第十一章 多元函数的微分学

11.1 偏导数

例 1 设$f(u,v)$是二阶偏导数连续的函数，$\dfrac{\partial^2 f}{\partial u^2}+\dfrac{\partial^2 f}{\partial v^2}=1$，令$F(x,y)=f\left(xy,\frac{x^2-y^2}{2}\right)$. 证明:

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}=x^2+y^2.$$

证 由链式法则，有$\dfrac{\partial F}{\partial x}=\dfrac{\partial f}{\partial u}\cdot y+\dfrac{\partial f}{\partial v}\cdot x$, $\dfrac{\partial F}{\partial y}=\dfrac{\partial f}{\partial u}\cdot x-\dfrac{\partial f}{\partial v}\cdot y$. 于是

$$\begin{aligned}\frac{\partial^2 F}{\partial x^2} &= \left(\frac{\partial^2 f}{\partial u^2}\cdot y+\frac{\partial^2 f}{\partial u\partial v}\cdot x\right)\cdot y+\left(\frac{\partial^2 f}{\partial u\partial v}\cdot y+\frac{\partial^2 f}{\partial v^2}\cdot x\right)\cdot x\\ &= \frac{\partial^2 f}{\partial u^2}\cdot y^2+\frac{\partial^2 f}{\partial u\partial v}\cdot 2xy+\frac{\partial^2 f}{\partial v^2}\cdot x^2.\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 F}{\partial y^2} &= \left(\frac{\partial^2 f}{\partial u^2}\cdot x-\frac{\partial^2 f}{\partial u\partial v}\cdot y\right)\cdot x-\left(\frac{\partial^2 f}{\partial u\partial v}\cdot x-\frac{\partial^2 f}{\partial v^2}\cdot y\right)\cdot y\\ &= \frac{\partial^2 f}{\partial u^2}\cdot x^2-\frac{\partial^2 f}{\partial u\partial v}\cdot 2xy+\frac{\partial^2 f}{\partial v^2}\cdot y^2.\end{aligned}$$

从而

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}=\left(\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}\right)\cdot(x^2+y^2)=x^2+y^2.$$

□

例 2 设$u=f(x,y)$, 其中是$f(x,y)$二阶偏导数连续的函数. 引进极坐标$x=r\cos\theta, y=r\sin\theta$, 求证:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}+\frac{1}{r}\frac{\partial u}{\partial r}.$$

证 由链式法则，有

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta,\\ \frac{\partial u}{\partial\theta} &= -\frac{\partial u}{\partial x}r\sin\theta+\frac{\partial u}{\partial y}r\cos\theta,\\ \frac{\partial^2 u}{\partial r^2} &= \frac{\partial^2 u}{\partial x^2}\cos^2\theta+2\frac{\partial^2 u}{\partial x\partial y}\cos\theta\sin\theta+\frac{\partial^2 u}{\partial y^2}\sin^2\theta,\\ \frac{\partial^2 u}{\partial\theta^2} &= \frac{\partial^2 u}{\partial x^2}r^2\sin^2\theta-2\frac{\partial^2 u}{\partial x\partial y}r^2\cos\theta\sin\theta+\frac{\partial^2 u}{\partial y^2}r^2\cos^2\theta-\frac{\partial u}{\partial x}r\cos\theta-\frac{\partial u}{\partial y}r\sin\theta.\end{aligned}$$

因此

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{1}{r}\frac{\partial u}{\partial r}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}.$$

□

例 3 设φ和ψ二阶导数连续, $z=x\varphi\left(\frac{y}{x}\right)+\psi\left(\frac{y}{x}\right)$, 求证:

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=0.$$

证　记$t=\frac{y}{x}$, 由链式法则，有

$$\begin{aligned}\frac{\partial z}{\partial x} &= \varphi(t)-\frac{y}{x}\varphi'(t)-\frac{y}{x^2}\psi'(t),\\ \frac{\partial z}{\partial y} &= \varphi'(t)+\frac{1}{x}\psi'(t),\\ \frac{\partial^2 z}{\partial x^2} &= \frac{y^2}{x^3}\varphi''(t)+\frac{2y}{x^3}\psi'(t)+\frac{y^2}{x^4}\psi''(t),\\ \frac{\partial^2 z}{\partial x\partial y} &= -\frac{y}{x^2}\varphi''(t)-\frac{1}{x^2}\psi'(t)-\frac{y}{x^3}\psi''(t),\\ \frac{\partial^2 z}{\partial y^2} &= \frac{1}{x}\varphi''(t)+\frac{1}{x^2}\psi''(t).\end{aligned}$$

因此

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=0.$$

□

例 4 $f(x_1,\cdots,x_n)$称为定义于$\mathbb{R}^n\setminus\{(0,\cdots,0)\}$内的$\lambda$次正齐次函数($\lambda$为实常数), 如果对任何$(x_1,\cdots,x_n)\in\mathbb{R}^n, x_1^2+\cdots+x_n^2\neq 0$, 和正实数$\alpha$均有

$$f(\alpha x_1,\cdots,\alpha x_n)=\alpha^\lambda f(x_1,\cdots,x_n).$$

求证欧拉定理: $\mathbb{R}^n\setminus\{(0,\cdots,0)\}$内一阶偏导数连续的函数$f(x_1,\cdots,x_n)$ 是λ次正齐次函数的充分必要条件是

$$\sum_{i=1}^{n}x_i\frac{\partial f}{\partial x_i}(x_1,\cdots,x_n)=\lambda f(x_1,\cdots,x_n).$$

证 “$\Rightarrow$”. $f(\alpha x_1,\cdots,\alpha x_n)=\alpha^\lambda f(x_1,\cdots,x_n)$两边对$\alpha$求导，得

$$\sum_{i=1}^n x_i\frac{\partial f}{\partial x_i}(\alpha x_1,\cdots,\alpha x_n)=\lambda\alpha^{\lambda-1}f(x_1,\cdots,x_n).$$

令$\alpha=1$即得

$$\sum_{i=1}^n x_i\frac{\partial f}{\partial x_i}(x_1,\cdots,x_n)=\lambda f(x_1,\cdots,x_n).$$

“$\Leftarrow$”. 令$\varphi(\alpha)=\frac{f(\alpha x_1,\cdots,\alpha x_n)}{\alpha^\lambda}$, 则

$$\begin{aligned}\varphi'(\alpha) &= \frac{\alpha^\lambda\sum\limits_{i=1}^n x_if_i'(\alpha x_1,\cdots,\alpha x_n)-\lambda\alpha^{\lambda-1}f(\alpha x_1,\cdots,\alpha x_n)}{\alpha^{2\lambda}}\\ &= \frac{\sum\limits_{i=1}^n \alpha x_if_i'(\alpha x_1,\cdots,\alpha x_n)-\lambda f(\alpha x_1,\cdots,\alpha x_n)}{\alpha^{\lambda+1}}\\ &= 0.\end{aligned}$$

故φ为常数函数，于是由$\varphi(\alpha)=\varphi(1)=f(x_1,\cdots,x_n)$得$f(\alpha x_1,\cdots,\alpha x_n)=\alpha^\lambda f(x_1,\cdots,x_n)$, 即$f$是$\lambda$次正齐次函数. □

例 5 设$f(x,y)$是$[a,b]\times[c,d]$上的二元函数，满足下列三个条件：

(1) $f(a,c)+f(b,d)=f(b,c)+f(a,d)$;

(2) 对任意$y\in[c,d]$, $f(x,y)$作为x的函数在$[a,b]$连续，在(a,b)可导；

(3) 对任意$x\in(a,b)$, $f_x'(x,y)$作为y的函数在$[c,d]$连续，在(c,d)可导，

证明：存在$(x_0,y_0)\in(a,b)\times(c,d)$, 使得$f_{xy}''(x_0,y_0)=0$.

证 令$\varphi(x)=f(x,d)-f(x,c)$, 则由条件(1)和(2)知$\varphi(a)=\varphi(b)$, $\varphi(x)$在$[a,b]$连续，在(a,b)可导. 因此由罗尔定理知存在$x_0\in(a,b)$, 使得$\varphi'(x_0)=0$, 即$f_x'(x_0,c)=f_x'(x_0,d)$. 再由条件(3), 应用罗尔定理即知存在$y_0\in(c,d)$, 使得$f_{xy}''(x_0,y_0)=0$. □

例 6 设$u(x,y)=f(x)+g(y)$, 其中f,g为导数连续的一元函数. 如果存在导数连续的一元函数$s(r)$使得$u(r\cos\theta,r\sin\theta)=s(r)$, 求$u(x,y)$.

证 因为$u(x,y)=f(x)+g(y)$, 所以等式$u(r\cos\theta, r\sin\theta)=s(r)$两边对$\theta$求导，得$f'(x)(-r\sin\theta)+g'(y)(r\cos\theta)=0$, 即$-yf'(x)+xg'(y)=0$. 因此当$x\neq 0$, $y\neq 0$时，就有$\dfrac{f'(x)}{x}=\dfrac{g'(y)}{y}$. 于是存在常数$C_1$, 使得$f'(x)=C_1x$, $g'(y)=C_1y$, 从而$f(x)=\frac{1}{2}C_1x^2+C_2$, $g(y)=\frac{1}{2}C_1y^2+C_3$, 其中$C_2$, C_3是常数. 故$u(x,y)=f(x)+g(y)=A(x^2+y^2)+B$, 其中$A$, B是常数. □

例 7 在自变量和因变量的变换下, 将$z=z(x,y)$的方程变换为$w=w(u,v)$的方程：$x=u$, $y=\dfrac{u}{1+uv}$, $z=\dfrac{u}{1+uw}$, 方程为$x^2\dfrac{\partial z}{\partial x}+y^2\dfrac{\partial z}{\partial y}=z^2$.

