Frobenius numbers, Sylvester numbers and sums associated with number of solutions

Takao Komatsu
Department of Mathematical Sciences, School of Science
Zhejiang Sci-Tech University
Hangzhou 310018 China
komatsu@zstu.edu.cn

MR Subject Classifications: Primary 11D07; Secondary 05A15, 05A17, 05A19, 11B68, 11D04, 11P81

Abstract

Let $a_1, a_2, \dots, a_k$ be positive integers with $\gcd(a_1, a_2, \dots, a_k) = 1$. Frobenius number is the largest positive integer that is NOT representable in terms of $a_1, a_2, \dots, a_k$. When $k \ge 3$, there is no explicit formula in general, but some formulae may exist for special sequences $a_1, a_2, \dots, a_k$, including, those forming arithmetic progressions and their modifications. In this paper, for general $k \ge 2$, we give formulae which the greatest integer, the total number and the sum of integers where the number of representable positive integers is less than or equal to a fixed positive integer. We also show general formulae for the power and weighted sum of the number of representable positive integers is less than or equal to a fixed positive integer. As an application, we show explicit expressions of these values for one triple set of triangular numbers.

Keywords: Frobenius problem, Frobenius numbers, Sylvester numbers, Sylvester sums, power sums, weighted sums

1 Introduction

The *Coin Exchange Problem* (or Postage Stamp Problem / Chicken McNugget Problem) has a long history and is one of the problems that has attracted many people as well as experts. Given positive integers $a_1, \dots, a_k$ with $\gcd(a_1, \dots, a_k) = 1$, it is well-known that all sufficiently large n can be represented as a nonnegative integer combination of $a_1, \dots, a_k$. Nowadays, it is most known as the *linear Diophantine problem of Frobenius*, which is to determine the largest positive integer that is NOT representable as a nonnegative integer combination of given positive integers that are coprime (see [21] for general references). This number is denoted by $g(a_1, \dots, a_k)$ and often called *Frobenius number*.

Let $n(a_1, \dots, a_k)$ be the number of positive integers with no nonnegative integer representation by $a_1, \dots, a_k$. It is sometimes called *Sylvester number*.

According to Sylvester, for positive integers a and b with $\gcd(a, b) = 1$,

$$g(a,b) = (a-1)(b-1) - 1 \quad [29]\,,$$
$$n(a,b) = \frac{1}{2}(a-1)(b-1) \quad [28]\,.$$

There are many kinds of problems related to the Frobenius problem. The problems for the number of solutions (e.g., [30]), and the sum of integer powers of values the gaps in numerical semigroups (e.g., [7, 11, 10]) are popular. One of other famous problems is about the so-called *Sylvester sums*

$$s(a_1, \dots, a_k) := \sum_{n \in \mathrm{NR}(a_1, \dots, a_k)} n$$

(see, e.g., [21, §5.5], [34] and references therein), where $\mathrm{NR}(a_1, \dots, a_k)$ denotes the set of positive integers without nonnegative integer representation by $a_1, \dots, a_k$. In addition, denote the set of positive integers with nonnegative integer representation by $a_1, \dots, a_k$ by $\mathrm{R}(a_1, \dots, a_k)$. It is harder to obtain the Sylvester number than the Frobenius number, and even harder to obtain the Sylvester sum. Finally, long time after Sylvester, Brown and Shiue [7] found the exact value for positive integers a and b with $\gcd(a, b) = 1$,

$$s(a,b) = \frac{1}{12}(a-1)(b-1)(2ab - a - b - 1)\,. \tag{1}$$

Rødseth [24] generalized Brown and Shiue's result by giving a closed form for $s_\mu(a, b) := \sum_{n \in \mathrm{NR}(a,b)} n^\mu$, where μ is a positive integer.

When $k = 2$, there exist beautiful closed forms for Frobenius numbers, Sylvester numbers and Sylvester sums, but when $k \geq 3$, exact determination of these numbers is extremely difficult. The Frobenius number cannot be given by closed formulas of a certain type (Curtis (1990) [9]), the problem to determine $F(a_1, \dots, a_k)$ is NP-hard under Turing reduction (see, e.g., Ramírez Alfonsín [21]). Nevertheless, one convenient formula is found by Johnson [14]. One analytic approach to the Frobenius number can be seen in [2, 15].

Though closed forms for general case are hopeless for $k \geq 3$, several formulae for Frobenius numbers, Sylvester numbers and Sylvester sums have been considered under special cases. For example, one of the best expositions for the Frobenius number in three variables can be seen in [32]. For general $k \geq 3$, the Frobenius number and the Sylvester number for some special cases are calculated, including arithmetic sequences and geometric-like sequences (e.g., [5, 19, 22, 25]).

