历年考题积分部分详解

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07级物理类

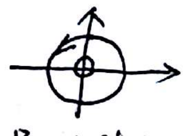
五. $z'_x=\frac{x}{\sqrt{x^2+y^2}}$ $\quad z'_y=\frac{y}{\sqrt{x^2+y^2}}$ 由于取里侧. 法向量取 $\left(\frac{-x}{\sqrt2\sqrt{x^2+y^2}},\frac{-y}{\sqrt2\sqrt{x^2+y^2}},\frac{1}{\sqrt2}\right)$

里侧 = 上侧, 第二型曲面积分转化二重积分 为正号.

$$\therefore I=\iint_{D_{xy}}\left(2y\cdot\frac{-x\sqrt2}{\sqrt2\sqrt{x^2+y^2}}-2x\frac{-y\sqrt2}{\sqrt2\sqrt{x^2+y^2}}+x^2+y^2\right)dxdy=\iint_{D_{xy}}(x^2+y^2)\,dxdy$$

采用极坐标 $I=\int_0^{2\pi}d\theta\int_1^3 r^2\cdot r\,dr=40\pi$

六 $P=\frac{-y}{2x^2+y^2}$ $\quad Q=\frac{x}{2x^2+y^2}$ $\quad Q'_x=P'_y=\frac{y^2-2x^2}{(2x^2+y^2)^2}$



有奇点 $(0,0)$ 挖洞 取 $L: 2x^2+y^2=\varepsilon^2$ $(\varepsilon\to0)$ 且取顺时针方向

由格林公式 $\oint_C+\oint_L=0$ $\quad\therefore\oint_C=-\oint_L=\oint_{L^-}$ $\quad L^-$ 即取逆时针方向

$$\oint_{L^-}\frac{-ydx+xdy}{2x^2+y^2}=\frac{1}{\varepsilon^2}\oint_{L^-}-ydx+xdy\ \to \text{再用格林公式}$$

$$=\frac{2}{\varepsilon^2}\boxed{\iint_D dxdy}\ \to \text{椭圆面积}$$

下面求椭圆面积 令 $x=\frac{\sqrt2}{2}r\cos\theta$ $\quad y=r\sin\theta$ $\quad S=\int_0^{2\pi}d\theta\int_0^{\varepsilon}\frac{\sqrt2}{2}r\,dr=\frac{\sqrt2}{2}\pi\varepsilon^2$

$$\therefore\oint_C\frac{-ydx+xdy}{2x^2+y^2}=\oint_{L^-}\frac{-ydx+xdy}{2x^2+y^2}=\frac{2}{\varepsilon^2}\times\frac{\sqrt2}{2}\pi\varepsilon^2=\sqrt2\pi.$$

七. $P=x+z^3$ $\quad R=-z$ $\quad P'_x+Q'_y+R'_z=0$ 由于未封闭 补面 $C\begin{cases}x^2+y^2\le4\\ z=2\end{cases}$ 取上侧

利用高斯公式 $\iint_S+\iint_C=0$ $\quad\iint_S(z^3+x)dydz-zdxdy=-\iint_C=\iint_{C^-}$ (C^- 即取下侧)

第二型曲面积分转化为二重积分 取负号 $\therefore\iint_{C^-}(z^3+x)dydz-zdxdy=-\iint_{D_{xy}}-2\,dxdy=8\pi$

$$\therefore I=\iint_S(z^3+x)dydz-zdxdy=8\pi$$

08级物理类

五. 第一型曲线积分 $\because y=x$ 代入 $x^2+y^2+z^2=a^2$ $\quad 2x^2+z^2=a^2$ 令 $x=y=\frac{\sqrt2}{2}a\cos\theta$, $z=a\sin\theta$

$x'_\theta=-\frac{\sqrt2}{2}a\sin\theta$ $\quad y'_\theta=-\frac{\sqrt2}{2}a\sin\theta$ $\quad z'_\theta=a\cos\theta$

$$I=\int_0^{2\pi}a^2\sqrt{\tfrac12a^2\sin^2\theta+\tfrac12a^2\sin^2\theta+a^2\cos^2\theta}=\int_0^{2\pi}a^3d\theta=2a^3\pi.$$