解 由$x=u$, $y=\dfrac{u}{1+uv}$得$u=x$, $v=\dfrac{1}{y}-\dfrac{1}{x}$. 于是

$$\frac{\partial u}{\partial x}=1,\ \frac{\partial u}{\partial y}=0,\ \frac{\partial v}{\partial x}=\frac{1}{x^2},\ \frac{\partial v}{\partial y}=-\frac{1}{y^2}.$$

由链式法则得

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\frac{\partial u}{\partial x}\cdot(1+uw)-u\left(\frac{\partial u}{\partial x}w+u\frac{\partial w}{\partial u}\frac{\partial u}{\partial x}+u\frac{\partial w}{\partial v}\frac{\partial v}{\partial x}\right)}{(1+uw)^2}\\
&= \frac{1-u^2\frac{\partial w}{\partial u}-\frac{\partial w}{\partial v}}{(1+uw)^2},\\
\frac{\partial z}{\partial y} &= \frac{\frac{\partial u}{\partial y}\cdot(1+uw)-u\left(\frac{\partial u}{\partial y}w+u\frac{\partial w}{\partial u}\frac{\partial u}{\partial y}+u\frac{\partial w}{\partial v}\frac{\partial v}{\partial y}\right)}{(1+uw)^2}\\
&= \frac{u^2\frac{\partial w}{\partial v}}{y^2(1+uw)^2}.
\end{aligned}$$

因此，方程$x^2\dfrac{\partial z}{\partial x}+y^2\dfrac{\partial z}{\partial y}=z^2$变换为

$$u^2\cdot\frac{1-u^2\frac{\partial w}{\partial u}-\frac{\partial w}{\partial v}}{(1+uw)^2}+y^2\cdot\frac{u^2\frac{\partial w}{\partial v}}{y^2(1+uw)^2}=\frac{u^2}{(1+uw)^2}.$$

整理化简得

$$u^4\frac{\partial w}{\partial u}=0,$$

进而简化为$\dfrac{\partial w}{\partial u}=0$. □

例 8 在自变量和因变量的变换下, 将$z=z(x,y)$的方程变换为$w=w(u,v)$的方程: $u=x^2+y^2$, $v=\dfrac{1}{x}+\dfrac{1}{y}$, $w=\ln z-(x+y)$, 方程为

$$y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=(y-x)z;$$

解 由$w=\ln z-(x+y)$得$z=\mathrm{e}^{w+x+y}$, 故

$$\frac{\partial z}{\partial x}=\mathrm{e}^{w+x+y}\left(2x\frac{\partial w}{\partial u}-\frac{1}{x^2}\frac{\partial w}{\partial v}+1\right)=z\left(2x\frac{\partial w}{\partial u}-\frac{1}{x^2}\frac{\partial w}{\partial v}+1\right),$$

$$\frac{\partial z}{\partial y}=\mathrm{e}^{w+x+y}\left(2y\frac{\partial w}{\partial u}-\frac{1}{y^2}\frac{\partial w}{\partial v}+1\right)=z\left(2y\frac{\partial w}{\partial u}-\frac{1}{y^2}\frac{\partial w}{\partial v}+1\right).$$

代入到$y\dfrac{\partial z}{\partial x}-x\dfrac{\partial z}{\partial y}=(y-x)z$，得

$$z\left(\frac{x}{y^2}\frac{\partial w}{\partial v}-\frac{y}{x^2}\frac{\partial w}{\partial v}+y-x\right)=(y-x)z,$$

化简得

$$(x^3-y^3)\frac{\partial w}{\partial v}=0.$$

又由w的连续可微性得

$$\frac{\partial w}{\partial v}=0.$$

□

11.2 全微分

例 1 设

$$f(x,y)=\begin{cases}|x|^\alpha|y|^\beta\sin\dfrac{1}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0,\end{cases}$$

其中$\alpha>0,\beta>0$. 讨论$f(x,y)$在$(0,0)$点的可微性及偏导数的连续性.

解 因为$\alpha>0$, $\beta>0$, 所以

$$\frac{\partial f}{\partial x}(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=0,$$

$$\frac{\partial f}{\partial y}(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y}=0.$$

于是$f(x,y)$在$(0,0)$点的可微当且仅当$|x|^\alpha|y|^\beta\sin\dfrac{1}{x^2+y^2}=o(\sqrt{x^2+y^2})$, 即$\dfrac{|x|^\alpha|y|^\beta}{\sqrt{x^2+y^2}}\sin\dfrac{1}{x^2+y^2}\to 0$ $((x,y)\to(0,0))$, 这又当且仅当$\displaystyle\lim_{(x,y)\to(0,0)}\frac{|x|^\alpha|y|^\beta}{\sqrt{x^2+y^2}}=0$, 令$x=r\cos\theta$, $y=r\sin\theta$, 该二元极

限等价于$\lim\limits_{r\to 0^+} r^{\alpha+\beta-1}|\cos^\alpha\theta\sin^\beta\theta| = 0$, 这当且仅当$\alpha+\beta>1$. 因此，$f(x,y)$在$(0,0)$点的可微当且仅当$\alpha+\beta>1$.

由函数在一点处连续的定义知$\dfrac{\partial f}{\partial x}$在$(0,0)$点连续当且仅当$\lim\limits_{(x,y)\to(0,0)}\dfrac{\partial f}{\partial x}(x,y) = \dfrac{\partial f}{\partial x}(0,0) = 0$. 因为$xy>0$ 时，

$$\frac{\partial f}{\partial x}(x,y) = \alpha x^{\alpha-1}y^\beta \sin\frac{1}{x^2+y^2} + x^\alpha y^\beta \cos\frac{1}{x^2+y^2}\cdot\frac{-2x}{(x^2+y^2)^2},$$

所以由对称性和上式可知$\dfrac{\partial f}{\partial x}$在$(0,0)$点连续当且仅当$\lim\limits_{(x,y)\to(0,0)} x^{\alpha-1}y^\beta = 0$且$\lim\limits_{(x,y)\to(0,0)}\dfrac{x^{\alpha+1}y^\beta}{(x^2+y^2)^2} = 0$, 这当且仅当$\alpha\geqslant 1$且$(\alpha+1)+\beta>4$. 同理可得$\dfrac{\partial f}{\partial y}$在$(0,0)$点连续当且仅当$\beta\geqslant 1$且$\alpha+(\beta+1)>4$. 因此$f(x,y)$在$(0,0)$点偏导数连续当且仅当$\alpha\geqslant 1$且$\beta\geqslant 1$且$\alpha+\beta>3$. □

例 2 设$f(x,y)$在$\mathbb{R}^2$上二次可微且恒不为*0*，证明*:* $f(x,y) = g(x)h(y)$的充分必要条件是$f(x,y)$满足方程$f\cdot f''_{xy} = f'_x\cdot f'_y$.

证 “$\Rightarrow$”. 设$f(x,y) = g(x)h(y)$, 则$f'_x = g'(x)h(y)$, $f'_y = g(x)h'(y)$, $f''_{xy} = g'(x)h'(y)$, 故$f\cdot f''_{xy} = f'_x\cdot f'_y$.

“$\Leftarrow$”. 因为$f(x,y)$在$\mathbb{R}^2$上恒不为0，所以不妨设$f(x,y)>0$. 令$F(x,y) = \ln f(x,y)$, 则有

$$F'_x = \frac{f'_x}{f}, \quad F''_{xy} = \frac{f''_{xy}f - f'_xf'_y}{f^2} = 0.$$

由$F''_{xy}(x,y)$在$\mathbb{R}^2$上恒等于0可得$F'_x(x,y) = p(x)$, 其中$p(x)$在$\mathbb{R}$上可微，因此

$$\ln f(x,y) = F(x,y) = P(x) + q(y),$$

其中$P(x)$是$p(x)$的一个原函数，$q(y)$在$\mathbb{R}$上可微，从而$f(x,y) = g(x)h(y)$, 其中$g(x) = \mathrm{e}^{P(x)}$, $h(y) = \mathrm{e}^{q(y)}$. □

例 3 设$f(x,y)$定义在(x_0,y_0)的一个邻域U内*,* $f'_x(x,y)$, $f'_y(x,y)$在U内处处存在且都在(x_0,y_0)可微*,* 求证*:*

$$\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0) = \frac{\partial^2 f}{\partial y\partial x}(x_0,y_0).$$

证 令

$$\varphi=\frac{1}{\Delta x\,\Delta y}[f(x_0+\Delta x,y_0+\Delta y)-f(x_0+\Delta x,y_0)-f(x_0,y_0+\Delta y)+f(x_0,y_0)].$$

设$\psi(x)=f(x,y_0+\Delta y)-f(x,y_0)$, 则

$$\begin{aligned}\varphi &= \frac{1}{\Delta x\,\Delta y}\left[\psi(x_0+\Delta x)-\psi(x_0)\right]\\ &= \frac{1}{\Delta y}\psi'(x_0+\theta_1\Delta x)\\ &= \frac{1}{\Delta y}\left[f'_x(x_0+\theta_1\Delta x,y_0+\Delta y)-f'_x(x_0+\theta_1\Delta x,y_0)\right].\end{aligned}$$

其中$0<\theta_1<1$. 由f'_x在(x_0,y_0)可微知

$$f'_x(x_0+\theta_1\Delta x,y_0+\Delta y)=f'_x(x_0,y_0)+f''_{xx}(x_0,y_0)\theta_1\Delta x+f''_{xy}(x_0,y_0)\Delta y+\rho_1\sqrt{(\theta_1\Delta x)^2+(\Delta y)^2},$$

$$f'_x(x_0+\theta_1\Delta x,y_0)=f'_x(x_0,y_0)+f''_{xx}(x_0,y_0)\theta_1\Delta x+\rho_2\theta_1\Delta x,$$

其中$\Delta x\to 0$, $\Delta y\to 0$时，有$\rho_1\to 0$, $\rho_2\to 0$. 于是

$$\varphi=f''_{xy}(x_0,y_0)+\rho_1\cdot\frac{\sqrt{(\theta_1\Delta x)^2+(\Delta y)^2}}{\Delta y}-\rho_2\theta_1\cdot\frac{\Delta x}{\Delta y}. \tag{1}$$

同理可得

$$\varphi=f''_{yx}(x_0,y_0)+\rho_3\cdot\frac{\sqrt{(\Delta x)^2+(\theta_2\Delta y)^2}}{\Delta x}-\rho_4\theta_2\cdot\frac{\Delta y}{\Delta x}, \tag{2}$$

其中$0<\theta_2<1$, 当$\Delta x\to 0$, $\Delta y\to 0$时，有$\rho_3\to 0$, $\rho_4\to 0$. 在(1)和(2)中令$\Delta x=\Delta y\to 0$, 得

$$f''_{xy}(x_0,y_0)=\lim_{\Delta x=\Delta y\to 0}\varphi=f''_{yx}(x_0,y_0).$$

□

例 4 设$f(x,y)$在(u_0,v_0)的一个邻域U内有定义, $g(x,y)=f(u(x,y),v(x,y))$, 其中$u(x,y)$和$v(x,y)$定义在(x_0,y_0)的一个邻域V内满足$u_0=u(x_0,y_0)$, $v_0=v(x_0,y_0)$且$\{(u,v)|\ u=u(x,y),v=v(x,y),(x,y)\in V\}\subseteq U$. 如果$f$在$(u_0,v_0)$可微, $u(x,y)$和$v(x,y)$在(x_0,y_0)的偏导数都存在, 求证: $g(x,y)$在(x_0,y_0)的偏导数也都存在且链式法则成立, 即

$$\begin{aligned}g'_x(x_0,y_0) &= f'_u(u_0,v_0)u'_x(x_0,y_0)+f'_v(u_0,v_0)v'_x(x_0,y_0),\\ g'_y(x_0,y_0) &= f'_u(u_0,v_0)u'_y(x_0,y_0)+f'_v(u_0,v_0)v'_y(x_0,y_0).\end{aligned}$$