In fact, by introducing the other numbers, it is possible to determine the functions $g(A)$, $n(A)$ and $s(A)$ for the set of positive integers $A := \{a_1, a_2, \dots, a_k\}$ with $\gcd(a_1, a_2, \dots, a_k) = 1$.

For each integer i with $1 \leq i \leq a_1 - 1$, there exists a least positive integer $m_i \equiv i \pmod{a_1}$ with $m_i \in \mathrm{R}(a_1, a_2, \dots, a_k)$. For convenience, we set $m_0 = 0$. With the aid of such a congruence consideration modulo a_1, very useful results are established.

Lemma 1. *We have*

$$g(a_1, a_2, \dots, a_k) = \left(\max_{1 \leq i \leq a_1 - 1} m_i \right) - a_1 \,, \quad \textit{[6]}$$

$$n(a_1, a_2, \dots, a_k) = \frac{1}{a_1} \sum_{i=1}^{a_1-1} m_i - \frac{a_1 - 1}{2} \,, \quad \textit{[25]}$$

$$s(a_1, a_2, \dots, a_k) = \frac{1}{2a_1} \sum_{i=1}^{a_1-1} m_i^2 - \frac{1}{2} \sum_{i=1}^{a_1-1} m_i + \frac{a_1^2 - 1}{12} \,. \quad \textit{[31]}$$

Note that the third formula appeared with a typo in [31], and it has been corrected in [20, 33].

There are many kinds of problems related to the Frobenius problem. The problems for the number of solutions (e.g., [30]), and the sum of integer powers of values the gaps in numerical semigroups (e.g., [7, 11, 10]) are popular. In [18], the various results within the cyclotomic polynomial and numerical semigroup communities are better unified.

There are several generalized problems. In particular, in [4], the s-Frobenius number $g_s = g_s(a_1, a_2, \dots, a_k)$, the largest integer that has exactly s representations, is studied and the explicit formula is shown when $k = 2$. However, for general k, the so-called natural order $g_0 < g_1 < g_2 < \cdots$ is not guaranteed [26]. In [3], a related function $g_s^* = g_s^*(a_1, a_2, \dots, a_k)$, the largest integer having at most s such representations, is so introduced that $g_0^* < g_1^* < g_2^* < \cdots$ is not guaranteed.

Let $d(n) = d(n; a_1, \dots, a_k)$ be the number of solutions, that is,

$$d(n; a_1, \dots .a_k) := \#\{(x_1, \dots, x_k) : n = x_1 a_1 + \cdots + x_k a_k, x_i \ge 0 \, (0 \le i \le k)\} .$$

This is actually the number of partitions of n whose summands are taken (repetitions allowed) from the sequence $a_1, \dots, a_k$. The generating function of $d(n; a_1, \dots, a_k)$ is given by

$$\sum_{n=0}^{\infty} d(n; a_1, \dots, a_k) z^n = \frac{1}{(1 - z^{a_1}) \cdots (1 - z^{a_k})} .$$

Then, we consider the set such that the number of solutions is at least a positive integer p, that is,

$$\mathrm{R}_p := \{n; d(n; a_1, \dots .a_k) \ge p + 1\} .$$

In this paper, we study the complimentary set $\mathrm{NR}_p := \mathbb{N} - \mathrm{R}_p$, where the number of solutions is at most p. So, if $p = 0$, $\mathrm{NR} = \mathrm{NR}_0$ is the set of positive integers which are not represented in terms of the linear combination $a_1 x_1 + a_2 x_2 + \cdots + a_k x_k = n$ for any nonnegative integers $x_1, x_2, \dots, x_k$. Then we give formulae for the largest integer $g_p(A)$, the number of the elements $n_p(A)$ and the sum of the elements $s_p(A)$ for such a set NR_p. Namely,

$$g_p(A) = \max \mathrm{NR}_p, \quad n_p(A) = \# \mathrm{NR}_p, \quad s_p(A) = \sum_{n \in \mathrm{NR}_p} n .$$

When $p = 0$, they are reduced to the well-known Frobenius number, Sylvester number and Sylvester sum. We also give general formulae for these power and weighted sums. For $k \ge 3$, that is, the number of variables exceeds 2, it is not easy to find an explicit formula. Nevertheless, as an application, we show explicit expressions of these values for one triple set of triangular numbers.

2 Main results

For general positive integers k and p, we consider the Frobenius-like number, the Sylvester-like number and the Sylvester-like sum

$$g(a_1, a_2, \dots, a_k; p) = \max_{d(n) \le p} n , \tag{2}$$

$$n(a_1, a_2, \dots, a_k; p) = \#\{n : d(n) \le p\} , \tag{3}$$

$$s(a_1, a_2, \dots, a_k; p) = \sum_{d(n) \le p} n\,. \tag{4}$$

More generally, we consider the Sylvester-like power sum

$$s_\mu(a_1, a_2, \dots, a_k; p) = \sum_{d(n) \le p} n^\mu \quad (\mu \ge 1)\,,$$

so that $s(a_1, a_2, \dots, a_k; p) = s_1(a_1, a_2, \dots, a_k; p)$ and $n(a_1, a_2, \dots, a_k; p) = s_0(a_1, a_2, \dots, a_k; p)$.