六 $P=\frac{x}{(x^2+y^2+z^2)^{3/2}}$ $\quad Q=\frac{y}{(x^2+y^2+z^2)^{3/2}}$ $\quad R=\frac{z}{(x^2+y^2+z^2)^{3/2}}$

$P'_x=\frac{1}{(x^2+y^2+z^2)^{3/2}}-\frac{3x^2}{(x^2+y^2+z^2)^{5/2}}$ $\quad Q'_y=\frac{1}{(x^2+y^2+z^2)^{3/2}}-\frac{3y^2}{(x^2+y^2+z^2)^{5/2}}$ $\quad R'_z=\frac{1}{(x^2+y^2+z^2)^{3/2}}-\frac{3z^2}{(x^2+y^2+z^2)^{5/2}}$

$P'_x + Q'_y + R'_z = 0$ 有奇点 $(0,0,0)$ 挖洞 C $x^2+y^2+z^2=\varepsilon^2$ $(\varepsilon\to 0)$ 取内侧

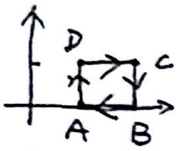
$\oiint_{\Sigma} + \oiint_{C} = 0$ (由高斯公式) $\oiint_{\Sigma} \dfrac{xdydz+ydzdx+zdxdy}{(x^2+y^2+z^2)^{3/2}} = -\oiint_{C} = \oiint_{C^-}$ (C^-取外侧)

$$I = \frac{1}{\varepsilon^3}\oiint_{C^-} \underline{xdydz+ydzdx+zdxdy} = \frac{3}{\varepsilon^3}\boxed{\iiint_V dxdydz} = \frac{3}{\varepsilon^3}\times\frac{4}{3}\pi\varepsilon^3 = 4\pi$$

再由高斯公式　　球体体积

09级物理类

三.



$I = \int_1^2 x^2dx + \int_0^1 (4-y^2)dy + \int_2^1 (x^2+1)dx + \int_1^0 (1-y^2)dy = 2$

或用格林公式 $\iint_D (2x-2y)\,dxdy = 2$

五. 由对称性. $\iint_S x^2ds = \iint_S y^2ds = \iint_S z^2ds$

$\because I = \frac{2}{3}\iint_S (x^2+y^2+z^2)ds = \frac{2}{3}\iint_S a^2ds = \frac{2}{3}a^2\boxed{\iint_S ds} = \frac{2}{3}a^2\times 4\pi a^2 = \frac{8}{3}\pi a^4$

球体表面积

六 $P = x+y+1$　$Q = y+z+1$　$R = z+x+1$　$P'_x + Q'_y + R'_z = 3$

曲面未封闭 补面 C: $\begin{cases} x^2+y^2\le 1 \\ z=0 \end{cases}$ 取下侧. $\iint_S + \iint_C = 3\boxed{\iiint_V dxdydz} = 3\times\frac{4}{3}\pi\times\frac{1}{2} = 2\pi$

半球体体积

$\iint_S = 2\pi - \iint_C = 2\pi + \iint_{C^-} (x+y+1)dydz + (y+z+1)dzdx + (z+x+1)dxdy$　C^- 取上侧

第二型曲面积分转二重积分 取正号　$\iint_{C^-} = \iint_{D_{xy}} (x+1)dxdy$　由对称性 $\iint_{D_{xy}} x\,dxdy = 0$

$\iint_{D_{xy}} dxdy = \pi$ 积分域面积　$\therefore \iint_S (x+y+1)dydz + (y+z+1)dzdx + (z+x+1)dxdy = 3\pi$

10级物理类

一. 5. $(x-\frac{a}{2})^2 + y^2 = (\frac{a}{2})^2$　令 $\begin{cases} x = \frac{a}{2} + \frac{a}{2}\cos\theta \\ y = \frac{a}{2}\sin\theta \end{cases}$　$x'_\theta = -\frac{a}{2}\sin\theta$　$y'_\theta = \frac{a}{2}\cos\theta$

$ds = \sqrt{\frac{a^2}{4}\sin^2\theta + \frac{a^2}{4}\cos^2\theta}\,d\theta = \frac{a}{2}d\theta$　$\int_L \sqrt{x^2+y^2}\,ds = \int_{-\pi}^{\pi} \frac{a}{\sqrt{2}}\sqrt{1+\cos\theta}\cdot\frac{a}{2}d\theta$