证 记$\Delta_x u = u(x_0+\Delta x, y_0) - u(x_0, y_0)$, $\Delta_x v = v(x_0+\Delta x, y_0) - v(x_0, y_0)$, $\Delta_x g = f(u_0 + \Delta_x u, v_0 + \Delta_x v) - f(u_0, v_0)$, 则由$f$在$(u_0, v_0)$可微得

$$\Delta_x g = f'_u(u_0, v_0)\Delta_x u + f'_v(u_0, v_0)\Delta_x v + \alpha \cdot \sqrt{(\Delta_x u)^2 + (\Delta_x v)^2},$$

其中$\lim\limits_{\Delta x \to 0} \alpha = 0$. 因此有

$$\frac{\Delta_x g}{\Delta x} = f'_u(u_0, v_0)\frac{\Delta_x u}{\Delta x} + f'_v(u_0, v_0)\frac{\Delta_x v}{\Delta x} + \alpha \cdot \sqrt{\left(\frac{\Delta_x u}{\Delta x}\right)^2 + \left(\frac{\Delta_x v}{\Delta x}\right)^2} \cdot \operatorname{sgn} \Delta x.$$

又由$u(x, y)$和$v(x, y)$在(x_0, y_0)的偏导数都存在知

$$\lim_{\Delta x \to 0} \frac{\Delta_x u}{\Delta x} = u'_x(x_0, y_0), \quad \lim_{\Delta x \to 0} \frac{\Delta_x v}{\Delta x} = v'_x(x_0, y_0),$$

故

$$\lim_{\Delta x \to 0} \frac{\Delta_x g}{\Delta x} = f'_u(u_0, v_0)u'_x(x_0, y_0) + f'_v(u_0, v_0)v'_x(x_0, y_0).$$

按偏导数的定义知

$$g'_x(x_0, y_0) = f'_u(u_0, v_0)u'_x(x_0, y_0) + f'_v(u_0, v_0)v'_x(x_0, y_0).$$

同理可证$g'_y(x_0, y_0)$的结果. □

11.3 方向导数及梯度的性质

若$f(X)$在X_0不可微，那么$f(X)$在X_0处的方向导数用定义计算；若$f(X)$在X_0不可微，那么$f(X)$在X_0处的方向导数可以通过梯度与单位方向向量的内积来计算.

例 1 设$f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x^2 + y^2}, & x^2 + y^2 \neq 0, \\ 0, & x^2 + y^2 = 0, \end{cases}$ $\vec{l} = (\cos\theta, \sin\theta)$, 求$\dfrac{\partial f}{\partial \vec{l}}(0, 0)$.

解 $\dfrac{\partial f}{\partial \vec{l}}(0, 0) = \lim\limits_{t \to 0^+} \dfrac{f(t\cos\theta, t\sin\theta) - f(0, 0)}{t} = \lim\limits_{t \to 0^+} \dfrac{\frac{t^3\cos^3\theta + t^3\sin^3\theta}{t^2} - 0}{t} = \cos^3\theta + \sin^3\theta.$ □

例 2 设$u = f(x, y, z)$的二阶偏导数连续, l的方向余弦为

$(\cos\alpha, \cos\beta, \cos\gamma)$, 试求$u$沿$l$的二阶方向导数$\frac{\partial^2 u}{\partial \vec{l}^2}$, 这里$\frac{\partial^2 u}{\partial \vec{l}^2} = \frac{\partial}{\partial \vec{l}}\left(\frac{\partial u}{\partial \vec{l}}\right)$.

解 记$\vec{l}=(\cos\alpha_1,\cos\alpha_2,\cos\alpha_3)$, 其中$\alpha_1=\alpha$, $\alpha_2=\beta$, $\alpha_3=\gamma$, 则

$$\frac{\partial u}{\partial\vec{l}}=\left\langle\nabla f,\vec{l}\right\rangle=\sum_{i=1}^{3}f_i'\cdot\cos\alpha_i.$$

于是

$$\frac{\partial^2 u}{\partial\vec{l}^2}=\frac{\partial}{\partial\vec{l}}\left(\frac{\partial u}{\partial\vec{l}}\right)=\left\langle\nabla\frac{\partial u}{\partial\vec{l}},\vec{l}\right\rangle=\sum_{i,j=1}^{3}f_{ij}''\cos\alpha_i\cos\alpha_j.$$

又$u=f(x,y,z)$的二阶偏导数连续, 故

$$\frac{\partial^2 u}{\partial\vec{l}^2}=\vec{l}\cdot H_f\cdot\vec{l}^T,$$

其中H_f是$f(x,y,z)$的黑塞矩阵. □

例 3 试确定常数a,b,c, 使得函数

$$f(x,y,z)=axy^2+byz+cz^2x^3.$$

在$(1,2,-1)$点沿z轴正向的方向导数取最大值*64*.

解 $\nabla f(x,y,z)=(ay^2+3cz^2x^2,2axy+bz,by+2czx^3)$, 故$\nabla f(1,2,-1)=(4a+3c,4a-b,2b-2c)$. 由题设知$\nabla f(1,2,-1)$与$z$轴正向同向，其长度为64, 故$4a+3c=0$, $4a-b=0$, $2b-2c=64$, 解得$a=6$, $b=24$, $c=-8$. □

例 4 设函数$f(x,y)$在$D=\{(x,y)|x^2+y^2\leqslant 1\}$上连续可微，对任意$(x,y)\in\partial D$，都有$f(x,y)=0$, 对任意$(x,y)\in D^\circ$和任意方向$\vec{l}$, 都有$\left|\frac{\partial f}{\partial\vec{l}}(x,y)\right|\leqslant 1$, 求证：

$$|f(x,y)|\leqslant 1-\sqrt{x^2+y^2}.$$

证 若$(x,y)\in\partial D$, 则$f(x,y)=0=1-\sqrt{x^2+y^2}$, 这时等式成立；若$(x,y)\in D^\circ$, 当$(x,y)\neq(0,0)$时，令$s=\frac{x}{\sqrt{x^2+y^2}}$, $t=\frac{y}{\sqrt{x^2+y^2}}$；当$(x,y)=(0,0)$时，令$(s,t)=(1,0)$, 那么就有$(s,t)\in\partial D$. 由微分中值定理知存在连结$(x,y)$, (s,t)的线段上一点(p,q), 使得$f(x,y)-f(s,t)=\frac{\partial f}{\partial\vec{l}}(p,q)(1-\sqrt{x^2+y^2})$, 其中$\vec{l}=(s,t)$. 于是

$$|f(x,y)|=|f(x,y)-f(s,t)|=\left|\frac{\partial f}{\partial\vec{l}}(p,q)\right|(1-\sqrt{x^2+y^2})\leqslant 1-\sqrt{x^2+y^2}.$$ □

例 5 设$u(x,y),v(x,y)$在开区域D内处处满足

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x},\ u^2+v^2=1,$$

求证: $u(x,y),v(x,y)$在D内为常数.

证 由$u^2+v^2=1$得

$$\begin{cases} u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}=0, \\ u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}=0. \end{cases}$$

又$\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y},\ \ \dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$, 故

$$\begin{cases} u\frac{\partial u}{\partial x}-v\frac{\partial u}{\partial y}=0, \\ v\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial y}=0. \end{cases} \tag{1}$$

因为方程组(1)的系数行列式等于1, 所以方程组(1)只有零解. 因此，在开区域D内$\dfrac{\partial u}{\partial x},\ \dfrac{\partial u}{\partial y}$恒等于0，进而$\dfrac{\partial v}{\partial x},\ \dfrac{\partial v}{\partial y}$恒等于0. 由11.3的定理3的推论1知$u(x,y),v(x,y)$在$D$内为常数. $\square$

例 6 设$F\in C^1(\mathbb{R}^n,\mathbb{R}^m)$, 求证: 对任意$X,\Delta X\in\mathbb{R}^n$, 存在$\theta\in(0,1)$使得

$$|F(X+\Delta X)-F(X)|\leqslant|J_F(X+\theta\Delta X)\Delta X^T|.$$

证 记$\Delta F=F(X+\Delta X)-F(X)$, 令

$$\varphi(t)=\langle\Delta F,F(X+t\Delta X)\rangle\,,\quad t\in[0,1].$$

一方面，$\varphi(1)-\varphi(0)=\langle\Delta F,\Delta F\rangle=|\Delta F|^2$, 另一方面，由微分中值定理，存在$\theta\in(0,1)$使得

$$\varphi(1)-\varphi(0)=\varphi'(\theta)=\left\langle\Delta F,J_F(X+\Delta X)\Delta X^T\right\rangle.$$

因此，由柯西不等式得

$$|\Delta F|^2=\left\langle\Delta F,J_F(X+\Delta X)\Delta X^T\right\rangle\leqslant|\Delta F|\cdot|J_F(X+\Delta X)\Delta X^T|,$$

从而

$$|F(X+\Delta X)-F(X)|=|\Delta F|\leqslant|J_F(X+\theta\Delta X)\Delta X^T|.$$

$\square$

例 7 设D是$\mathbb{R}^n$中的凸区域，映射$F: D \to \mathbb{R}^m$可微, k是常数，求证：对任何$X, Y \in D$, 都有$|F(X) - F(Y)| \leqslant k|X - Y|$的充分必要条件是对任何$X \in D$和任何$\mathbb{R}^n$中的单位向量$V$, 都有$\left|J_F(X)V^T\right| \leqslant k.$

证 “$\Leftarrow$”. 对任何$X, Y \in D$, 若$X = Y$, 则$|F(X) - F(Y)| = 0 = k|X - Y|$; 若$X \neq Y$, 则由拟微分中值定理知存在$\theta \in (0,1)$, 使得

$$|F(X) - F(Y)| \leqslant \left|J_F(X + \theta(Y - X))(Y - X)^T\right| = \left|J_F(X)V^T\right| |Y - X| \leqslant k|Y - X| = k|X - Y|,$$

其中$V = \dfrac{Y - X}{|Y - X|}$是$\mathbb{R}^n$中的单位向量.

“$\Rightarrow$”. 反证. 若不然，则存在$X_0 \in D$和$\mathbb{R}^n$中的单位向量V_0, 使得$\left|J_F(X_0)V_0^T\right| > k$. 由$F$可微知

$$\lim_{t \to 0^+} \frac{\left|F(X_0 + tV_0) - F(X_0) - J_F(X_0)(tV_0)^T\right|}{t} = 0.$$

记$\lambda = \left|J_F(X_0)V_0^T\right|$, 对于$\varepsilon_0 = \dfrac{\lambda - k}{2} > 0$, 存在$\delta > 0$, 使得对任何$t \in (0, \delta)$, 有$X_0 + tV_0 \in D$且$\left|F(X_0 + tV_0) - F(X_0) - J_F(X_0)(tV_0)^T\right| < t\varepsilon_0$, 从而

$$|F(X_0 + tV_0) - F(X_0)| \geqslant \left|J_F(X_0)(tV_0)^T\right| - t\varepsilon_0 = t\lambda - t\varepsilon_0 > tk = k|tV_0|,$$

与$|F(X_0 + tV_0) - F(X_0)| \leqslant k|tV_0|$矛盾！ □

接下来的三个例题有一定难度，大家可以体会一下辅助函数的构造及微分中值定理的应用.