For this purpose, for each $0 \le i \le a_1 - 1$, we introduce the positive integer $m_i^{(p)}$ congruent to i modulo a_1 such that the number of representations of $m_i^{(p)}$ is bigger than or equal to $p+1$ and that of $m_i - a_1$ is less than or equal to p. Note that $m_0^{(0)} = 0$.

Our main theorem is stated as follows.

Theorem 1. *Let k, p and μ be integers with $k \ge 2$, $p \ge 0$ and $\mu \ge 1$. Assume that* $\gcd(a_1, a_2, \dots, a_k) = 1$. *We have*

$$\begin{aligned} s_\mu(a_1, a_2, \dots, a_k; p) &:= \sum_{d(n) \le p} n^\mu \\ &= \frac{1}{\mu+1} \sum_{\kappa=0}^{\mu} \binom{\mu+1}{\kappa} B_\kappa a_1^{\kappa-1} \sum_{i=0}^{a_1-1} \bigl(m_i^{(p)}\bigr)^{\mu+1-\kappa} + \frac{B_{\mu+1}}{\mu+1}(a_1^{\mu+1} - 1)\,. \end{aligned}$$

Proof. For convenience, put $a = a_1$. Since

$$\sum_{j=1}^{\ell} j^n = \sum_{\kappa=0}^{n} \binom{n}{\kappa} (-1)^\kappa B_\kappa \frac{\ell^{n+1-\kappa}}{n+1-\kappa}$$

for $\ell = (m_i^{(p)} - i)/a$ $(1 \le i \le a-1)$ and $\ell_0 = (m_i^{(p)} - a)/a$ $(i = 0)$, we have

$$s_\mu(a_1, a_2, \dots, a_k; p) = \sum_{i=1}^{a-1} \sum_{j=1}^{\ell} (m_i^{(p)} - ja)^\mu + \sum_{j=1}^{\ell} (m_i^{(p)} - ja)^\mu\,. \tag{5}$$

First, consider the first term for $1 \le i \le a-1$.

$$\sum_{i=1}^{a-1} \sum_{j=1}^{\ell} (m_i^{(p)} - ja)^\mu$$

$$= \sum_{i=1}^{a-1}\sum_{j=1}^{\ell}\sum_{n=0}^{\mu}\binom{\mu}{n}(m_i^{(p)})^{\mu-n}j^n(-a)^n$$
$$= \sum_{i=1}^{a-1}\sum_{n=0}^{\mu}\binom{\mu}{n}(m_i^{(p)})^{\mu-n}(-a)^n\sum_{\kappa=0}^{n}\binom{n}{\kappa}(-1)^\kappa B_\kappa\frac{1}{n+1-\kappa}\left(\frac{m_i^{(p)}-i}{a}\right)^{n+1-\kappa}$$
$$= \sum_{n=0}^{\mu}\binom{\mu}{n}\sum_{\kappa=0}^{n}\binom{n}{\kappa}\frac{(-1)^{n-\kappa}a^{\kappa-1}B_\kappa}{n+1-\kappa}\sum_{i=1}^{a-1}\sum_{j=0}^{n+1-\kappa}\binom{n+1-\kappa}{j}(m_i^{(p)})^{\mu+1-\kappa-j}(-i)^j\,.$$

Consider the terms for $j=0$. Since

$$\sum_{n=\kappa}^{\mu}\binom{\mu-\kappa+1}{n+1-\kappa}(-1)^{n-\kappa} = 1+\sum_{j=0}^{\mu-\kappa+1}\binom{\mu-\kappa+1}{j}(-1)^{j-1} = 1\,,$$

we get

$$\sum_{n=\kappa}^{\mu}\binom{\mu}{n}\binom{n}{\kappa}\frac{(-1)^{n-\kappa}}{n+1-\kappa} = \frac{1}{\mu+1}\binom{\mu+1}{\kappa}\,.$$

Hence,

$$\sum_{n=0}^{\mu}\binom{\mu}{n}\sum_{\kappa=0}^{n}\binom{n}{\kappa}\frac{(-1)^{n-\kappa}a^{\kappa-1}B_\kappa}{n+1-\kappa}\sum_{i=1}^{a-1}(m_i^{(p)})^{\mu+1-\kappa}$$
$$= \sum_{\kappa=0}^{\mu}\sum_{n=\kappa}^{\mu}\binom{\mu}{n}\binom{n}{\kappa}\frac{(-1)^{n-\kappa}a^{\kappa-1}B_\kappa}{n+1-\kappa}\sum_{i=1}^{a-1}(m_i^{(p)})^{\mu+1-\kappa}$$
$$= \frac{1}{\mu+1}\sum_{\kappa=0}^{\mu}\binom{\mu+1}{\kappa}B_\kappa a^{\kappa-1}\sum_{i=1}^{a_1-1}(m_i^{(p)})^{\mu+1-\kappa}\,.$$

