第11.2节

全微分

可微的定义

一元函数$y = f(x)$在x_0处可微:

$$\Delta y = f(x_0 + \Delta x) - f(x_0) = A(x_0)\Delta x + o(|\Delta x|).$$

类似于一元函数我们问何时有

$$\Delta z = A\Delta x + B\Delta y + o(\sqrt{|\Delta x|^2 + |\Delta y|^2}).$$

Definition

设$f(x,y)$为区域D上的函数, 如果存在实数A和B使得

$$\lim_{\rho\to 0}\frac{1}{\rho}\left[f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)-A\Delta x-B\Delta y\right]=0,$$

其中$\rho=\sqrt{\Delta x^2+\Delta y^2}$, 即

$$f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)=A\Delta x+B\Delta y+o(\rho)\ (\rho\to 0).$$

则称f在(x_0,y_0)**可微**.

定义中有序数对(A,B)被称为$f(x,y)$在点(x_0,y_0)处的梯度，记为$\mathrm{grad}\, f(x_0,y_0)=\nabla f(x_0,y_0)$. 我们称线性函数$A\Delta x+B\Delta y$为$f(x,y)$在点$(x_0,y_0)$处的**全微分**，记

$$\mathrm{d}f(x_0,y_0)=A\Delta x+B\Delta y=\langle \nabla f(x_0,y_0),\Delta X\rangle.$$

若记$\Delta x=\mathrm{d}x$，$\Delta y=\mathrm{d}y$，即$\Delta X=\mathrm{d}X=(\mathrm{d}x,\mathrm{d}y)$，从而

$$\mathrm{d}f(x_0,y_0)=A\mathrm{d}x+B\mathrm{d}y.$$

设$X_0 = (x_{01}, \cdots, x_{0n}) \in D$. 如果存在$L = (A_1, \cdots, A_n) \in \mathbb{R}^n$, 使得

$$f(X_0 + \Delta X) - f(X_0) = \langle L, \Delta X \rangle + o(|\Delta X|) \ \ (|\Delta X| \to 0),$$

其中$\Delta X = (\Delta x_1, \cdots, \Delta x_n) \in \mathbb{R}^n, X_0 + \Delta X \in D$, 则称$f$在$X_0$可微, L称为f在X_0的梯度, 记为$\nabla f(X_0)$或$\text{grad} f(X_0)$. 通常称$\sum_{i=1}^n A_i \Delta x_i$ 为$f(X)$在X_0的全微分. 与二元函数相同写$\Delta x_i = \mathrm{d}x_i$, 则

$$\mathrm{d}f(X_0) = \sum_{i=1}^n A_i \,\mathrm{d}x_i = \langle \nabla f(X_0), \Delta X \rangle = \langle \nabla f(X_0), \mathrm{d}X \rangle.$$

$$f(X_0+\Delta X)-f(X_0)=\langle L,\Delta X\rangle+o(|\Delta X|)\ \ (|\Delta X|\to 0).$$

$$f(X_0+\Delta X)-f(X_0)=\langle L,\Delta X\rangle+o(|\Delta X|)\ \ (|\Delta X|\to 0).$$

$\Rightarrow L=\nabla f=(\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n}).$

Example

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0.\end{cases}$$

$$f'_x(x,y)=\begin{cases}\dfrac{y^3}{(\sqrt{x^2+y^2})^3}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0,\end{cases}$$

$$f'_y(x,y)=\begin{cases}\dfrac{x^3}{(\sqrt{x^2+y^2})^3}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0,\end{cases}$$

所有偏导数处处存在且有界.

但$f(x,y)$在$(0,0)$点不可微.

事实上, 若可微,由于$f'_x(0,0)=f'_y(0,0)=0$, 从而$\frac{xy}{\sqrt{x^2+y^2}}=o\left(\sqrt{x^2+y^2}\right)$, 即

$$\lim_{x^2+y^2\to 0}\frac{xy}{x^2+y^2}=0,$$

但$\lim_{x^2+y^2\to 0}\frac{xy}{x^2+y^2}$不存在, 矛盾!

Theorem

设$f(X)$在X_0的某邻域内所有偏导数存在且都在X_0连续,
则$f(X)$在X_0可微.