例 8 设$X_0 \in \mathbb{R}^n$, $f(X)$在X_0的某邻域中两次连续可微, $\nabla f(X_0) = 0$, 黑塞矩阵$H_f(X_0)$正定, 求证：存在$\delta > 0$, 使得对任意$X \in B_\delta(X_0) \setminus \{X_0\}$, 都有

$$\langle \nabla f(X), X - X_0 \rangle > 0.$$

证 因为$f(X)$在X_0的某邻域中两次连续可微, 黑塞矩阵$H_f(X_0)$正定, 所以存在$\delta > 0$, 对任意$X \in B_\delta(X_0)$, 有$H_f(X)$正定(请自行给出证明). 对任意$X \in B_\delta(X_0) \setminus \{X_0\}$, 令

$$\varphi(t) = f(X_0 + t(X - X_0)), \quad t \in [0,1],$$

则

$$\varphi'(t) = \langle \nabla f(X_0 + t(X - X_0)), X - X_0 \rangle,$$

$$\varphi''(t) = (X - X_0) \cdot H_f(X_0 + t(X - X_0)) \cdot (X - X_0)^T > 0.$$

因此$\varphi'(t)$在$[0,1]$严格递增，从而$\varphi'(1) > \varphi'(0)$. 又$\varphi'(0) = \langle \nabla f(X_0), X - X_0 \rangle = 0$, $\varphi'(1) = \langle \nabla f(X), X - X_0 \rangle$, 故

$$\langle \nabla f(X), X - X_0 \rangle > 0.$$

□

例 9 设$f(x,y)$在$\mathbb{R}^2$上连续可微且对任意实数x, y, 都有

$$f(x,y) = a\frac{\partial f}{\partial x}(x,y) + b\frac{\partial f}{\partial y}(x,y),$$

其中a, b是常数, 求证: 如果$f(x,y)$在$\mathbb{R}^2$上有界，那么$f(x,y)$在$\mathbb{R}^2$上恒等于0.

证 不妨设$a^2 + b^2 \neq 0$, 任取(x_0, y_0), 令$\varphi(t) = f(x_0 + at, y_0 + bt)$, $t \in \mathbb{R}$, 则

$$\varphi'(t) = a\frac{\partial f}{\partial x}(x_0 + at, y_0 + bt) + b\frac{\partial f}{\partial y}(x_0 + at, y_0 + bt) = \varphi(t).$$

由此可得$(\mathrm{e}^{-t}\varphi(t))' \equiv 0$, 故$\varphi(t) = C\mathrm{e}^t$, $t \in \mathbb{R}$, 其中C是常数. 由$f(x,y)$在$\mathbb{R}^2$上有界得$\varphi(t)$在$\mathbb{R}$上有界，从而$C = 0$, 故$f(x_0, y_0) = \varphi(0) = 0$, 再由$(x_0, y_0)$的任意性得$f(x,y)$在$\mathbb{R}^2$上恒等于0. □

例 10 设$D = [0,1] \times [0,1]$, 函数$f(x,y)$在D上可微，$f(0,0) + f(0,1) + f(1,0) + f(1,1) = 0$, 对任何$(x,y) \in D$, 有$\left|\frac{\partial f}{\partial x}(x,y)\right| + \left|\frac{\partial f}{\partial y}(x,y)\right| \leqslant 1$, 求证：对任何$(x,y) \in D$, 有$|f(x,y)| \leqslant \frac{3}{4}$.

证 首先证明“对任何$(x_1, y_1), (x_2, y_2) \in D$, 有$|f(x_1, y_1) - f(x_2, y_2)| \leqslant \max\{|x_1 - x_2|, |y_1 - y_2|\}$”. 若$(x_1, y_1) = (x_2, y_2)$, 则显然等式成立，故下设$(x_1, y_1) \neq (x_2, y_2)$. 由多元函数的微分中值定理知在$(x_1, y_1)$, (x_2, y_2)为端点的线段上存在一点(ξ, η), 使得

$$f(x_1, y_1) - f(x_2, y_2) = \frac{\partial f}{\partial x}(\xi, \eta)(x_1 - x_2) + \frac{\partial f}{\partial y}(\xi, \eta)(y_1 - y_2).$$

记$M=\max\{|x_1-x_2|,|y_1-y_2|\}$, 则由上式以及条件$\left|\dfrac{\partial f}{\partial x}(x,y)\right|+\left|\dfrac{\partial f}{\partial y}(x,y)\right|\leqslant 1$得

$$|f(x_1,y_1)-f(x_2,y_2)|\leqslant\left|\frac{\partial f}{\partial x}(\xi,\eta)\right|\cdot|x_1-x_2|+\left|\frac{\partial f}{\partial y}(\xi,\eta)\right|\cdot|y_1-y_2|\leqslant M\left|\frac{\partial f}{\partial x}(\xi,\eta)\right|+M\left|\frac{\partial f}{\partial y}(\xi,\eta)\right|\leqslant M.$$

这就证明了"对任何$(x_1,y_1),(x_2,y_2)\in D$, 有$|f(x_1,y_1)-f(x_2,y_2)|\leqslant\max\{|x_1-x_2|,|y_1-y_2|\}$".

对任何$(x,y)\in D$, 由上面的命题以及条件$f(0,0)+f(0,1)+f(1,0)+f(1,1)=0$得

$$\begin{aligned}4|f(x,y)| &= |4f(x,y)-f(0,0)-f(0,1)-f(1,0)-f(1,1)|\\ &\leqslant |f(x,y)-f(0,0)|+|f(x,y)-f(0,1)|+|f(x,y)-f(1,0)|+|f(x,y)-f(1,1)|\\ &\leqslant \max\{x,y\}+\max\{x,1-y\}+\max\{1-x,y\}+\max\{1-x,1-y\}.\end{aligned}$$

由对称性，不妨设$0\leqslant x\leqslant y\leqslant\dfrac{1}{2}$, 则由上式得

$$4|f(x,y)|\leqslant y+(1-y)+(1-x)+(1-x)=3-2x\leqslant 3.$$

因此，对任何$(x,y)\in D$, 有$|f(x,y)|\leqslant\dfrac{3}{4}$. □

11.4 多元函数的泰勒公式

例 1 写出函数$f(x,y)=\mathrm{e}^x\ln(1-y)$在$(0,0)$点的二阶泰勒展开式.

解 借助一元函数的泰勒公式，得

$$\begin{aligned}f(x,y) &= \mathrm{e}^x\ln(1-y)\\ &= \left(1+x+\frac{x^2}{2}+o(x^2)\right)\left(-y-\frac{y^2}{2}+o(y^2)\right)\\ &= -y-xy-\frac{1}{2}y^2+o(x^2+y^2).\end{aligned}$$

□

例 2 设$f(x_1,x_2,\cdots,x_n)=\ln(1+x_1+x_2+\cdots+x_n)$, 将$f$在$(0,\cdots,0)$点$m$阶泰勒展开, 并证明对于$|x_1+\cdots+x_n|<1$, 当$m\to+\infty$时展开式余项趋于*0*.

解 记$t=x_1+x_2+\cdots+x_n$, 由带柯西余项的泰勒公式得

$$f(x_1,x_2,\cdots,x_n)=\ln(1+t)=\sum_{k=1}^{m}(-1)^{k-1}\frac{t^k}{k}+R_m(t),$$

其中余项为

$$R_m(t)=(-1)^m\frac{(1-\theta)^m t^{m+1}}{(1+\theta t)^{m+1}},\ 0<\theta<1.$$

因为$|t|<1$, 所以$0<\dfrac{1-\theta}{1+\theta t}<1$, 从而

$$0\leqslant|R_m(t)|=\frac{|t|^{m+1}}{1+\theta t}\cdot\left(\frac{1-\theta}{1+\theta t}\right)^m\leqslant\frac{|t|^{m+1}}{1-|t|}.$$

由$m\to\infty$时$|t|^{m+1}\to 0$, 根据两边夹定理得$\lim\limits_{m\to\infty}R_m(t)=0$. □

注 这里要用柯西余项，用拉格朗日余项解决不了问题.

11.5 隐函数存在定理

例 1 设$u=\dfrac{x+z}{y+z}$, 其中z为由方程

$$z\,e^z=x\,e^x+y\,e^y$$

确定的x,y的隐函数, 求$\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y}$.

解 在$z\mathrm{e}^z=x\mathrm{e}^x+y\mathrm{e}^y$两边对$x$求导，得$(1+z)\mathrm{e}^z\dfrac{\partial z}{\partial x}=(1+x)\mathrm{e}^x$, 故$\dfrac{\partial z}{\partial x}=\dfrac{1+x}{1+z}\mathrm{e}^{x-z}$. 同理可得$\dfrac{\partial z}{\partial y}=\dfrac{1+y}{1+z}\mathrm{e}^{y-z}$. 于是

$$\frac{\partial u}{\partial x}=\frac{\left(1+\frac{\partial z}{\partial x}\right)(y+z)-(x+z)\frac{\partial z}{\partial x}}{(y+z)^2}=\frac{1}{y+z}+\frac{(y-x)(1+x)}{(y+z)^2(1+z)}\mathrm{e}^{x-z}.$$

同理可得

$$\frac{\partial u}{\partial y}=-\frac{x+z}{(y+z)^2}+\frac{(y-x)(1+y)}{(y+z)^2(1+z)}\mathrm{e}^{y-z}.$$ □

例 2 设u,v,w是由方程组

$$\begin{cases}x=f(u,v,w),\\ y=g(u,v,w),\\ z=h(u,v,w)\end{cases}$$

确定的x,y和z的函数, 求$\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y},\dfrac{\partial u}{\partial z}$.