If $0<j<\mu+1-\kappa$, by

$$\sum_{n=\kappa}^{\mu}\binom{\mu}{n}\binom{n}{\kappa}\frac{(-1)^{n-\kappa}}{n+1-\kappa}\binom{n+1-\kappa}{j}$$
$$= \frac{(-1)^{j-1}\mu!}{j!\kappa!(\mu+1-\kappa-j)!}\sum_{h=0}^{\mu-\kappa}\binom{\mu+1-\kappa-j}{n}(-1)^{\mu+1-\kappa-j-n}$$
$$= 0\,,$$

all the terms for $(m_i^{(p)})^{\mu+1-\kappa-j}(-i)^j$ with $\mu+1-\kappa-j>0$ and $j>0$ are vanished. If $n=\mu$ and $j=\mu+1-\kappa$,

$$\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\frac{(-1)^{\mu-\kappa}a^{\kappa-1}B_\kappa}{\mu+1-\kappa}\sum_{i=1}^{a-1}(-i)^{\mu+1-\kappa}$$

$$
\begin{aligned}
&= -\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\frac{a^{\kappa-1}B_\kappa}{\mu+1-\kappa}\sum_{i=1}^{a-1} i^{\mu+1-\kappa}\\
&= -\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\frac{a^{\kappa-1}B_\kappa}{\mu+1-\kappa}\sum_{k=0}^{\mu+1-\kappa}\binom{\mu+1-\kappa}{k}B_{\mu+1-\kappa-k}\frac{a^{k+1}}{k+1}\\
&= -\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\frac{B_\kappa B_{\mu+1-\kappa}}{\mu+1-\kappa}a^\kappa\\
&\quad -\sum_{\kappa=0}^{\mu}\sum_{k=1}^{\mu+1-\kappa}\binom{\mu}{\kappa}\binom{\mu+1-\kappa}{k}\frac{B_\kappa B_{\mu+1-\kappa-k}}{(\mu+1-\kappa)(k+1)}a^{\kappa+k}\\
&= -\sum_{\ell=0}^{\mu}\binom{\mu}{\ell}\frac{B_\ell B_{\mu+1-\ell}}{\mu+1-\ell}a^\ell\\
&\quad -\sum_{\ell=1}^{\mu+1}\sum_{\kappa=0}^{\ell-1}\binom{\mu}{\kappa}\binom{\mu+1-\kappa}{\ell-\kappa}\frac{B_\kappa B_{\mu+1-\ell}}{(\mu+1-\kappa)(\ell+1-\kappa)}a^\ell\,. \qquad (6)
\end{aligned}
$$

By using the recurrence relation

$$
\sum_{\kappa=0}^{\mu+1}\binom{\mu+2}{\kappa}B_\kappa=0\,,
$$

the term of $a^{\mu+1}$ yields from the second sum in (6) as

$$
-\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\frac{B_\kappa a^{\mu+1}}{(\mu+1-\kappa)(\mu+2-\kappa)}=\frac{B_{\mu+1}}{\mu+1}a^{\mu+1}\,.
$$

The constant term yields from the first sum in (6) as

$$
-\frac{B_{\mu+1}}{\mu+1}\,.
$$

Other terms of a^ℓ $(1\le\ell\le\mu)$ are cancelled, so vanished, because by

$$
\sum_{\kappa=0}^{\ell}\binom{\ell+1}{\kappa}B_\kappa=0\,,
$$

we get

$$
\sum_{\kappa=0}^{\ell-1}\binom{\mu}{\kappa}\binom{\mu+1-\kappa}{\ell-\kappa}\frac{B_\kappa}{(\mu+1-\kappa)(\ell+1-\kappa)}=-\binom{\mu}{\ell}\frac{B_\ell}{\mu+1-\ell}\,.
$$

Hence, we obtain the term

$$\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\frac{(-1)^{\mu-\kappa}a^{\kappa-1}B_\kappa}{\mu+1-\kappa}\sum_{i=1}^{a-1}(-i)^{\mu+1-\kappa}=\frac{B_{\mu+1}}{\mu+1}(a^{\mu+1}-1)\,.$$

Next, consider the second term for $i=0$ in (5). In a similar manner, for $j=0$ we have

$$\frac{1}{\mu+1}\sum_{\kappa=0}^{\mu}\binom{\mu+1}{\kappa}B_\kappa a^{\kappa-1}\bigl(m_0^{(p)}\bigr)^{\mu+1-\kappa}\,.$$

