{n+1} = (q_{n+1} + 1)^{3^{-n-1}} < ((q_n + 1)^3 - 1 + 1)^{3^{-n-1}} = (q_n + 1)^{3^{-n}} = v_n.$$

Also, from (4a), $u_n<v_n$. Hence $u_n<u_{n+1}<v_{n+1}<v_n$ for all $n=1, 2, \cdots$. It follows that both sequences are monotone and bounded and hence converge: let

$$\theta = \lim_{n\to\infty} u_n \quad \text{and} \quad \phi = \lim_{n\to\infty} v_n.$$

From $u_n<\theta\leqq\phi<v_n$ follows

$$u_n^{3^n} < \theta^{3^n} \leqq \phi^{3^n} < v_n^{3^n} \tag{5a}$$

or, from (4a),

$$q_n < \theta^{3^n} < q_n + 1. \tag{6a}$$

Thus $q_n = [\theta^{3^n}]$ and the theorem is proved.

After the proof, we see that the result is less astounding: to construct $\{u_n\}$ and hence θ, we need to be able to recognize arbitrarily large primes. If we could do that, we would already implicitly have a formula for primes. But the result is still picturesque, the idea pleasing, and the details neatly worked out.

The theorem is also susceptible to generalization. The first to do so was Kuipers [6] who proved, quite simply, that the 3 in Theorem A is not essential:

THEOREM B. *If $c\geqq 3$ is an integer, then there is a real number θ such that $[\theta^{c^n}]$ is a prime for all n, $n=1, 2, \cdots$.*

The modifications in the proof of Theorem A which are needed to prove this are as follows. If $c \geqq 3$ is an integer, let $a=3c-4$ and $b=3c-1$. Then $a/b \geqq 5/8$, so it follows from Ingham's result that there is a constant k_1 such that

$$p_{n+1} - p_n < k_1 p_n^{a/b} \tag{1b}$$

for $n=1, 2, \cdots$. If $N > k_1^b$ and p_n is the smallest prime $< N^c$, then

$$p_n < N^c < p_{n+1} < p_n + k_1 p_n^{a/b} < N^c + N^{1/b} N^{ca/b} = N^c + N^{c-1} < (N+1)^c - 1 \tag{2b}$$

(in the equality, we used the fact that $ca+1=(c-1)b$). So, given $N > k_1^b$, there is a prime between N^c and $(N+1)^c-1$. Hence we can construct a sequence of primes $\{q_n\}$ such that

$$q_n^c < q_{n+1} < (q_n + 1)^c - 1, \tag{3b}$$

and the proof proceeds as in Theorem A to show that $q_n = [\theta^{c^n}]$ for $n=1, 2, \cdots$.

Ansari [1] had the same idea as Kuipers and carried it out in the same way, except he noted that there is no need to require c to be an integer. He proved

THEOREM C. *If $c > 8/3$, then there is a real number θ such that $[\theta^{c^n}]$ is a prime for all n, $n=1, 2, \cdots$.*

To prove this, we note that $c > 8/3$ implies $\delta = 3c/8 - 1 > 0$. Then, if $N > k^{1/\delta}$ (the k in (1a)) and p_n is the smallest prime $< N^c$, we have

$$p_n < N^c < p_{n+1} < p_n + k p_n^{5/8} < N^c + N^\delta N^{5c/8} = N^c + N^{c-1} < (N+1)^c - 1. \tag{2c}$$

The proof then proceeds as before.

$c > 8/3$ can be improved further. Actually, Ansari proved Theorem C with $c > 77/29$, using an improvement of (1a) due to Titchmarsh (unpublished). Titchmarsh showed that 5/8 in (1a) could be replaced by any number larger than 48/77, which leads to the lower bound $c > 77/29$. This has since been improved. The exponent in (1a) is connected with the rate of growth of the Riemann ζ-function along the line $t=1/2$. In fact, if $\zeta(\frac{1}{2}+it) < k_2 t^\alpha$ ($k_2, k_3, \cdots$ are constants), then $p_{n+1} - p_n < k_3 p_n^\beta$ with $\beta = (1+4\alpha)/(2+4\alpha) + \epsilon$ for any $\epsilon > 0$. Currently, the best estimate of α is $6/37 + \epsilon$, for any $\epsilon > 0$, by Haneke [3]. This gives $\beta = 61/98 + \epsilon$ and lets us state Theorem C with $c > 98/37$. If the conjecture that $\zeta(\frac{1}{2}+it) < k_4 t^\epsilon$ for any $\epsilon > 0$ is true, then we could take $\beta = 1/2 + \epsilon$ and c to be any real number > 2. The conjecture seems far from being verified though: Haneke's paper is 74 pages long.

All of the preceding results, though themselves elementary, have the esthetic defect of depending on (1a), which is far from elementary. Wright [14] remedied this by replacing (1a) with Bertrand's Postulate:

$$p_{n+1} - p_n < p_n, \tag{1d}$$

which is true for all n, $n=1, 2, \cdots$, and has an elementary proof (see, for example, [4] Theorem 418). Thus, if p_n is the largest prime $<N$,

(2d) $$p_n < N < p_{n+1} < 2p_n < 2N,$$

and there is a prime between N and $2N$ for any integer $N \geqq 2$. Hence we can construct the sequence $\{q_n\}$ by taking q_1 to be any prime and requiring that