解 每个方程求微分，得

$$\begin{cases} \mathrm{d}x = \dfrac{\partial f}{\partial u}\mathrm{d}u + \dfrac{\partial f}{\partial v}\mathrm{d}v + \dfrac{\partial f}{\partial w}\mathrm{d}w, \\ \mathrm{d}y = \dfrac{\partial g}{\partial u}\mathrm{d}u + \dfrac{\partial g}{\partial v}\mathrm{d}v + \dfrac{\partial g}{\partial w}\mathrm{d}w, \\ \mathrm{d}z = \dfrac{\partial h}{\partial u}\mathrm{d}u + \dfrac{\partial h}{\partial v}\mathrm{d}v + \dfrac{\partial h}{\partial w}\mathrm{d}w, \end{cases}$$

故由线性方程组的Cramer法则得

$$\mathrm{d}u = \frac{\begin{vmatrix} \mathrm{d}x & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \\ \mathrm{d}y & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \\ \mathrm{d}z & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \end{vmatrix}} = \frac{I_1}{I}\mathrm{d}x + \frac{I_2}{I}\mathrm{d}y + \frac{I_3}{I}\mathrm{d}z,$$

其中$I = \dfrac{D(f,g,h)}{D(u,v,w)}$, $I_1 = \dfrac{D(g,h)}{D(v,w)}$, $I_2 = \dfrac{D(h,f)}{D(v,w)}$, $I_3 = \dfrac{D(f,g)}{D(v,w)}$. 于是

$$\frac{\partial u}{\partial x} = \frac{I_1}{I}, \quad \frac{\partial u}{\partial y} = \frac{I_2}{I}, \quad \frac{\partial u}{\partial z} = \frac{I_3}{I}.$$

□

例 3 设函数$f(x,y)$定义在$D = \{(x,y)|\ x^2+y^2<1\}$内且满足$f(0,0)=0$, $|f(x,y)| \leqslant 1$, $\forall (x,y)\in D$. 又函数

$$F(x,y,z) = z^3 + z(x^2+y^2) + f(x,y)$$

定义在$G = \{(x,y,z)|\ x^2+y^2<1, |z|\leqslant 1\}$上, 求证: 在$D$上存在唯一的函数$z=\varphi(x,y)$满足方程$F(x,y,z)=0$, 且$|\varphi(x,y)|<1$, $\quad \forall (x,y)\in D$.

证 因为$F_z'(x,y,z) = 3z^2+x^2+y^2>0$, $\forall (x,y,z)\in G\setminus\{(0,0,0)\}$, $F_z'(0,0,0)=0$, 所以在G上，$F(x,y,z)$作为z的函数严格递增. 任意固定$(x,y)\in D$,

$$F(x,y,1) = 1+(x^2+y^2)+f(x,y)>0, \quad F(x,y,-1) = -1-(x^2+y^2)+f(x,y)<0.$$

由连续性和严格递增性知存在唯一的$\varphi(x,y)\in(-1,1)$, 使得$F(x,y,\varphi(x,y))=0$. □

例 4 设$x = y+\varphi(y)$, 其中$\varphi(0)=0$且当$-a<y<a$时, $\varphi'(y)$连续, $|\varphi'(0)|<1$, 求证: 存在$\delta>0$, 使得当$-\delta<x<\delta$时有唯一的可微函数$y=y(x)$满足

$$x = y(x)+\varphi(y(x)), \text{ 且 } y(0)=0.$$

证 令$f(x,y)=y+\varphi(y)-x$, 则f在$\mathbb{R}\times(-a,a)$上连续可微，又$f(0,0)=\varphi(0)=0$, $f'_y(0,0)=1+\varphi'(0)\neq 0$, 由隐函数定理知存在$\delta>0$, 使得当$-\delta<y<\delta$时, 存在唯一可微的$y=y(x)$满足方程$x=y+\varphi(y)$且$y(0)=0$. $\square$

例 5 设$z=z(x,y)$由方程组

$$\begin{cases} z=\alpha x+\varphi(\alpha)y+\psi(\alpha), \\ 0=x+\varphi'(\alpha)y+\psi'(\alpha) \end{cases}$$

所确定, 求证: $z''_{xx}z''_{yy}-(z''_{xy})^2=0$.

证 第一个方程两边对x求偏导，得

$$z'_x=\alpha'_x x+\alpha+\varphi'(\alpha)\alpha'_x y+\psi'(\alpha)\alpha'_x,$$

再结合第二个方程，得$z'_x=\alpha$. 第一个方程两边对y求偏导，得

$$z'_y=\alpha'_y x+\varphi'(\alpha)\alpha'_y y+\varphi(\alpha)+\psi'(\alpha)\alpha'_y,$$

再结合第二个方程，得$z'_y=\varphi(\alpha)$. 于是

$$z''_{xx}=\alpha'_x,\ z''_{xy}=\alpha'_y,\ z''_{yx}=\varphi'(\alpha)\alpha'_x,\ z''_{yy}=\varphi'(\alpha)\alpha'_y.$$

第二个方程分别对x, y求导得

$$1+\varphi''(\alpha)\alpha'_x y+\psi''(\alpha)\alpha'_x=0,\quad \varphi''(\alpha)\alpha'_y y+\varphi'(\alpha)+\psi''(\alpha)\alpha'_y=0,$$

于是有

$$\alpha'_y=-\frac{\varphi'(\alpha)}{\varphi''(\alpha)y+\psi''(\alpha)}=\varphi'(\alpha)\alpha'_x,$$

故$z''_{xy}=z''_{yx}$, 从而

$$z''_{xx}z''_{yy}-(z''_{xy})^2=z''_{xx}z''_{yy}-z''_{xy}z''_{yx}=\alpha'_x\cdot\varphi'(\alpha)\alpha'_y-\alpha'_y\cdot\varphi'(\alpha)\alpha'_x=0.$$

$\square$

例 6 设$z(x,y)\in C^2$,且$z''_{xx}z''_{yy}-(z''_{xy})^2\neq 0$. 从方程组

$$\begin{cases} p=z'_x(x,y) \\ q=z'_y(x,y) \end{cases}$$

中解出$x=x(p,q),y=y(p,q)$. 记

$$w(p,q)=px(p,q)+qy(p,q)-z(x(p,q),y(p,q)),$$

求证: $w'_p(p,q)=x(p,q),w'_q(p,q)=y(p,q)$, 且

$$\begin{pmatrix} z''_{xx}(x,y) & z''_{xy}(x,y) \\ z''_{xy}(x,y) & z''_{yy}(x,y) \end{pmatrix}\begin{pmatrix} w''_{pp}(p,q) & w''_{pq}(p,q) \\ w''_{pq}(p,q) & w''_{qq}(p,q) \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

其中$x=x(p,q),y=y(p,q)$.

证 因为$w(p,q)=px(p,q)+qy(p,q)-z(x(p,q),y(p,q))$, 所以

$$\begin{aligned} w'_p(p,q) &= x+px'_p+qy'_p-(z'_x\cdot x'_p+z'_y\cdot y'_p)=x(p,q), \\ w'_q(p,q) &= px'_q+y+qy'_q-(z'_x\cdot x'_q+z'_y\cdot y'_q)=y(p,q). \end{aligned}$$

在$p=z'_x(x(p,q),y(p,q))$两边对p求导，得$1=z''_{xx}\cdot x'_p+z''_{xy}\cdot y'_p$, 两边对$q$求导，得$0=z''_{xx}\cdot x'_q+z''_{xy}\cdot y'_q$. 同理可证，$0=z''_{yx}\cdot x'_p+z''_{yy}\cdot y'_p$, $1=z''_{yx}\cdot x'_q+z''_{yy}\cdot y'_q$. 于是

$$\begin{pmatrix} z''_{xx} & z''_{xy} \\ z''_{yx} & z''_{yy} \end{pmatrix}\begin{pmatrix} w''_{pp} & w''_{pq} \\ w''_{qp} & w''_{qq} \end{pmatrix}=\begin{pmatrix} z''_{xx} & z''_{xy} \\ z''_{yx} & z''_{yy} \end{pmatrix}\begin{pmatrix} x'_p & x'_q \\ y'_p & y'_q \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

□

例 7 设$F:\mathbb{R}^n\to\mathbb{R}^n$为$C^1$映射且存在$\lambda>0$使得

$$|F(X)-F(Y)|\geqslant\lambda|X-Y|,\quad\forall X,Y\in\mathbb{R}^n.$$

求证: 对任意$Y_0\in\mathbb{R}^n$, 存在唯一的$X_0\in\mathbb{R}^n$使得

$$F(X_0)=Y_0.$$

证　要证明的结论实际上是$F:\mathbb{R}^n\to\mathbb{R}^n$是双射. 由$|F(X)-F(Y)|\geqslant\lambda|X-Y|$不难得出$F$是单射. 为证明$F$是满射，即$F(\mathbb{R}^n)=\mathbb{R}^n$, 只需证明$F(\mathbb{R}^n)$既开又闭. 先证明$F(\mathbb{R}^n)$闭，这只需证明$F(\mathbb{R}^n)$列闭. 任取$F(\mathbb{R}^n)$中收敛到点$Y_0$的点列$\{F(X_m)\}$, 则$\{F(X_m)\}$是柯西列，由$|F(X_m)-F(X_k)|\geqslant\lambda|X_m-X_k|$可知$\{X_m\}$也是柯西列，从而$\{X_m\}$收敛于$X_0$. 由连续性得$Y_0=\lim\limits_{m\to\infty}F(X_m)=F(X_0)$, 故$Y_0\in F(\mathbb{R}^n)$. 这就证明了$F(\mathbb{R}^n)$列闭. 再证明$F(\mathbb{R}^n)$开. 对任意$X\in\mathbb{R}^n$, $J_F(X)$非奇异. 反证. 若存在X使得$J_F(X)$奇异，则存在$\Delta X\in\mathbb{R}^n$, $\Delta X\neq O$, 使得$J_F(X)\Delta X^T=O$. 于是由F的可微性得

$$F(X+t\Delta X)-F(X)=J_F(X)\cdot t\Delta X^T+o(t)=o(t)\quad(t\to0^+),$$

与$|F(X+t\Delta X)-F(X)|\geqslant\lambda|\Delta X|\cdot t$矛盾！因为$F:\mathbb{R}^n\to\mathbb{R}^n$为$C^1$映射且对任意$X\in\mathbb{R}^n$, $J_F(X)$非奇异，所以由11.5节的定理4知$F(\mathbb{R}^n)$开. □

11.6 曲线的切线与曲面的切平面

例 1 在曲线$x=t,y=t^2,z=t^3$上求一点, 使得该曲线在此点的切线平行于平面$x+2y+z=4$;

解　因为点(t,t^2,t^3)处曲线的切线平行于平面$x+2y+z=4$当且仅当点(t,t^2,t^3)处曲线的切向量$(1,2t,3t^2)$与平面$x+2y+z=4$的法向量$(1,2,1)$垂直，所以

$$1\cdot1+2\cdot2t+1\cdot3t^2=0.$$

解得$t=-1$或$t=-\dfrac{1}{3}$, 故所求的曲线上的点为$(-1,1,-1)$, $\left(-\dfrac{1}{3},\dfrac{1}{9},-\dfrac{1}{27}\right)$. □

例 2 设$f(x,y)$在$\mathbb{R}^2$上连续可微，对任何实数x, y, t, 有$f(tx,ty)=t^2f(x,y)$, 已知点$P_0(1,-2,2)$在曲面$S:z=f(x,y)$上，且$\dfrac{\partial f}{\partial x}(1,-2)=6$, 求$\dfrac{\partial f}{\partial y}(1,-2)$的值，并写出曲面$S$在点$P_0$处的切平面方程.