For $0<j<\mu+1-\kappa$, all the terms are vanished. For $n=\mu$ and $j=\mu+1-\kappa$, we have

$$\begin{aligned}&\sum_{\kappa=0}^{\mu}\binom{\mu}{\kappa}\frac{(-1)^{\mu-\kappa}a^{\kappa-1}B_\kappa}{\mu+1-\kappa}(-a)^{\mu+1-\kappa}\\&=-\frac{a^\mu}{\mu+1}\sum_{\kappa=0}^{\mu}\binom{\mu+1}{\kappa}B_\kappa\\&=0\,.\end{aligned}$$

Combining all the terms, we get the desired formula. $\square$

When $\mu=0,1$ in Theorem 1, together with $g(a_1,a_2,\dots,a_k;p)$ we have formulae for the Frobenius-like number, the Sylvester-like number and the Sylvester-like sum.

Corollary 1. *Let k, p and μ be integers with $k\ge 2$, $p\ge 0$ and $\mu\ge 1$. Assume that* $\gcd(a_1,a_2,\dots,a_k)=1$. *We have*

$$g(a_1,a_2,\dots,a_k;p)=\max_{0\le i\le a_1-1}m_i^{(p)}-a_1\,, \tag{7}$$

$$n(a_1,a_2,\dots,a_k;p)=\frac{1}{a_1}\sum_{i=0}^{a_1-1}m_i^{(p)}-\frac{a_1+1}{2}\,, \tag{8}$$

$$s(a_1,a_2,\dots,a_k;p)=\frac{1}{2a_1}\sum_{i=0}^{a_1-1}\bigl(m_i^{(p)}\bigr)^2-\frac{1}{2}\sum_{i=0}^{a_1-1}m_i^{(p)}+\frac{a_1^2-1}{12}\,. \tag{9}$$

Remark. When $p=0$ in Theorem 1 with Corollary 1, the formulae are reduced to

$$g(a_1,a_2,\dots,a_k)=\left(\max_{1\le i\le a_1-1}m_i\right)-a_1\,,\quad [6]$$

$$n(a_1,a_2,\dots,a_k)=\frac{1}{a_1}\sum_{i=1}^{a_1-1}m_i-\frac{a_1-1}{2}\,,\quad [25]$$

$$s(a_1,a_2,\dots,a_k)=\frac{1}{2a_1}\sum_{i=1}^{a_1-1}m_i^2-\frac{1}{2}\sum_{i=1}^{a_1-1}m_i+\frac{a_1^2-1}{12}\,.\quad [31]$$

Notice that $m_0=m_0^{(1)}=0$ is applied in the classical formulae. Hence, the sum runs from $i=1$.

Proof of Corollary 1. (7) It is trivial that if $r=\max_{0\le i\le a_1-1}m_i^{(p)}$ is the largest integer whose number of representations is $p+1$, then $r-a_1$ is the largest positive integer whose number of representations is p.

The formulae (8) and (9) are reduced from Theorem 1 for $\mu=0$ and $\mu=1$, respectively. $\square$

In the case of two variables, namely, $a_1=a$ and $a_2=b$, by $\{m_i|0\le i\le a-1\}=\{b(pa+i)|0\le i\le a-1\}$, Corollary 1 is further reduced to the following.

Corollary 2 ([1])**.** *For a nonnegative integer p, we have*

$$\begin{aligned}g_p(a,b)&=(p+1)ab-a-b\,,\\ n_p(a,b)&=\frac{1}{2}\big((2p+1)ab-a-b+1\big)\,,\\ s_p(a,b)&=\frac{1}{12}\big(2(3p^2+3p+1)a^2b^2-3(2p+1)ab(a+b)\\ &\qquad+a^2+b^2+3ab-1\big)\,.\end{aligned}$$

2.1 Examples

Consider the sequence $5,7,11$. The numbers of representations of each n are given by the table.

Since $m_0^{(4)}=50$, $m_1^{(4)}=51$, $m_2^{(4)}=47$, $m_3^{(4)}=53$ and $m_4^{(4)}=49$, by Corollary 1 we have

$$\begin{aligned}&g_4(5,7,11)=53-5=48\,,\\ &n_4(5,7,11)=\frac{50+51+47+53+49}{5}-\frac{5+1}{2}=47\,,\\ &s_4(5,7,11)\end{aligned}$$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$d(n)$	0	0	0	0		0		0	0				0							
					1		1			1	1	1		1	1	1	1	1	1	1
n	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
$d(n)$	2	2			2	2	2	2	2	2	2			2						4
			1	1								3	3		3	3	3	3	3	
n	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
$d(n)$		4	4	4	4	4		4						6	6	6	6	6	6	
	3						5		5	5	5	5	5							7
n	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
$d(n)$	7	7	7	7						9	9	9	9	9			11			11
					8	8	8	8	8						10	10		10	10	
n	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100
$d(n)$	11		11				13	13	13	13					15	15	15			
		12		12	12	12					14	14	14	14				16	16	16

Table 1.

$$
\begin{aligned}
&= \frac{50^2+51^2+47^2+53^2+49^2}{2\cdot 5} - \frac{50+51+47+53+49}{2} + \frac{5^2-1}{12} \\
&= 1129\,.
\end{aligned}
$$

It is clear that the values of g_4 and n_4 are valid from Table 1. Since $1+2+\cdots+46+48=1129$, the value of s_4 is valid.