$\because \cos\theta = 2\cos^2\frac{\theta}{2} - 1$　$I = \int_{-\pi}^{\pi} \frac{a}{\sqrt{2}}\cdot\sqrt{2}\cos\frac{\theta}{2}\cdot\frac{a}{2}d\theta = 2a^2$　$-\pi\sim\pi$上积分

若取积分上下限 $0\sim 2\pi$　$\sqrt{2\cos^2\frac{\theta}{2}} = \sqrt{2}\left|\cos\frac{\theta}{2}\right|$ (讨论)

一. 6　$y = \pi - x$　$I = \int_0^{\pi}\sin(\pi-x)\,dx + \int_0^{\pi}\sin x\,d(\pi - x)$

$= \int_0^{\pi}\sin x\,dx - \int_0^{\pi}\sin x\,dx = 0$

二. 投影到 xOz 面上　仅看 $y>0$ 部分　$y'_z = 0$　$y'_x = \frac{-x}{\sqrt{R^2-x^2}}$

$\sqrt{1+(y'_x)^2+(y'_z)^2} = \frac{R}{\sqrt{R^2-x^2}}$　$I = 2\iint_{D_{xz}} (R^4+z^2)\frac{R}{\sqrt{R^2-x^2}}dxdz$

$$I = 2\int_{-R}^{R}\frac{R}{\sqrt{R^2-x^2}}dx\int_0^h (R^4+z^2)dz = 2\pi R^5h + \frac{2}{3}\pi Rh^3$$

五. $\oint_L \frac{(e^{x^2}-x^2y)dx+(xy^2-\sin y^2)dy}{x^2+y^2}$ L: $x^2+y^2=1$ ∴ $I=\oint_L (e^{x^2}-x^2y)dx+(xy^2-\sin y^2)dy$

$P=e^{x^2}-x^2y$ $Q=xy^2-\sin y^2$ $P'_y=-x^2$ $Q'_x=y^2$

由格林公式 $I=-\iint_{D_{xy}}(x^2+y^2)dxdy \xrightarrow{\text{极坐标}} =-\int_0^{2\pi}d\theta\int_0^1 r^3dr=-\frac{\pi}{2}$

六 有奇点挖洞 C: $x^2+y^2+z^2=\varepsilon^2\ (\varepsilon\to 0)$ 取内侧

$P=\frac{x}{(x^2+y^2+z^2)^{3/2}}$ $Q=\frac{y}{(x^2+y^2+z^2)^{3/2}}$ $R=\frac{z}{(x^2+y^2+z^2)^{3/2}}$ $P'_x+Q'_y+R'_z=0$

$\iint_S+\iint_C=0$ ∴ $\iint_S \frac{xdydz+ydzdx+zdxdy}{\sqrt{(x^2+y^2+z^2)^3}}=-\iint_C=\iint_{C^-}$ C^- 取外侧.

$\iint_{C^-}=\frac{1}{\varepsilon^3}\iint_{C^-}\underbrace{xdydz+ydzdx+zdxdy}_{\text{再用高斯公式}}=\frac{3}{\varepsilon^3}\underbrace{\boxed{\iiint dxdydz}}_{\text{球体体积}}=\frac{3}{\varepsilon^3}\times\frac{4}{3}\pi\varepsilon^3=4\pi$

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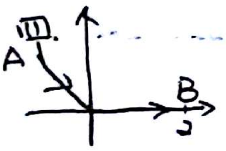
三.3. $\int_\Gamma(x^2+y^2+z^2)ds=\int_\Gamma(a^2+1)ds=(a^2+1)\underbrace{\boxed{\int_\Gamma ds}}_{\text{弧长}}=(a^2+1)2\pi a=2\pi(a^3+a)$

三.4 $\iint_\Sigma 4ds=4\boxed{\iint_\Sigma ds}$ $z=4-\frac{4}{3}y-2x$ $z'_x=-2$ $z'_y=-\frac{4}{3}$ $ds=\frac{\sqrt{61}}{3}dxdy$

∴ $I=4\times\frac{\sqrt{61}}{3}\underbrace{\boxed{\iint_{D_{xy}}dxdy}}_{\text{积分域面积}}=\frac{4}{3}\sqrt{61}\times\frac{1}{2}\times2\times3=4\sqrt{61}$

四.