解　因为$f(tx,ty)=t^2f(x,y)$, 所以由齐次函数的欧拉定理知

$$x\frac{\partial f}{\partial x}(x,y)+y\frac{\partial f}{\partial y}(x,y)=2f(x,y).$$

由点$P_0(1,-2,2)$在曲面$S: z=f(x,y)$上知$f(1,-2)=2$, 又$\dfrac{\partial f}{\partial x}(1,-2)=6$, 故$1\cdot 6-2\dfrac{\partial f}{\partial y}(1,-2)=2\cdot 2$, 解得$\dfrac{\partial f}{\partial y}(1,-2)=1$. 因为曲面$S$在点$P_0$处的法向量为

$$\left(\frac{\partial f}{\partial x}(1,-2),\frac{\partial f}{\partial y}(1,-2),-1\right)=(6,1,-1),$$

所以曲面S在点P_0处的切平面方程为

$$6(x-1)+(y+2)-(z-2)=0,\ \text{即}6x+y-z-2=0.$$

□

例 3 设Γ是曲线$\begin{cases} x+y+z=1, \\ z-xy=0 \end{cases}$在$x,y$平面上的投影, 求$\Gamma$在其上一点$(x_0,y_0)$的切线方程.

解 Γ的方程是$x+y+xy-1=0$, 故在(x_0,y_0)处的法向量是$(1+y_0,1+x_0)$, 从而切线方程是$(1+y_0)(x-x_0)+(1+x_0)(y-y_0)=0$. □

例 4 求证: 曲面$\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{a}(a>0)$上任一点的切平面在各坐标轴上的截距之和等于$a$.

证 曲面$\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{a}(a>0)$上任一点$(x_0,y_0,z_0)$处的法向量是$\left(\frac{1}{2\sqrt{x_0}},\frac{1}{2\sqrt{y_0}},\frac{1}{2\sqrt{z_0}}\right)$, 于是该点处的切平面是$\frac{1}{\sqrt{x_0}}(x-x_0)+\frac{1}{\sqrt{y_0}}(y-y_0)+\frac{1}{\sqrt{z_0}}(z-z_0)=0$. 切平面在各坐标轴上的截距分别是$\sqrt{x_0}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})=\sqrt{ax_0}$, $\sqrt{y_0}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})=\sqrt{ay_0}$, $\sqrt{z_0}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})=\sqrt{az_0}$. 因此截距之和为$\sqrt{ax_0}+\sqrt{ay_0}+\sqrt{az_0}=\sqrt{a}\cdot\sqrt{a}=a$. □

11.7 极值理论

例 1 求函数$f(x,y)=4\ln y+\dfrac{(x-1)^2+(y-2)^2}{y^2}$的极值;

解 函数$f(x,y)$的定义域是$D=\big\{(x,y)\big|y>0\big\}$. 由$\begin{cases} \frac{\partial f}{\partial x}=0, \\ \frac{\partial f}{\partial y}=0 \end{cases}$得

$$\begin{cases} \frac{2(x-1)}{y^2}=0, \\ \frac{4}{y}-\frac{2(x-1)^2}{y^3}+\frac{4}{y^2}-\frac{8}{y^3}=0. \end{cases}$$

解得$x=1,\ y=1$或$x=1,\ y=-2$. 因为$(1,-2)\notin D$, 所以函数$f(x,y)$有唯一临界点$(1,1)$. 因为

$$H_f(1,1)=\begin{pmatrix} f''_{xx}(1,1) & f''_{xy}(1,1) \\ f''_{yx}(1,1) & f''_{yy}(1,1) \end{pmatrix}=\begin{pmatrix} 2 & 0 \\ 0 & 12 \end{pmatrix}$$

是正定矩阵，所以由极值的充分条件知$(1,1)$是$f(x,y)$的极小值点，函数$f(x,y)$有极小值$f(1,1)=1$. □

例 2 求函数$f(x,y)=x+4y$在条件$x^2+2y^2=1$下的极值.

解 令$L(x,y)=x+4y+\lambda(x^2+2y^2-1)$, 则由拉格朗日乘子法得方程组

$$\begin{cases} 1+2\lambda x=0, \\ 4+4\lambda y=0, \\ x^2+2y^2-1=0. \end{cases}$$

由前两个方程得$x=-\dfrac{1}{2\lambda}$, $y=-\dfrac{1}{\lambda}$, 代入到第三个方程中，得$\dfrac{1}{4\lambda^2}+\dfrac{2}{\lambda^2}-1=0$, 即$4\lambda^2=9$, 解得$\lambda=\dfrac{3}{2}$或$\lambda=-\dfrac{3}{2}$. 由$L(x,y)=x+4y+\lambda(x^2+2y^2-1)$得

$$H_L(x,y)=\begin{pmatrix} 2\lambda & 0 \\ 0 & 4\lambda \end{pmatrix}.$$

若$\lambda=\dfrac{3}{2}$, 则$x=-\dfrac{1}{2\lambda}=-\dfrac{1}{3}$, $y=-\dfrac{1}{\lambda}=-\dfrac{2}{3}$. 由$H_L\left(-\dfrac{1}{3},-\dfrac{2}{3}\right)=\begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}$正定知$\left(-\dfrac{1}{3},-\dfrac{2}{3}\right)$是条件极小值点，相应的条件极小值为$f\left(-\dfrac{1}{3},-\dfrac{2}{3}\right)=-3$.
若$\lambda=-\dfrac{3}{2}$, 则$x=-\dfrac{1}{2\lambda}=\dfrac{1}{3}$, $y=-\dfrac{1}{\lambda}=\dfrac{2}{3}$. 由$H_L\left(\dfrac{1}{3},\dfrac{2}{3}\right)=\begin{pmatrix} -3 & 0 \\ 0 & -6 \end{pmatrix}$负定知$\left(\dfrac{1}{3},\dfrac{2}{3}\right)$是条件极大值点，相应的条件极大值为$f\left(\dfrac{1}{3},\dfrac{2}{3}\right)=3$. □

例 3 设$f(x,y)$在$\mathbb{R}^2$上两次连续可微, 且对任意$(x,y)\in\mathbb{R}^2$, 都有

$$\frac{\partial^2 f}{\partial x^2}(x,y)+\frac{\partial^2 f}{\partial y^2}(x,y)>0,$$

求证: $f(x,y)$在$\mathbb{R}^2$上无极大值.

证 反证. 若$f(x,y)$在(x_0,y_0)处取得极大值，则由极值的必要条件知$H_f(x_0,y_0)\leqslant 0$, 从而

$$\frac{\partial^2 f}{\partial x^2}(x_0,y_0)+\frac{\partial^2 f}{\partial y^2}(x_0,y_0)=\mathrm{Tr}\ H_f(x_0,y_0)\leqslant 0,$$

与题设矛盾！ □

例 4 设$f(x) \in C^1(\mathbb{R}^n)$且在$\mathbb{R}^n$上是凸函数, 如果$X_0 \in \mathbb{R}^n$且$X_0$是$f(x)$的临界点, 求证: X_0是$f(x)$的最小点.

证 因为X_0是$f(x)$的临界点，所以$\nabla f(X_0) = O$. 由11.4的例1知，对任意$X \in \mathbb{R}^n$, 有

$$f(X) - f(X_0) \geqslant \langle \nabla f(X_0), X - X_0 \rangle = 0.$$

故X_0是$f(x)$的最小点. □

例 5 设$f(x,y) = ax^2 + 2bxy + cy^2 + 2dx + 2ey + f$, 求证: 如果$f(x,y)$在$(x_0, y_0)$达到极大(小)值, 则它必在$(x_0, y_0)$达到最大(小)值.

证 记$X = (x,y)$, $X_0 = (x_0, y_0)$, $\Delta X = X - X_0$, 则由泰勒公式，有

$$f(X) - f(X_0) = \langle \nabla f(X_0), \Delta X \rangle + \frac{1}{2}\Delta X \cdot H_f(X_0) \cdot \Delta X^T.$$

因为f在X_0处达到极大值, 所以$\nabla f(X_0) = O$, $H_f(X_0) \leqslant 0$, 从而

$$f(X) - f(X_0) = \frac{1}{2}\Delta X \cdot H_f(X_0) \cdot \Delta X^T \leqslant 0,$$

故$f(X) \leqslant f(X_0)$, 由X的任意性知$f(X_0)$是f的最大值. □

例 6 设函数$f(X)$在有界闭区域D上有连续偏导数，且有常数a使得

$$f(X) = a, \quad \forall X \in \partial D,$$

求证: 存在$X_0 \in D^\circ$使得

$$\nabla f(X_0) = O.$$

证 若$f(X)$在D上恒为a, 则任取$X_0 \in D$, 有$\nabla f(X_0) = 0$; 若$f(X)$在D上不恒为a, 由D是有界闭区域知f在D上有最大值和最小值，则f的最大值和最小值至少有一个在D的内部取得，设$X_0 \in D^\circ$是f的一个最值点，则X_0是f的极值点，从而$\nabla f(X_0) = O$. □

例 7 设$D=\{(x,y)|x^2+y^2\leqslant 1\}$, $f\in C^1(D)$, $|f(x,y)|\leqslant 1$, $\forall(x,y)\in D$, 证明: 存在$(x_0,y_0)\in D^\circ$, 使得$|\nabla f(x_0,y_0)|<4$.

证 令$g(x,y)=f(x,y)+2(x^2+y^2)$, $\forall(x,y)\in D$, 则$g\in C^1(D)$. 因为$|f(x,y)|\leqslant 1$, 所以对任意$(x,y)\in\partial D$, 有$g(x,y)=f(x,y)+2\geqslant 1$. 又$g(0,0)=f(0,0)\leqslant 1$, 故在$D^\circ$中必有$g$的最小值点. 设$(x_0,y_0)\in D^\circ$是$g$的一个最小值点，则$(x_0,y_0)$是$g$的一个极值点，故$\nabla g(x_0,y_0)=O$. 再由$\nabla g(x,y)=\nabla f(x,y)+(4x,4y)$得$\nabla f(x_0,y_0)=(-4x_0,-4y_0)$, 于是$|\nabla f(x_0,y_0)|=4\sqrt{x_0^2+y_0^2}<4$. □

例 8 设$D=\{(x,y):x^2+y^2<1\}$, $f(x,y)$是D上两次连续可微的有界正值函数，且满足

$$\Delta\ln f(x,y)\geqslant f^2(x,y)\ \left(\Delta\text{是拉普拉斯算子，即}\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right).$$