When $\mu=6$ in Theorem 1, we have

$$
\begin{aligned}
&s_6(a_1,\dots,a_k;p) \\
&= \frac{1}{7a_1}\sum_{i=1}^{a_1-1}\bigl(m_i^{(p)}\bigr)^7 - \frac{1}{2}\sum_{i=1}^{a_1-1}\bigl(m_i^{(p)}\bigr)^6 + \frac{a_1}{2}\sum_{i=1}^{a_1-1}\bigl(m_i^{(p)}\bigr)^5 - \frac{a_1^3}{6}\sum_{i=1}^{a_1-1}\bigl(m_i^{(p)}\bigr)^3 \\
&\quad + \frac{a_1^5}{42}\sum_{i=1}^{a_1-1} m_i^{(p)}\,.
\end{aligned}
$$

So, for $(5,7,11)$ and $p=4$, we have

$$
\begin{aligned}
&s_6(5,7,11;4) \\
&= \frac{50^7+51^7+47^7+53^7+49^7}{7\cdot 5} - \frac{50^6+51^6+47^6+53^6+49^6}{2} \\
&\quad + \frac{5(50^5+51^5+47^5+53^5+49^5)}{2} - \frac{5^3(50^3+51^3+47^3+53^3+49^3)}{6}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{5^5(50+51+47+53+49)}{42}\\
&=79330369495\,.
\end{aligned}
$$

Indeed,

$$1^6+2^6+\cdots+46^6+48^6=79330369495\,.$$

3 Weighted sums

In this section, we study the weighted sums whose numbers of representations are less than or equal to p:

$$s_\mu^{(\lambda)}(a_1,\dots,a_k;p):=\sum_{d(p)\le p}\lambda^n n^\mu\,,$$

where $\lambda\ne 0,1$ and μ is a positive integer.

For this purpose, we need Eulerian numbers $\left\langle{n \atop m}\right\rangle$, appearing in the generating function

$$\sum_{k=0}^\infty k^n x^k=\frac{1}{(1-x)^{n+1}}\sum_{m=0}^{n-1}\left\langle{n \atop m}\right\rangle x^{m+1}\quad(n\ge 1) \tag{10}$$

with $0^0=1$ and $\left\langle{0 \atop 0}\right\rangle=1$ ([8, p.244]), and have an explicit formula

$$\left\langle{n \atop m}\right\rangle=\sum_{k=0}^m(-1)^k\binom{n+1}{k}(m-k+1)^n$$

([8, p.243],[12]). Similarly, for $0\le i\le a_1-1$, let $m_i^{(p)}$ be the least positive integer in $\mathrm{R}_p(a_1,a_2,\dots,a_k)$ among those congruent to i modulo a_1. Then, except $m_0^{(0)}=0$, $m_i^{(p)}-a_1$ is the largest positive integer in $\mathrm{NR}_p(a_1,a_2,\dots,a_k)$ among those congruent to i modulo a_1.

Theorem 2. *Assume that $\lambda\ne 0$ and $\lambda^{a_1}\ne 1$. Then for a positive integer μ,*

$$
\begin{aligned}
&s_\mu^{(\lambda)}(a_1,\dots,a_k;p)\\
&=\sum_{n=0}^\mu\frac{(-a_1)^n}{(\lambda^{a_1}-1)^{n+1}}\binom{\mu}{n}\sum_{j=0}^n\left\langle{n \atop n-j}\right\rangle\lambda^{ja_1}\sum_{i=0}^{a_1-1}\bigl(m_i^{(p)}\bigr)^{\mu-n}\lambda^{m_i^{(p)}}\\
&\quad+\frac{(-1)^{\mu+1}}{(\lambda-1)^{\mu+1}}\sum_{j=0}^\mu\left\langle{\mu \atop \mu-j}\right\rangle\lambda^j\,.
\end{aligned}
$$