$\int_{-1}^0(12x^3+e^{x^2})dx-(\cos x^2-xe^{x^2})d(x^2)+\int_0^2e^0dx$

$=\int_{-1}^0(12x^3-2x\cos x^2+e^{x^2}+2x^2e^{x^2})dx+2$

$=(3x^4-\sin x^2+xe^{x^2})\Big|_{-1}^0+2$

$=e+\sin 1-1$

五. 法一 $P=x$ $Q=y$ $R=z$ $P'_x+Q'_y+R'_z=3$ (注意 P.Q.R 对应关系)

补面 $C_1\begin{cases}x^2+y^2\le 1\\ z=0\end{cases}$ 取下侧 $C_2\begin{cases}x^2+y^2\le 1\\ z=3\end{cases}$ 取上侧

$\iint_\Sigma+\iint_{C_1}+\iint_{C_2}=3\underbrace{\boxed{\iiint dxdydz}}_{\text{体积}}=3\times\pi\times1^2\times3=9\pi$

$\iint_{C_1}zdxdy+ydzdx+xdydz=0$ $\iint_{C_2}zdxdy+ydzdx+xdydz=3\iint_{D_{xy}}dxdy=3\pi$

∴ $\iint_\Sigma zdxdy+ydzdx+xdydz=6\pi$

法二、分别投影到三个坐标面上 直接计算第二型曲面积分

① 投影到 xOy 面上 $\iint_{\Sigma} z\,dxdy = 0$ （投影域后一条闭合曲线）

② 投影到 xOz 面上 $y>0$ 时 $y=\sqrt{1-x^2}$ 第二型曲面积分转二重积分取正号

$y<0$ 时 $y=-\sqrt{1-x^2}$ 第二型曲面积分转二重积分取负号

$$\iint_{\Sigma} y\,dzdx = \underbrace{\int_0^3 dz \int_{-1}^{1} \sqrt{1-x^2}\,dx}_{y>0\text{部分}} + \underbrace{\left(-\int_0^3 dz \int_{-1}^{1} -\sqrt{1-x^2}\,dx\right)}_{y<0\text{部分}}$$

$$= 2\int_0^3 dz \int_{-1}^{1} \sqrt{1-x^2}\,dx = 3\pi$$

③ 投影到 yOz 面上 $x>0$ 时 $x=\sqrt{1-y^2}$ 第二型曲面积分转二重积分取正号

$x<0$ 时 $x=-\sqrt{1-y^2}$ 第二型曲面积分转二重积分取负号

$$\iint_{\Sigma} x\,dydz = \underbrace{\int_0^3 dz \int_{-1}^{1} \sqrt{1-y^2}\,dy}_{x>0\text{部分}} + \underbrace{\left(-\int_0^3 dz \int_{-1}^{1} -\sqrt{1-y^2}\,dy\right)}_{x<0\text{部分}}$$

$$= 2\int_0^3 dz \int_{-1}^{1} \sqrt{1-y^2}\,dy = 3\pi$$

$\therefore I = 0 + 3\pi + 3\pi = 6\pi$

12级物理类

八.(1) $z>0$ 部分 $z=\sqrt{R^2-x^2-y^2}$ $z'_x = \dfrac{-x}{\sqrt{R^2-x^2-y^2}}$ $z'_y = \dfrac{-y}{\sqrt{R^2-x^2-y^2}}$

$\sqrt{1+z'^2_x+z'^2_y} = \dfrac{R}{\sqrt{R^2-x^2-y^2}}$ 同理 $z<0$ 亦可得 $\sqrt{1+z'^2_x+z'^2_y} = \dfrac{R}{\sqrt{R^2-x^2-y^2}}$

$\therefore I = 2\iint_{D_{xy}} (R^2-x^2-y^2)\dfrac{R}{\sqrt{R^2-x^2-y^2}}\,dxdy = 2\int_0^{2\pi} d\theta \int_0^R (R^2-r^2)\dfrac{R}{\sqrt{R^2-r^2}}\cdot r\,dr$