证明 偏导数存在且在(x_0, y_0)连续, 则

$$
\begin{aligned}
\Delta f &= f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0) \\
&= f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0+\Delta y) \\
&\quad + f(x_0, y_0+\Delta y) - f(x_0, y_0) \\
&= f'_x(x_0+\theta_1\Delta x, y_0+\Delta y)\Delta x + f'_y(x_0, y_0+\theta_2\Delta y)\Delta y.
\end{aligned}
$$

其中$0<\theta_1<1,\ 0<\theta_2<1$.

$$
\begin{aligned}
&\left|\frac{\Delta f - f'_x(x_0, y_0)\Delta x - f'_y(x_0, y_0)\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}}\right| \\
&\leq \left|f'_x(x_0 + \theta_1 \Delta x, y_0 + \Delta y) - f'_x(x_0, y_0)\right| \frac{|\Delta x|}{\sqrt{\Delta x^2 + \Delta y^2}} \\
&+ \left|f'_y(x_0, y_0 + \theta_2 \Delta y) - f'_y(x_0, y_0)\right| \frac{|\Delta y|}{\sqrt{\Delta x^2 + \Delta y^2}} \\
&\leq \left|f'_x(x_0 + \theta_1 \Delta x, y_0 + \Delta y) - f'_x(x_0, y_0)\right| \\
&+ \left|f'_y(x_0, y_0 + \theta_2 \Delta y) - f'_y(x_0, y_0)\right| \\
&\to 0.
\end{aligned}
$$

$$\text{函数} f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0,\end{cases}$$

在$(0,0)$点邻近偏导数无界，但在$(0,0)$点可微.

连续　　偏导数存在

可微

偏导数连续

X自变量: 设$f(X)$在$\Omega\subseteq\mathbb{R}^n$内处处可微, 则

$$\mathrm{d}f(X)=\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(X)\,\mathrm{d}x_i,\quad X\in\Omega.$$

X中间变量: 记$g(T)=f(x_1(t_1,\cdots,t_k),\cdots,x_n(t_1,\cdots,t_k))$,

$$\mathrm{d}g(T)=\sum_{j=1}^{k}\frac{\partial g}{\partial t_j}(T)\,\mathrm{d}t_j.$$

由链式法则知

$$\frac{\partial g}{\partial t_j}(T)=\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(X(T))\frac{\partial x_i}{\partial t_j}(T),\quad j=1,\cdots,k.$$

$$
\begin{aligned}
\mathrm{d}g(T) &= \sum_{j=1}^{k}\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(X(T))\frac{\partial x_i}{\partial t_j}(T)\,\mathrm{d}t_j \\
&= \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(X(T))\sum_{j=1}^{k}\frac{\partial x_i}{\partial t_j}(T)\,\mathrm{d}t_j \\
&= \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(X(T))\,\mathrm{d}x_i(T).
\end{aligned}
$$

由此可知，不论X是自变量还是中间变量,函数$f(X)$的全微分都可统一表示. 这称为一阶全微分的形式不变性.

设$u(X), v(X)$是可微函数, λ为常数, 则

(1) $\mathrm{d}(u(X) \pm v(X)) = \mathrm{d}u(X) \pm \mathrm{d}v(X)$,

(2) $\mathrm{d}(\lambda u(X)) = \lambda \mathrm{d}u(X)$,

(3) $\mathrm{d}(u(X)v(X)) = u(X)\mathrm{d}v(X) + v(X)\mathrm{d}u(X)$,

(4) $\mathrm{d}\left(\dfrac{u(X)}{v(X)}\right) = \dfrac{v(X)\mathrm{d}u(X) - u(X)\mathrm{d}v(X)}{v^2(X)}$.

$$\mathrm{d}f(X)=\sum_{i=1}^{n} f'_{x_i}(X)\Delta x_i$$ 中，$\Delta x_1,\cdots,\Delta x_n$ 不依赖 X 的变量，而 $f'_{x_1}(X),\cdots,f'_{x_n}(X)$ 是 X 的函数. 如果它们在 X_0 都可微，称

$$\mathrm{d}(\mathrm{d}f)(X_0)=\mathrm{d}f'_{x_1}(X_0)\Delta x_1+\cdots+\mathrm{d}f'_{x_n}(X_0)\Delta x_n$$

为函数 $f(X)$ 在 X_0 处的二阶全微分，记为 $\mathrm{d}^2f(X_0)$，即

$$\begin{aligned}\mathrm{d}^2f(X_0) &= \mathrm{d}f'_{x_1}(X_0)\mathrm{d}x_1+\cdots+\mathrm{d}f'_{x_n}(X_0)\mathrm{d}x_n\\ &= \sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial x_i\partial x_j}(X_0)\mathrm{d}x_i\mathrm{d}x_j.\end{aligned}$$

$$H_f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

称为f在X_0的黑塞(Hesse)矩阵.