证明：

$$f(x,y)\leqslant\frac{2}{1-x^2-y^2}\ (\forall(x,y)\in D).$$

证 令$g(x,y)=\dfrac{2}{1-x^2-y^2}$, 则

$$\Delta\ln g(x,y)=\frac{4}{(1-x^2-y^2)^2}=g^2(x,y).$$

记$F(x,y)=\ln g(x,y)-\ln f(x,y)=\ln\dfrac{g(x,y)}{f(x,y)}$, 则$\Delta F(x,y)\leqslant g^2(x,y)-f^2(x,y)$. 由$\lim\limits_{x^2+y^2\to 1}g(x,y)=+\infty$和$f$有界知$\lim\limits_{x^2+y^2\to 1}F(x,y)=+\infty$, 从而$F(x,y)$在$D$中取得最小值. 设$(x_0,y_0)$是$F$的一个最小值点，则由$H_F(x_0,y_0)$半正定知$\dfrac{\partial^2 F}{\partial x^2}(x_0,y_0)\geqslant 0$, $\dfrac{\partial^2 F}{\partial y^2}(x_0,y_0)\geqslant 0$, 从而$\Delta F(x_0,y_0)\geqslant 0$, 由此得到$g^2(x_0,y_0)-f^2(x_0,y_0)\geqslant\Delta F(x_0,y_0)\geqslant 0$, 进而得到$F(x_0,y_0)=\ln\dfrac{g(x_0,y_0)}{f(x_0,y_0)}\geqslant\ln 1=0$. 因为$(x_0,y_0)$是$F$的一个最小值点，所以

$$\ln\frac{g(x,y)}{f(x,y)}=F(x,y)\geqslant F(x_0,y_0)\geqslant 0.$$

由此即得

$$f(x,y)\leqslant g(x,y)=\frac{2}{1-x^2-y^2}\ (\forall(x,y)\in D).$$

□

例 9 已知x,y平面上n个点$(x_i,y_i),i=1,\cdots,n$, 求实数a,b, 使得直线$y=ax+b$与n个已给点的"偏差平方和"

$$\sigma(a,b)=\sum_{i=1}^{n}(ax_i+b-y_i)^2$$

达到最小.

解 由极值的必要条件得

$$\begin{cases} 2\sum\limits_{i=1}^{n}(ax_i+b-y_i)x_i=0,\\ 2\sum\limits_{i=1}^{n}(ax_i+b-y_i)=0. \end{cases}$$

整理得

$$\begin{cases} \left(\sum x_i^2\right)a+\left(\sum x_i\right)b=\sum x_iy_i,\\ \left(\sum x_i\right)a+nb=\sum y_i. \end{cases}$$

不妨设x_i不全相等，则由柯西不等式得$\left(\sum x_i\right)^2<n\cdot\sum x_i^2$, 故上面的线性方程组有唯一解

$$a_0=\frac{\begin{vmatrix}\sum x_iy_i & \sum x_i\\ \sum y_i & n\end{vmatrix}}{\begin{vmatrix}\sum x_i^2 & \sum x_i\\ \sum x_i & n\end{vmatrix}},\quad b_0=\frac{\begin{vmatrix}\sum x_i^2 & \sum x_iy_i\\ \sum x_i & \sum y_i\end{vmatrix}}{\begin{vmatrix}\sum x_i^2 & \sum x_i\\ \sum x_i & n\end{vmatrix}}.$$

可以证明$\sigma(a,b)$在$\mathbb{R}^2$上取得最小值，$\sigma(a,b)$的最小值点必为极值点，故唯一临界点(a_0,b_0)就是最小值点，上面的a_0, b_0即为所求的a, b. 这时，直线$y=ax+b$的方程可写为

$$\begin{vmatrix} x & y & 1\\ \sum x_i^2 & \sum x_iy_i & \sum x_i\\ \sum x_i & \sum y_i & n \end{vmatrix}=0.$$

下面证明$\sigma(a,b)$在$\mathbb{R}^2$上取得最小值. 取$x_i\neq x_j$, 则当$a^2+b^2\to+\infty$时，$(ax_i+b-y_i)^2+(ax_j+b-y_j)^2\to+\infty$, 从而$\sigma(a,b)=\sum\limits_{i=1}^{n}(ax_i+b-y_i)^2\to+\infty$. 由练习10.3的第1题知$\sigma(a,b)$在$\mathbb{R}^2$上取得最小值. □

例 10 设Ω是$\mathbb{R}^n$的有界开区域, $u(X)\in C^2(\Omega)\cap C(\overline{\Omega})$. 若

$$\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}(X)+\sum_{i=1}^{n}b_i(X)\frac{\partial u}{\partial x_i}(X)\geqslant 0,\quad \forall X\in\Omega,$$

其中$b_i(X)$, $i=1,\cdots,n$均为连续函数, 求证:

$$u(X)\leqslant\max_{X\in\partial\Omega}u(X),\ \ \forall X\in\Omega.$$

证 令$v(X)=u(X)-\max\limits_{\partial\Omega} u$, 则有

$$\sum_{i=1}^{n}\frac{\partial^2 v}{\partial x_i^2}(X)+\sum_{i=1}^{n}b_i(X)\frac{\partial v}{\partial x_i}(X)\geqslant 0,\quad \forall X\in\Omega.$$

只要证$v(X)\leqslant 0,\quad \forall X\in\Omega$. 为此，在适当对坐标系进行平移之后，可设$\Omega\subseteq\{X=(x_1,\cdots,x_n)|\ 0<x_1<d\}$. 取$r>0$充分大，使$r+b_1(X)>0, \forall X\in\Omega$. 令$z(X)=\mathrm{e}^{2rd}-\mathrm{e}^{rx_1}$, 则有$z(X)>0$并且

$$\sum_{i=1}^{n}\frac{\partial^2 z}{\partial x_i^2}(X)+\sum_{i=1}^{n}b_i(X)\frac{\partial z}{\partial x_i}(X)=-(r^2+b_1(X)r)\mathrm{e}^{rx_1}<0,\ \ \forall X\in\Omega.$$

令$v(X)=z(X)w(X)$. 从而有

$$\begin{aligned} z\sum_{i=1}^{n}\frac{\partial^2 w}{\partial x_i^2}(X)\quad &+\sum_{i=1}^{n}\left(b_i(X)z(X)+2\frac{\partial z}{\partial x_i}(X)\right)\frac{\partial w}{\partial x_i}(X)\\ &\qquad w(X)\left(\sum_{i=1}^{n}\frac{\partial^2 z}{\partial x_i^2}(X)+\sum_{i=1}^{n}b_i(X)\frac{\partial z}{\partial x_i}(X)\right)\geqslant 0.\end{aligned}$$

显然$\max\limits_{\partial\Omega} w=0$. 只需要证明$w(X)\leqslant 0$. 用反证法. 设$X_0\in\Omega$是$w(X)$的最大值点且$w(X_0)>0$, 则由定理1可知$\dfrac{\partial w}{\partial x_i}(X_0)=0,\ i=1,\cdots,n$, 且

$$H_w(X_0)=\begin{pmatrix}\dfrac{\partial^2 w}{\partial x_1^2}(X_0) & \cdots & \dfrac{\partial^2 w}{\partial x_1\partial x_n}(X_0)\\ & \cdots & \\ \dfrac{\partial^2 w}{\partial x_n\partial x_1}(X_0) & \cdots & \dfrac{\partial^2 w}{\partial x_n^2}(X_0)\end{pmatrix}$$

为半负定对称方阵, 从而$\sum\limits_{i=1}^{n}\dfrac{\partial^2 w}{\partial x_i^2}(X_0)\leqslant 0$. 由此$w(X)\left(\sum\limits_{i=1}^{n}\dfrac{\partial^2 z}{\partial x_i^2}(X)+\sum\limits_{i=1}^{n}b_i(X)\dfrac{\partial z}{\partial x_i}(X)\right)\geqslant 0$. 所以$w(X_0)\leqslant 0$与$w(X_0)>0$矛盾. □

注 这里展示了极值理论在偏微分方程中的应用，尽管较难，但其思想值得体会.

例 11 求函数$f(x,y,z)=xyz$在条件$x^2+y^2+z^2=1, x+y+z=0$下的极值.

解 拉格朗日函数是$L(x,y,z)=xyz+\lambda(x^2+y^2+z^2-1)+\mu(x+y+z)$, 由拉格朗日乘子法，解$\begin{cases} yz+2\lambda x+\mu=0,\\ xz+2\lambda y+\mu=0,\\ xy+2\lambda z+\mu=0,\\ x^2+y^2+z^2=1,\\ x+y+z=0\end{cases}$ 得$\lambda=-\frac{3}{2}xyz$, $\mu=\frac{1}{6}$. 由前三个方程得$\begin{vmatrix} yz & x & 1\\ xz & y & 1\\ xy & z & 1\end{vmatrix}=0$, 故$(x-y)(y-$

$z)(z-x)=0$, 从而$x=y$或$y=z$或$z=x$. 解得条件临界点$\left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}}\right)$, $\left(-\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}}\right)$, $\left(-\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)$, $\left(\frac{2}{\sqrt{6}},-\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)$, $\left(\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)$, $\left(-\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)$. 由$S=\{(x,y,z)|x^2+y^2+z^2=1,\ x+y+z=0\}$是球面上的大圆知$S$是一个有界闭集，从而连续函数$f(x,y,z)$在$S$上取得最大值和最小值. 这里，条件最值点都是条件极值点，从而是条件临界点. 因为条件临界值只有$-\frac{\sqrt{6}}{18}$和$\frac{\sqrt{6}}{18}$两个取值，所以它们就是条件最小值和条件最大值. 故6个条件临界点全是条件最值点，从而全是条件极值点. $\left(-\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)$, $\left(\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)$, $\left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}}\right)$是条件极小点，条件极小值为$-\frac{\sqrt{6}}{18}$, $\left(\frac{2}{\sqrt{6}},-\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)$, $\left(-\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)$, $\left(-\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}}\right)$是条件极大点，条件极大值为$\frac{\sqrt{6}}{18}$. □

例 12 求$x_1^2+\cdots+x_n^2$在条件$a_1x_1+\cdots+a_nx_n=1$下的极值, 其中实数$a_1,\cdots,a_n$不全为*0*.

解 拉格朗日函数是

$$L(x_1,\cdots,x_n)=x_1^2+\cdots+x_n^2+\lambda(a_1x_1+\cdots+a_nx_n-1),$$

由拉格朗日乘子法得

$$\begin{cases} 2x_1+\lambda a_1=0, \\ \cdots, \\ 2x_n+\lambda a_n=0, \\ a_1x_1+\cdots+a_nx_n-1=0. \end{cases}$$

解得$x_i=\dfrac{a_i}{a_1^2+\cdots+a_n^2}$. 不难看到$x_1^2+\cdots+x_n^2$在条件$a_1x_1+\cdots+a_nx_n=1$下取最小值，故唯一的条件临界点就是条件极小值点. $x_i=\dfrac{a_i}{a_1^2+\cdots+a_n^2}$ $(i=1,\cdots,n)$时，$x_1^2+\cdots+x_n^2$有条件极小值$\dfrac{1}{a_1^2+\cdots+a_n^2}$. □

例 13 求函数$f(x,y,z)=x+y+z$在区域$D=\{(x,y,z)\big|\ x^2+y^2\leqslant z\leqslant 1\}$上的最大值和最小值.