$= -2\pi R\times\dfrac{2}{3}(R^2-r^2)^{3/2}\Big|_0^R = \dfrac{4}{3}\pi R^4$

另解：由对称性 $\iint_{\Sigma} x^2 ds = \iint_{\Sigma} y^2 ds = \iint_{\Sigma} z^2 ds$

$$\therefore I = \frac{1}{3}\iint_{\Sigma} \underbrace{(x^2+y^2+z^2)}_{=R^2} ds = \frac{1}{3}R^2 \overbrace{\boxed{\iint_{\Sigma} ds}}^{\text{球表面积}} = \frac{1}{3}R^2\times 4\pi R^2 = \frac{4}{3}\pi R^4$$

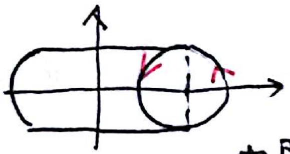
八(2) $P=(y-z)x$ $Q=0$ $R=(x-y)$ $P'_x+Q'_y+R'_z = y-z$

$$\oiint_{\Sigma} (x-y)dxdy + (y-z)x\,dydz = \iiint (y-z)\,dxdydz$$

$\because$ 柱面关于 xOz 面对称 $\iiint y\,dxdydz = 0$.

$I = -\iiint z\,dxdydz$ 柱坐标得 $-\int_0^{2\pi} d\theta \int_0^1 r\,dr \int_0^3 z\,dz = -\dfrac{9}{2}\pi$

九. (1) L: 圆心 $(2,0)$ $r=1$ 未包含原点



设左半圆 L_1 右半圆 L_2.

补一段曲线 L_3 使 L_1与L_3围成正向曲线 C_1 L_2与L_3围成正向曲线 C_2.

由题干条件可知 $\oint_{C_1} \frac{2xy\,dx+\varphi(x)\,dy}{x^4+y^2} = \oint_{C_2} \frac{2xy\,dx+\varphi(x)\,dy}{x^4+y^2}$.

$$\oint_L \frac{2xy\,dx+\varphi(x)\,dy}{x^4+y^2} = \oint_{C_2} - \oint_{C_1} = 0$$

(2) 仿照 (1) 证明已得 在任意不围绕原点的简单闭曲线 L 上 $\oint_L \frac{2xy\,dx+\varphi(x)\,dy}{x^4+y^2} = 0$

$$\because \frac{\partial}{\partial x}\left(\frac{\varphi(x)}{x^4+y^2}\right) = \frac{\partial}{\partial y}\left(\frac{2xy}{x^4+y^2}\right) \qquad \begin{cases} x^4\varphi'(x) - 4x^3\varphi(x) = 2x^5 \\ \varphi'(x)y^2 = -2xy^2 \end{cases}$$

$\therefore \varphi(x) = -x^2$

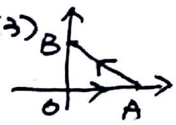
(3) 令 C: 由 $x=\pm1$ $y=\pm1$ 围成 $I = \oint_C \frac{2xy\,dx - x^2\,dy}{x^4+y^2}$

$$I = \int_{-1}^{1} \frac{-2x}{x^4+1}dx + \cancel{\int_{-1}^{1} \frac{-1}{1+y^2}dy} + \int_{1}^{-1} \frac{2x}{x^4+1}dx + \cancel{\int_{1}^{-1} \frac{-1}{1+y^2}dy}$$

$$= 2\int_{-1}^{1} \frac{-2x}{x^4+1}dx = -2\int_{-1}^{1} \frac{1}{x^4+1}d(x^2) = -2\arctan x^2 \Big|_{-1}^{1} = 0$$

13级物理类

四 (3)



OA: $y=0$
AB: $y=1-x$

$$I = \int_0^1 x\,dx + \int_0^1 (x+1-x)\cdot\sqrt{2}\,dx$$
$$= \frac{1}{2} + \sqrt{2}$$

第一型曲线积分! 不是第二型

四 (4) 投影到 xOy 面 $z=\sqrt{R^2-x^2-y^2}$ $z'_x = \frac{-x}{\sqrt{R^2-x^2-y^2}}$ $z'_y = \frac{-y}{\sqrt{R^2-x^2-y^2}}$

$$\therefore \sqrt{1+z_x'^2+z_y'^2} = \frac{R}{\sqrt{R^2-x^2-y^2}} \qquad I = \iint\limits_{0\le x^2+y^2\le R^2-h^2} \left(x+y+\sqrt{R^2-x^2-y^2}\right)\frac{R}{\sqrt{R^2-x^2-y^2}}\,dxdy$$