Remark. If one wants to avoid $0^0 = 1$ (e.g., in computational calculations), we use the formula

$$
\begin{aligned}
&s_\mu^{(\lambda)}(a_1, \dots, a_k; p)\\
&= \sum_{n=0}^{\mu-1} \frac{(-a_1)^n}{(\lambda^{a_1}-1)^{n+1}} \binom{\mu}{n} \sum_{j=0}^{n} \left\langle {n \atop n-j} \right\rangle \lambda^{j a_1} \sum_{i=0}^{a_1-1} \bigl(m_i^{(p)}\bigr)^{\mu-n} \lambda^{m_i^{(p)}}\\
&\quad + \frac{(-a_1)^\mu}{(\lambda^{a_1}-1)^{\mu+1}} \sum_{j=0}^{\mu} \left\langle {\mu \atop \mu-j} \right\rangle \lambda^{j a_1} \sum_{i=0}^{a_1-1} \lambda^{m_i^{(p)}}\\
&\quad + \frac{(-1)^{\mu+1}}{(\lambda-1)^{\mu+1}} \sum_{j=0}^{\mu} \left\langle {\mu \atop \mu-j} \right\rangle \lambda^{j}\,.
\end{aligned}
$$

Proof. From the definition of $m_i^{(p)}$, for each i, there exists a nonnegative integer ℓ such that

$$
m_i^{(p)} - a_1, m_i^{(p)} - 2a_1, \dots, m_i^{(p)} - \ell a_1 \in \mathrm{NR}_p(a_1, a_2, \dots, a_k)
$$

with $m_i^{(p)} - \ell a_1 > 0$ and $m_i^{(p)} - (\ell+1)a_1 < 0$. Since $m_i^{(p)} \equiv i \pmod{a_1}$, we see $\ell = (m_i^{(p)} - i)/a_1$ for $1 \le i \le a_1 - 1$, $\ell_0 = (m_0^{(p)} - a_1)/a_1$ for $i = 0$. Let $\lambda \ne 0$ and $\lambda^{a_1} \ne 1$. For simplicity, put $a = a_1$. Then, the weighted sum of elements in $\mathrm{R}_p(a_1, a_2, \dots, a_k)$ congruent to i modulo a_1 is given by

$$
\sum_{j=1}^{\ell} \lambda^{m_i^{(p)} - ja} (m_i^{(p)} - ja)^\mu\,.
$$

Notice that for $n \ge 1$,

$$
\begin{aligned}
\sum_{j=1}^{\infty} \lambda^{m_i^{(p)} - ja} j^n &= \frac{\lambda^{m_i^{(p)}}}{(1-\lambda^{-a})^{n+1}} \sum_{h=0}^{n-1} \left\langle {n \atop h} \right\rangle \lambda^{-(h+1)a}\\
&= \frac{\lambda^{m_i^{(p)}}}{(\lambda^{a}-1)^{n+1}} \sum_{h=0}^{n-1} \left\langle {n \atop h} \right\rangle \lambda^{(n-h)a}\\
&= \frac{\lambda^{m_i^{(p)}}}{(\lambda^{a}-1)^{n+1}} \sum_{h=1}^{n} \left\langle {n \atop n-h} \right\rangle \lambda^{ha}\,.
\end{aligned}
$$

So, for $n \ge 0$, we get

$$
\sum_{j=1}^{\infty} \lambda^{m_i^{(p)} - ja} j^n = \frac{\lambda^{m_i^{(p)}}}{(\lambda^{a}-1)^{n+1}} \sum_{h=0}^{n} \left\langle {n \atop n-h} \right\rangle \lambda^{ha}\,.
$$

Hence, for $\ell = (m_i^{(p)} - i)/a$ $(1 \le i \le a-1)$ and $\ell_0 = (m_0^{(p)} - a)/a$,

$$
\begin{aligned}
&s_\mu^{(\lambda)}(a_1, \dots, a_k; p)\\
&\qquad = \sum_{i=1}^{a-1} \sum_{j=1}^{\ell} \lambda^{m_i^{(p)} - ja} (m_i^{(p)} - ja)^\mu + \sum_{j=1}^{\ell_0} \lambda^{m_0^{(p)} - ja} (m_0^{(p)} - ja)^\mu\,. \qquad (11)
\end{aligned}
$$