注：二阶可微 $\Rightarrow$ 对称矩阵.

令 $\varphi = \frac{1}{\Delta x\,\Delta y}[f(x_0+\Delta x, y_0+\Delta y) - f(x_0+\Delta x, y_0) - f(x_0, y_0+\Delta y) + f(x_0, y_0)]$.
$H(y) = f(x_0+\Delta x, y) - f(x_0, y)$.

$$\varphi = \frac{1}{\Delta x\,\Delta y}[H(y_0+\Delta y) - H(y_0)] = \frac{1}{\Delta x}H'(y_0+\theta_1\Delta y)$$
$$= \frac{1}{\Delta x}[f'_y(x_0+\Delta x, y_0+\theta_1\Delta y) - f'_y(x_0, y_0+\theta_1\Delta y)].$$

$$第1项 = f'_y(x_0, y_0) + f''_{yx}\Delta x + f''_{yy}\theta_1\Delta y + o(\sqrt{(\Delta x)^2 + (\theta_1\Delta y)^2})$$

$$第2项 = f'_y(x_0, y_0) + f''_{yy}\theta_1\Delta y + o(\theta_1|\Delta y|)$$

$$\varphi = f''_{yx}(x_0, y_0) + \frac{o(\sqrt{(\Delta x)^2 + (\theta_1\Delta y)^2})}{\Delta x} - \frac{o(\theta_1|\Delta y|)}{\Delta x}$$

$$\varphi = f''_{yx}(x_0, y_0) + \frac{o(\sqrt{(\Delta x)^2 + (\theta_1 \Delta y)^2})}{\Delta x} - \frac{o(\theta_1 |\Delta y|)}{\Delta x}$$

同理 $\varphi = f''_{xy}(x_0, y_0) + \dfrac{o(\sqrt{(\theta_2 \Delta x)^2 + (\Delta y)^2})}{\Delta y} - \dfrac{o(\theta_2 |\Delta x|)}{\Delta y}$

在上两式中上$\Delta x = \Delta y \to 0$得到

$$f''_{yx}(x_0, y_0) = \lim_{\Delta x = \Delta y \to 0} \varphi = f''_{xy}(x_0, y_0).$$

如果函数$f(X)$三次可微，则其三阶全微分为

$$\mathrm{d}^3 f(X) = \sum_{i,j,k=1}^{n} \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} \mathrm{d}x_i \mathrm{d}x_j \mathrm{d}x_k.$$

一般地，若函数$f(X)$ k次可微，则其k阶全微分为

$$\mathrm{d}^k f(X) = \mathrm{d}(\mathrm{d}^{k-1} f)(X) = \sum_{i_1,\cdots,i_k=1}^{n} \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}} \mathrm{d}x_{i_1} \cdots \mathrm{d}x_{i_k}.$$

记$I = (i_1, \cdots, i_k)$，$\partial^k x_I = \partial x_{i_1} \cdots \partial x_{i_k}$，$\mathrm{d}^k x_I = \mathrm{d}x_{i_1} \cdots \mathrm{d}x_{i_k}$，则

$$\mathrm{d}^k f(X) = \sum_{I} \frac{\partial^k f}{\partial^k x_I} \mathrm{d}^k x_I.$$

设$F:\ D\subseteq\mathbb{R}^n\to\mathbb{R}^m$表示为

$$F(X)=\begin{pmatrix} f_1(X)\\ \vdots \\ f_m(X)\end{pmatrix},\quad X=(x_1,\cdots,x_n)\in D.$$

$$\begin{aligned} F'_{x_i}(X_0)=\frac{\partial F}{\partial x_i}(X_0) &= \lim_{\Delta x_i\to 0}\frac{F(X_0+\Delta x_i\cdot\vec{e}_i)-F(X_0)}{\Delta x_i},\\ &= \begin{pmatrix} \dfrac{\partial f_1}{\partial x_i}(X_0)\\ \vdots \\ \dfrac{\partial f_m}{\partial x_i}(X_0)\end{pmatrix}. \end{aligned}$$