积分域关于x轴 y轴对称 $\iint\limits_{D_{xy}} x\frac{R}{\sqrt{R^2-x^2-y^2}}\,dxdy = \iint\limits_{D_{xy}} y\cdot\frac{R}{\sqrt{R^2-x^2-y^2}} = 0$

$$\therefore I = \iint\limits_{D_{xy}} R\,dxdy = R\cdot\underbrace{\iint\limits_{D_{xy}} dxdy}_{\text{积分域面积}} = R\cdot\pi(R^2-h^2)$$

四 (5) 斯托克斯公式 习题12.3 (1) 原题

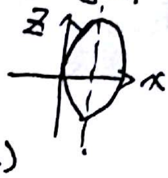
$P=y$ $Q=z$ $R=x$ $I = \iint\limits_S \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{vmatrix} dydz + \begin{vmatrix} \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\ R & P \end{vmatrix} dzdx + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} dxdy$

$I = \iint\limits_S -dydz - dzdx - dxdy$ (S为Γ在平面 $x+y+z=0$ 上所围圆, 取上侧)

法向量 $(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3})$ $I = \iint\limits_S \{-1,-1,-1\}\cdot\{\frac{1}{\sqrt3},\frac{1}{\sqrt3},\frac{1}{\sqrt3}\}\,dS = -\sqrt3\iint\limits_S dS = -\sqrt3\pi a^2$

七. $P=x^2$ $Q=y^2$ $R=z^2$ 并不是一个封闭曲面 补面C $\begin{cases}(x-1)^2+\frac{z^2}{4}\le 1\\ y=1\end{cases}$ 取左侧

$\iint_\Sigma + \iint_C = \iiint_\Omega 2(x+y+z)\,dxdydz$ 将Ω投影到xoz面



∵ $\iiint_\Omega 2(x-1)\,dxdydz=0$ $\iiint_\Omega 2z\,dxdydz=0$ (由对称性)

∴ $\iiint_\Omega (2x+2y+2z)\,dxdydz = \iiint_\Omega (2+2y)\,dxdydz$

$\iiint_\Omega (2+2y)\,dxdydz$ 采用广义柱坐标换元 令 $x=r\cos\theta+1$, $z=2r\sin\theta$

$= \int_0^{2\pi}d\theta\int_0^1 dr\int_1^{1+\sqrt{1-r^2}}(2+2y)\cdot 2r\,dy = \frac{19}{3}\pi.$

$\iint_C x^2dydz+y^2dzdx+z^2dxdy = \iint_C 0+y^2dzdx+0 = -\iint_{D_{xz}} dzdx = -2\pi$

(第二型曲面积分转换为二重积分时 由于C取左侧 带负号)

∴ $\iint_\Sigma x^2dydz+y^2dzdx+z^2dxdy = \frac{19}{3}\pi-(-2\pi) = \frac{25}{3}\pi.$

八. $F'_x = zF_1'+F_2'$ $F'_y = -F_1'-zF_2'$ $F'_z = xF_1'-yF_2'$

$z'_x = \frac{zF_1'+F_2'}{yF_2'-xF_1'}$. $z'_y = \frac{-F_1'-zF_2'}{yF_2'-xF_1'}$.

$P=-2xz-yz^2$ $Q=xz^2+2yz$.

$P'_y = \frac{(xz^2+2yz+2x)F_1'+(2xz+yz^2)F_2'}{yF_2'-xF_1'}$

$Q'_x = \frac{(xz^2+2yz)F_1'+(2xz+yz^2+2y)F_2'}{yF_2'-xF_1'}$

$Q'_x-P'_y = 2$

由格林公式 $I=\iint_D 2\,dxdy = 2\iint_D dxdy$. 令 $x=ar\cos\theta$, $y=br\sin\theta$.

$I=2\int_0^{2\pi}d\theta\int_0^1 abr\,dr = 2\pi ab.$

椭圆 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 面积：πab