解 由$f'_x=f'_y=f'_z=1$知f在D的内部没有临界点，故只需在D的边界上讨论. D的边界是$\{(x,y,z)|z=1,x^2+y^2\leqslant 1\}\cup\{(x,y,z)|0\leqslant z<1,x^2+y^2=z\}$. 函数$f(x,y,z)=x+y+z$在条件$z=1$, $x^2+y^2\leqslant 1$下的最值问题归为$f(x,y,1)=x+y+1$在区域$\{(x,y)|x^2+y^2\leqslant$

$1\}$上的最值问题. 该区域内部没有临界点，由拉格朗日乘子法解$\begin{cases}1+2\lambda x=0,\\ 1+2\lambda y=0,\\ x^2+y^2=1\end{cases}$得条件临界点$\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},1\right)$, $\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},1\right)$, 相应的条件临界值为$\sqrt{2}+1$, $-\sqrt{2}+1$. 对于$f(x,y,z)=x+y+z$在$x^2+y^2=z(0\leqslant z<1)$条件下的条件临界点，由拉格朗日乘子法解方程组$\begin{cases}1+2\lambda x=0,\\ 1+2\lambda y=0,\\ 1-\lambda=0,\\ x^2+y^2-z=0\end{cases}$得条件临界点$(-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$, 相应的条件临界值为$-\frac{1}{2}$. 因此$f(x,y,z)$在$D$上的最大值为$\sqrt{2}+1$, 最小值为$-\dfrac{1}{2}$; $\square$

例 14 椭球面$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$在第*1*卦限中点的切平面与坐标平面围成一个四面体, 在这些四面体中, 体积的最小值为多少?

解 在椭球面$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$上任取一点$(x_0,y_0,z_0)$, 其中$x_0,y_0,z_0>0$, 过该点的切平面为

$$\frac{x_0x}{a^2}+\frac{y_0y}{b^2}+\frac{z_0z}{c^2}=1,$$

切平面在坐标轴的截距分别为$\dfrac{a^2}{x_0}$, $\dfrac{b^2}{y_0}$, $\dfrac{c^2}{z_0}$, 故四面体的体积为$V=\dfrac{a^2b^2c^2}{6x_0y_0z_0}$. 因此，问题可以通过求目标函数$xyz$在约束条件$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$下的最大值来解决. 由拉格朗日乘子法解方程组$\begin{cases}\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\\ yz+\frac{2\lambda x}{a^2}=0,\\ xz+\frac{2\lambda y}{b^2}=0,\\ xy+\frac{2\lambda z}{c^2}=0\end{cases}$得唯一条件临界点$\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)$. 不难看到$xyz$在第一卦限的椭球面上取得最大值，故$\left(\frac{a}{\sqrt{3}},\frac{b}{\sqrt{3}},\frac{c}{\sqrt{3}}\right)$为条件最大值点，相应的四面体体积的最小值为$\dfrac{\sqrt{3}abc}{2}$. $\square$

例 15 求函数$u=x_1^2\cdots x_n^2$在条件$x_1^2+\cdots+x_n^2=r^2$下的最大值, 并由此导出均值不等式: $\sqrt[n]{a_1\cdots a_n}\leqslant\dfrac{a_1+\cdots+a_n}{n}$, 其中$a_i\geqslant 0, i=1,\cdots,n$.

解 令

$$L(x_1,\cdots,x_n)=x_1^2\cdots x_n^2+\lambda(x_1^2+\cdots+x_n^2-r^2),$$

则由拉格朗日乘子法得

$$\begin{cases}2x_1x_2^2\cdots x_n^2+2\lambda x_1=0,\\ \cdots,\\ 2x_1^2\cdots x_{n-1}^2x_n+2\lambda x_n=0,\\ x_1^2+\cdots+x_n^2-r^2=0.\end{cases}$$

解得$x_i=\dfrac{r^2}{n}$, $i=1,2,\cdots,n$. 因为u在S上取得最大值，故当$x_i=\dfrac{r^2}{n}$ $(i=1,2,\cdots,n)$时，u取得最大值$\dfrac{r^{2n}}{n^n}$.

令$a_i=x_i^2$, $i=1,2,\cdots,n$, 则$a_1a_2\cdots a_n\leqslant\dfrac{(a_1+a_2+\cdots+a_n)^n}{n^n}$, 由此即得均值不等式

$$\sqrt[n]{a_1a_2\cdots a_n}\leqslant\frac{a_1+a_2+\cdots+a_n}{n}.$$

□

例 16 设$n\geqslant 2$, $f(X,Y)=f(x_1,\cdots,x_n,y_1,\cdots,y_n)=\langle X,Y\rangle$,

$$S=\left\{(X,Y)\in\mathbb{R}^{2n}\big|\ |X|_p=1,|Y|_q=1,x_i\geqslant 0,y_i\geqslant 0,i=1,\cdots,n\right\},$$

$\dfrac{1}{p}+\dfrac{1}{q}=1$, $p>1,q>1$, 这里对$s>1$, $X\in\mathbb{R}^n$, 定义$|X|_s=\left(\displaystyle\sum_{i=1}^n|x_i|^s\right)^{\frac{1}{s}}$, 求证:

$$f(X,Y)\leqslant 1,\forall\,(X,Y)\in S.$$

并由此证明赫尔德不等式:

$$\sum_{i=1}^n a_ib_i\leqslant\left(\sum_{i=1}^n a_i^p\right)^{1/p}\left(\sum_{i=1}^n b_i^q\right)^{1/q},a_i\geqslant 0,b_i\geqslant 0.$$

证 显然$S\subseteq\mathbb{R}^{2n}$是有界闭集, 从而$f(X,Y)$在$S$上有最大值和最小值, 显然最小值为零, 下面证明最大值为1. S是一个带边界的有界闭集，其边界$\partial S=\{(X,Y)\in S|\ \exists\ i=1,\cdots,n,$ 使得 $x_i=0$ 或 $y_i=0\}$, 内部$S^\circ=\{(X,Y)\in\mathbb{R}^{2n}|\ |X|_p=1,|Y|_q=1,x_i>0,y_i>0,i=1,\cdots,n\}$. 需要证明: $f(X,Y)\leqslant 1$, $(X,Y)\in S^\circ$ 和$f(X,Y)\leqslant 1$, $(X,Y)\in\partial S$均成立.

若$(X,Y)\in\partial S$, 则x_i或y_i中有一个等于零, 例如$x_n=0$, 这时若$y_n=1$, 则$f(X,Y)=0$为最小值. 若$y_n\neq 1$, 则由$\displaystyle\sum_{i=1}^n y_i^q=1$, 可得$\displaystyle\sum_{i=1}^{n-1}\tilde{y}_i^q=1$, $\tilde{y}_i=\dfrac{y_i}{(1-y_n^q)^{1/q}}$. 若有$\displaystyle\sum_{i=1}^{n-1}x_i\tilde{y}_i\leqslant 1$, 则$\displaystyle\sum_{i=1}^n x_iy_i\leqslant(1-y_n^q)^{1/q}\leqslant 1$. 从而可在低维空间中考虑相同的问题, 由于$n=1$时不等式显然成立, 对维数$n$用数学归纳法, 从而$f(X,Y)\leqslant 1$, $(X,Y)\in\partial S$成立.

若$(X,Y)\in S^\circ$, 即有$x_i>0,y_i>0$, $i=1,\cdots,n$. 考虑$f(X,Y)=\displaystyle\sum_{i=1}^n x_iy_i$在条件$g_1(X,Y)=\displaystyle\sum_{i=1}^n x_i^p-1=0$, $g_2(X,Y)=\displaystyle\sum_{i=1}^n y_i^q-1=0$之下的条件极值问题. 令

$$L(X,Y)=f(X,Y)+\lambda_1g_1(X,Y)+\lambda_2g_2(X,Y).$$

由条件极值的必要条件, 有

$$\begin{array}{l} y_i + \lambda_1 p x_i^{p-1} = 0, \\ x_i + \lambda_2 q y_i^{q-1} = 0, \end{array} \ i = 1, \cdots, n.$$

由此可知$\lambda_1 < 0, \lambda_2 < 0$, 并且

$$\sum_{i=1}^{n} x_i y_i + \lambda_1 p = 0, \sum_{i=1}^{n} x_i y_i + \lambda_2 q = 0.$$

由此得$\lambda_1 p = \lambda_2 q$. 把这个结果代入上面方程组并消元, 可得$\lambda_1 p = \lambda_2 q = -1$以及$x_i^{1/q} = y_i^{1/p}$, 从而这时$f(X,Y) = \sum\limits_{i=1}^{n} x_i y_i = 1$, 这是唯一的条件临界值, 它在$x_{01} = x_{02} = \cdots = x_{0n} = \left(\frac{1}{n}\right)^{1/p}$, $y_{01} = y_{02} = \cdots = y_{0n} = \left(\frac{1}{n}\right)^{1/q}$达到, 因此$f(X,Y)$ 在S上的最大值为这点的函数值$f(X_0, Y_0) = 1$, 所以

$$f(X,Y) \leqslant 1, \forall\, (X,Y) \in S.$$

取

$$x_i = \frac{a_i}{\left(\sum\limits_{i=1}^{n} a_i^p\right)^{1/p}}, \quad y_i = \frac{b_i}{\left(\sum\limits_{i=1}^{n} b_i^q\right)^{1/q}},$$

则得赫尔德不等式. 等号成立的充要条件为$x_i^{1/q} = y_i^{1/p}$. □

例 17 在曲线$17x^2 + 12xy + 8y^2 = 100$上求到原点距离最近的点和最远的点.

解 拉格朗日函数是$L(x,y) = x^2 + y^2 + \lambda(17x^2 + 12xy + 8y^2 - 100)$, 由拉格朗日乘子法，得到方程组$\begin{cases} 17x^2 + 12xy + 8y^2 - 100 = 0, \\ 2x + 34\lambda x + 12\lambda y = 0, \\ 2y + 12\lambda x + 16\lambda y = 0 \end{cases}$, 由后两个方程知$\begin{vmatrix} 17\lambda + 1 & 6\lambda \\ 6\lambda & 8\lambda + 1 \end{vmatrix} = 0$, 解得$\lambda = -\frac{1}{20}$或$\lambda = -\frac{1}{5}$. 对于$\lambda = -\frac{1}{20}$, 解得两个条件临界点$(2,1)$, $(-2,-1)$; 对于$\lambda = -\frac{1}{5}$, 解得两个条件临界点$(2,4)$, $(-2,-4)$. 再由$x^2 + y^2$在曲线$17x^2 + 12xy + 8y^2 = 100$上取得最大值和最小值知曲线$17x^2 + 12xy + 8y^2 = 100$上到原点距离最近的点为$(2,1)$和$(-2,-1)$, 到原点最远的点为$(2,4)$和$(-2,-4)$. □