如果存在矩阵$L=(a_{ij})_{m\times n}$使得

$$\lim_{|\Delta X|\to 0}\frac{1}{|\Delta X|}\left|F(X_0+\Delta X)-F(X_0)-L\Delta X^T\right|=0,$$

或者

$$F(X_0+\Delta X)-F(X_0)=L\Delta X^T+o(|\Delta X|).$$

则称F在X_0可微, L仅依赖于F及X_0, 不依赖于ΔX. 称L为向量值函数F在X_0的雅可比(Jacobi)矩阵,记为$J_F(X_0)$.

称线性部分$L\Delta X^T$为映射F的全微分, 记为$\mathrm{d}F(X_0)$.

Theorem

设$F: D \subseteq \mathbb{R}^n \to \mathbb{R}^m$在$X_0$可微 $\Leftrightarrow$ 函数$f_1(X), \cdots, f_m(X)$都在X_0可微, 且当F在X_0可微时有

$$\mathrm{d}F(X_0) = J_F(X_0)\Delta X^T,$$

其中F的雅可比矩阵$J_F(X)$为$m \times n$的函数矩阵

$$J_F(X) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ & \cdots & \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{pmatrix}(X) \equiv \frac{\partial(f_1, \cdots, f_m)}{\partial(x_1, \cdots, x_n)}.$$

证明 $\Rightarrow$ 设F在X_0可微，从而存在唯一的$m \times n$矩阵$L = (a_{ij})_{m\times n}$使得

$$F(X_0 + \Delta X) - F(X_0) = L\Delta X^T + o(|\Delta X|).$$

即对于任意$i = 1, \cdots, m$,都有

$$\lim_{|\Delta X|\to 0} \frac{1}{|\Delta X|}\left[f_i(X_0 + \Delta X) - f_i(X_0) - \sum_{j=1}^{n} a_{ij}\Delta x_j\right] = 0.$$

即知函数$f_i, i = 1, 2, \cdots m$在X_0可微，且$a_{ij} = \dfrac{\partial f_i}{\partial x_j}(X_0)$，即

$$L = J_F(X_0).$$

证明 $\Leftarrow$ 设$f_1, \cdots, f_m$都是在X_0可微的函数，则

$$\lim_{|\Delta X| \to 0} \frac{1}{|\Delta X|} \left[f_i(X_0 + \Delta X) - f_i(X_0) - \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(X_0) \Delta x_j \right] = 0.$$

由此可得

$$\lim_{|\Delta X| \to 0} \frac{1}{|\Delta X|} |F(X_0 + \Delta X) - F(X_0) - J_F(X_0) \Delta X^T| = 0.$$

于是F在X_0可微且

$$\mathrm{d}F(X_0) = J_F(X_0) \Delta X^T.$$

$$J_F(X)=\begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n}\\ & \cdots & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n}\end{pmatrix}(X)\equiv\frac{\partial(f_1,\cdots,f_m)}{\partial(x_1,\cdots,x_n)}.$$

$$J_F(X)=\nabla F(X)=\begin{pmatrix}\nabla f_1(X)\\ \vdots\\ \nabla f_m(X)\end{pmatrix}=\left(\frac{\partial F}{\partial x_1},\cdots,\frac{\partial F}{\partial x_n}\right)(X).$$

$$\mathrm{d}F(X)=\nabla F(X)\mathrm{d}X^T.$$

Theorem (链式法则)

设$G:\Omega\subseteq\mathbb{R}^l\to\mathbb{R}^n$, $F:D\subseteq\mathbb{R}^n\to\mathbb{R}^m$, 记$U=F\circ G$, 即$U(X)=F(G(X)),\forall X\in\Omega$. 对于$X_0\in\Omega$, $Y_0=G(X_0)\in D$, 如果F在Y_0可微, G在X_0可微, 则U在X_0可微且

$$\mathrm{d}U(X_0)=J_F(Y_0)dG(X_0)=J_F(Y_0)J_G(X_0)\Delta X^T. \tag{1}$$

从而有

$$J_U(X_0)=J_F(Y_0)J_G(X_0),\ \text{或}\ \nabla U(X_0)=\nabla F(Y_0)\nabla G(X_0).$$