(3d) $$2^{q_n} < q_{n+1} < 2^{q_n+1}$$

If we then let

(4d) $$u_n = \log^{(n)} q_n \quad \text{and} \quad v_n = \log^{(n)} (q_n + 1),$$

(where $\log^{(n)}$ denotes the n-times iterated logarithm to the base 2), then from (3d),

$$q_n < \log^{(1)} q_{n+1} < \log^{(1)} (q_{n+1} + 1) < q_n + 1.$$

If we take logarithms to the base 2 of the preceding inequalities n times, we have $u_n < u_{n+1} < v_{n+1} < v_n$. So, as before,

$$\theta = \lim_{n\to\infty} u_n \quad \text{and} \quad \phi = \lim_{n\to\infty} v_n$$

exist. If $\exp^{(n)}$ denotes the n-times iterated exponential to the base 2, we have

(5d) $$\exp^{(n)} u_n < \exp^{(n)} \theta < \exp^{(n)} v_n$$

or

(6d) $$q_n < \exp^{(n)} \theta < q_n + 1.$$

Restating (6d), we have proved

THEOREM D. *There exists a real number θ such that $[2^{2^{\cdot^{\cdot^{\cdot^{2^\theta}}}}}]$ is a prime for any number of iterations of the exponential.*

Of course, 2 could be replaced by any number >1. To balance the virtue of being completely elementary, this theorem has the small defect of being more awkward to state than Theorems A–C.

Ore [9] was able to put both Mills's and Wright's results in one theorem. Ore defines a selection function f for a sequence $\{p_n\}$ to be a function which is continuous, eventually increasing, and such that there exists a subsequence $\{q_n\}$ of $\{p_n\}$ with the property that for all n, $n=1, 2, \cdots$,

(3e) $$f(q_n - \delta_n) < q_{n+1} - \delta_{n+1} \leqq q_{n+1} + \epsilon_{n+1} < f(q_n + \epsilon_n)$$

for some $\delta_n \geqq 0$ and $\epsilon_n \geqq 0$. If we let

(4e) $$u_n = f^{(-n)}(q_n - \delta_n) \quad \text{and} \quad v_n = f^{(-n)}(q_n + \epsilon_n)$$

(where $f^{(n)}$ and $f^{(-n)}$ denote the nth iterates of f and f^{-1} respectively), then, as in Theorems A–D, $\theta = \lim_{n\to\infty} u_n$ exists, and we can prove

THEOREM E. *With the above notation,* $q_n = [f^{(n)}(\theta)]$.

The details of the proof can be traced through as in Theorem A. Ore abstracted in (3e) the idea in (3d), (3b), and (3a); choosing $f(x) = x^3$ gives Mills's result, $f(x) = x^c$ gives Kuiper's and Ansari's, and $f(x) = 2^x$ gives Wright's.

Niven [8] looked at θ^{c^n} in a different way and proved

THEOREM F. *Given any* $c > 1$, *there exists a real number* θ *such that* $[c^{\theta^n}]$ *is a prime for all* n, $n = 1, 2, \cdots$.

To prove this, choose q_1 to be a prime satisfying $q_1 > c^8$ and $q_1 > k$, where k is the constant in (1a). Then, using (1a), Niven shows, with great cleverness, that it is possible to find a prime satisfying

$$q_n^{(\log q_n)^{1/n}} < q_{n+1} < (1 + q_n)^{(\log(1+q_n))^{1/n}} - 1 \tag{3f}$$

for $n = 1, 2, \cdots$. The logarithms are to the base c. (3f) is established by induction. It is the same as

$$c^{(\log q_n)^{(n+1)/n}} < c^{\log q_{n+1}} < c^{(\log(1+q_n))^{(n+1)/n}} - 1,$$

whence

$$(\log q_n)^{1/n} < (\log q_{n+1})^{1/(n+1)} < (\log(1 + q_{n+1}))^{1/(n+1)} < \log(1 + q_n)^{1/n}.$$

If

$$u_n = (\log q_n)^{1/n} \quad \text{and} \quad v_n = (\log(1 + q_n))^{1/n}, \tag{4f}$$

then the preceding inequalities show that

$$u_n < u_{n+1} < v_{n+1} < v_n.$$

Putting $\theta = \lim_{n\to\infty} u_n$ we have, as before, $q_n = [c^{\theta^n}]$.

Mills's original idea has three branches—Theorems C, D, and F—and these were brought back together in a satisfying paper by Wright [15]. In it he shows

THEOREM G. *The set of suitable* θ *in Theorems* C, D, *and* F *has cardinality* c, *measure* 0, *and is nowhere dense.*

The proof is not easy.

After Wright's effort, no further results of the preceding type have appeared, perhaps because they would seem anticlimactic. But formulas for primes, more or less interesting, continue to appear. The interested reader can find some recent ones in papers by Bang [2], Sierpinski [10], Srinivasan [11, 12], and Willans [13].

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RATIONAL EXPONENTIAL EXPRESSIONS AND A CONJECTURE CONCERNING π AND e

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Abstract. One of the most controversial and least well defined of mathematical problems is the problem of simplification. The recent upsurge of interest in mechanized mathematics has lent new urgency to this problem, but so far very little has been accomplished. This paper attempts to shed light on the situation by introducing the class of rational exponential expressions, defining simplification within this class, and showing constructively how to achieve it. It is shown that the only simplified rational exponential expression equivalent to 0 is 0 itself, provided that an easily stated conjecture is true. However the conjecture, if true, will surely be difficult to prove, since it asserts as a special case that π and e are algebraically independent, and no one has yet been able to prove even the much weaker conjecture that $\pi+e$ is irrational.

1. Introduction. Basically simplification means the application of mathematical identities to transform a given expression into an equivalent expression satisfying some desired criteria.

If the class of admissible expressions is too broad, many dilemmas, of which the following is typical, arise. A basic identity for simplification of expressions involving the logarithm function is

$$\log z_1 z_2 = \log z_1 + \log z_2 \tag{1}$$

which holds everywhere on the Riemann surface whose points have the form

$$z = re^{i\theta},\ r > 0,\ -\infty < \theta < \infty. \tag{2}$$