证明 记$\Delta G = G(X_0 + \Delta X) - G(X_0)$, 由于$G$在$X_0$可微, 则

$$\Delta G = \mathrm{d}G(X_0) + \rho_1 = J_G(X_0)\Delta X^T + \rho_1, \tag{2}$$

其中$\rho_1 = \rho_1(X_0, \Delta X)$且$\lim_{|\Delta X|\to 0} \dfrac{|\rho_1|}{|\Delta X|} = 0$. 由于$J_G(X_0) : \mathbb{R}^l \to \mathbb{R}^n$线性映射, 从而存在$M > 0$使得

$$|J_G(X_0)\Delta X^T| \leqslant M|\Delta X|, \ \forall \Delta X \in \mathbb{R}^l.$$

于是由$\lim_{|\Delta X|\to 0} \dfrac{|\rho_1|}{|\Delta X|} = 0$, 可知存在$\delta > 0$, 使得当$|\Delta X| < \delta$时

$$|\Delta G| \leqslant 2M\,|\Delta X|. \tag{3}$$

$$\begin{aligned}\Delta U &= U(X_0+\Delta X)-U(X_0)\\ &= F(G(X_0+\Delta X))-F(G(X_0))\\ &= F(Y_0+\Delta G)-F(Y_0),\end{aligned}$$

又F在Y_0可微, 并由(2),可得

$$\begin{aligned}\Delta U &= J_F(Y_0)\Delta G+\rho_2\\ &= J_F(Y_0)\mathrm{d}G(X_0)+J_F(Y_0)\rho_1+\rho_2,\end{aligned}$$

其中$\rho_2=\rho_2(Y_0,\Delta G)$且$\lim_{|\Delta G|\to 0}\dfrac{|\rho_2|}{|\Delta G|}=0.$

由于$J_F(Y_0):\mathbb{R}^n\to\mathbb{R}^m$线性映射, 从而存在$N>0$,使得

$$|J_F(Y_0)\rho_1|\leqslant N\,|\rho_1|.$$

从而$\lim_{|\Delta X|\to 0}\dfrac{|J_F(Y_0)\rho_1|}{|\Delta X|}=0$. 由(3)可得

$$\frac{|\rho_2|}{|\Delta X|}=\frac{|\rho_2|}{|\Delta G|}\frac{|\Delta G|}{|\Delta X|}\leqslant 2M\frac{|\rho_2|}{|\Delta G|},$$

又当$|\Delta X|\to 0$ 时, $|\Delta G|\to 0$, 所以$\lim_{|\Delta X|\to 0}\dfrac{|\rho_2|}{|\Delta X|}=0$. 于是

$$J_F(Y_0)\rho_1+\rho_2=o(|\Delta X|)\ \ (|\Delta X|\to 0).$$

故U在X_0可微且(1)式成立.

设$f:\mathbb{R}^n\to\mathbb{R}$, $G:\mathbb{R}^l\to\mathbb{R}^n$, 令$U(X)=f(G(X))$,则

$$G(X)=\begin{pmatrix} y_1(x_1,\cdots,x_l)\\ \vdots \\ y_n(x_1,\cdots,x_l)\end{pmatrix},\quad (x_1,\cdots,x_l)\in\mathbb{R}^l.$$

$$\mathrm{d}U(X_0)=J_f(Y_0)dG(X_0)=J_f(Y_0)J_G(X_0)\Delta X^T.$$

$$\Rightarrow \frac{\partial U}{\partial x_i}(X_0)=\sum_{j=1}^n\frac{\partial f}{\partial y_j}(Y_0)\frac{\partial y_j}{\partial x_i}(X_0),\ i=1,\cdots,l,$$

记向量$Z=(X,Y)$, $U(T)=F(X(T),Y(T))$, 则

$$J_F(Z)=\begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n}\\ & \cdots & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n}\end{pmatrix}=\left(\frac{\partial F}{\partial X},\frac{\partial F}{\partial Y}\right).$$

设$F(Z)$,$X(T)$,$Y(T)$都可微，则$U(T)$可微且有链式法则:

$$\frac{\partial U}{\partial T}=\frac{\partial F}{\partial X}\frac{\partial X}{\partial T}+\frac{\partial F}{\partial Y}\frac{\partial Y}{\partial T}.$$

$$J_F(Z)J_G(T)=\begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n}\\ & \cdots & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n}\end{pmatrix}\begin{pmatrix}\frac{\partial x_1}{\partial t_1} & \cdots & \frac{\partial x_1}{\partial t_l}\\ & \cdots & \\ \frac{\partial x_n}{\partial t_1} & \cdots & \frac{\partial x_n}{\partial t_l}\end{pmatrix}.$$

定义向量值函数$F : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$的$k$阶可微性. 设$F(X)$的分量为$f_1, \cdots, f_m$, 则

$$\mathrm{d}^k F(X) = \begin{pmatrix} \mathrm{d}^k f_1(X) \\ \vdots \\ \mathrm{d}^k f_m(X) \end{pmatrix}.$$

- $C^0(D,\mathbb{R}^m)$: D到$\mathbb{R}^m$的连续向量值函数.
- $C^r(D,\mathbb{R}^m)$: D到$\mathbb{R}^m$的r阶连续可微映射; 如果$F, \mathrm{d}F, \cdots, \mathrm{d}^r F$均为$x \in D$的连续映射.
- $C^0(D,\mathbb{R}^1)$, $C^r(D,\mathbb{R}^1)$分别简记为$C^0(D), C^r(D)$.
- 如果F在D内r阶可微, 则$F \in C^{r-1}(D,\mathbb{R}^m)$.
- $F \in C^r(D,\mathbb{R}^m)$等价于$F$的分量函数$f_1, \cdots, f_m$都属于$C^r(D)$.
- 函数$f_k \in C^r(D)$等价于f_k及其直到r阶所有偏导数都在D内连续(零阶导数约定为函数本身).

练习1

设$f(X)$是$\mathbb{R}^n$上两次连续可微的函数，$X_0 \in \mathbb{R}^n$, $Y \in \mathbb{R}^n$, 令$\varphi_Y(t) = f(X_0 + tY)$ $(t \in \mathbb{R})$, 求证: 如果对任意$Y \neq O$, 都有$\varphi''_Y(0) > 0$, 那么黑塞矩阵$H_f(X_0)$正定.

练习2

设$f(x,y)=\begin{cases} |x|^\alpha|y|^\beta\sin\dfrac{1}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0,\end{cases}$ 其中$\alpha>0,\beta>0$. 讨论$f(x,y)$在$(0,0)$点的可微性及偏导数的连续性.

设D是$\mathbb{R}^n$中的一个开集, $F : D \to \mathbb{R}^m$是一个映射, 对于$X_0 \in D$, F在X_0可微, 求证: 存在X_0的邻域U和正数λ, 使得

$$|F(X) - F(X_0)| \leqslant \lambda |X - X_0|, \quad \forall X \in U.$$

设$f(x,y)$在$\mathbb{R}^2$上连续可微且对任意实数x, y, 都有

$$f(x,y) = a\frac{\partial f}{\partial x}(x,y) + b\frac{\partial f}{\partial y}(x,y),$$

其中a, b是常数, 求证: 如果$f(x,y)$在$\mathbb{R}^2$上有界，那么$f(x,y)$在$\mathbb{R}^2$上恒等于0.

设$F:\mathbb{R}^n\to\mathbb{R}^n$为$C^1$映射，$J_F(X)$处处非奇且

$$\lim_{|X|\to+\infty}|F(X)|=+\infty,$$

求证：对任意$Y_0\in\mathbb{R}^n$，存在$X_0\in\mathbb{R}^n$使得$F(X_0)=Y_0$.

设$F:\mathbb{R}^n\to\mathbb{R}^n$为$C^1$映射且存在$\lambda>0$使得

$$|F(X)-F(Y)|\geqslant\lambda|X-Y|,\quad\forall X,Y\in\mathbb{R}^n.$$

求证：对任意$Y_0\in\mathbb{R}^n$, 存在唯一的$X_0\in\mathbb{R}^n$使得

$$F(X_0)=Y_0.$$