Concerning the first term for $1 \le i \le a-1$, we have

$$
\begin{aligned}
&\sum_{i=1}^{a-1} \sum_{j=1}^{\ell} \lambda^{m_i^{(p)} - ja} (m_i^{(p)} - ja)^\mu\\
&= \sum_{i=1}^{a-1} \sum_{j=1}^{\ell} \lambda^{m_i^{(p)} - ja} \sum_{n=0}^{\mu} \binom{\mu}{n} \bigl(m_i^{(p)}\bigr)^{\mu-n} j^n (-a)^n\\
&= \sum_{i=1}^{a-1} \sum_{n=0}^{\mu} \binom{\mu}{n} \bigl(m_i^{(p)}\bigr)^{\mu-n} (-a)^n \sum_{j=1}^{\infty} \lambda^{m_i^{(p)} - ja} j^n\\
&\quad - \sum_{i=0}^{a-1} \sum_{n=0}^{\mu} \binom{\mu}{n} \bigl(m_i^{(p)}\bigr)^{\mu-n} (-a)^n \sum_{j=\ell+1}^{\infty} \lambda^{m_i^{(p)} - ja} j^n\\
&= \sum_{i=1}^{a-1} \sum_{n=1}^{\mu} \binom{\mu}{n} \bigl(m_i^{(p)}\bigr)^{\mu-n} (-a)^n \sum_{j=1}^{\infty} \lambda^{m_i^{(p)} - ja} j^n\\
&\quad + \sum_{i=1}^{a-1} \bigl(m_i^{(p)}\bigr)^{\mu} \sum_{j=1}^{\infty} \lambda^{m_i^{(p)} - ja}\\
&\quad - \sum_{i=1}^{a-1} \sum_{n=0}^{\mu} \binom{\mu}{n} \bigl(m_i^{(p)}\bigr)^{\mu-n} (-a)^n \sum_{j=1}^{\infty} \lambda^{i - ja} \left(j + \frac{m_i^{(p)} - i}{a} \right)^n\\
&= \sum_{i=1}^{a-1} \sum_{n=1}^{\mu} \binom{\mu}{n} \bigl(m_i^{(p)}\bigr)^{\mu-n} (-a)^n \frac{\lambda^{m_i^{(p)}}}{(\lambda^a - 1)^{n+1}} \sum_{h=1}^{n} \left\langle {n \atop n-h} \right\rangle \lambda^{ha}\\
&\quad + \sum_{i=1}^{a-1} \bigl(m_i^{(p)}\bigr)^{\mu} \frac{\lambda^{m_i^{(p)}}}{\lambda^a - 1}\\
&\quad - \sum_{i=1}^{a-1} \sum_{n=0}^{\mu} \binom{\mu}{n} \bigl(m_i^{(p)}\bigr)^{\mu-n} (-a)^n \sum_{j=1}^{\infty} \lambda^{i - ja} \left(j + \frac{m_i^{(p)} - i}{a} \right)^n\\
&= \sum_{n=0}^{\mu} \frac{(-a)^n}{(\lambda^a - 1)^{n+1}} \binom{\mu}{n} \sum_{h=0}^{n} \left\langle {n \atop n-h} \right\rangle \lambda^{ha} \sum_{i=1}^{a-1} \bigl(m_i^{(p)}\bigr)^{\mu-n} \lambda^{m_i^{(p)}}
\end{aligned}
$$

$$-\sum_{i=1}^{a-1}\sum_{j=1}^{\infty}\lambda^{i-ja}(i-ja)^{\mu}\,.$$

Concerning the second term for $i=0$ in (11), the sum

$$\sum_{j=1}^{\infty}\lambda^{i-ja}\left(j+\frac{m_i^{(p)}-i}{a}\right)^n$$

is replaced by the sum

$$\sum_{j=1}^{\infty}\lambda^{a-ja}\left(j+\frac{m_0^{(p)}-a}{a}\right)^n\,.$$

Hence, we obtain

$$\begin{aligned}
&s_{\mu}^{(\lambda)}(a_1,\dots,a_k;p)\\
&=\sum_{n=0}^{\mu}\frac{(-a)^n}{(\lambda^a-1)^{n+1}}\binom{\mu}{n}\sum_{h=0}^{n}\left\langle {n \atop n-h} \right\rangle\lambda^{ha}\sum_{i=0}^{a-1}\bigl(m_i^{(p)}\bigr)^{\mu-n}\lambda^{m_i^{(p)}}\\
&\quad-\sum_{i=1}^{a-1}\sum_{j=1}^{\infty}\lambda^{i-ja}(i-ja)^{\mu}-\sum_{j=1}^{\infty}\lambda^{a-ja}(a-ja)^{\mu}\,.
\end{aligned}$$

Since the last two terms are equal to

$$\begin{aligned}
-\sum_{k=0}^{\infty}\lambda^{-k}(-k)^{\mu}&=\frac{(-1)^{\mu+1}}{(1-\lambda^{-1})^{\mu+1}}\sum_{m=0}^{\mu-1}\left\langle {\mu \atop m} \right\rangle\lambda^{-(m+1)}\\
&=\frac{(-1)^{\mu+1}}{(\lambda-1)^{\mu+1}}\sum_{j=0}^{\mu}\left\langle {\mu \atop \mu-j} \right\rangle\lambda^{j}\,,
\end{aligned}$$

we have the desired result. $\square$

3.1 Examples

Consider the sequence $14,17,20,23,26,29$. Then, $a=14$, $d=3$, $k=6$, $q=2$ and $r=3$. By Theorem 2, we have

$$\begin{aligned}
&s_2^{(\sqrt[3]{2})}(14,17,20,23,26,29)\\
&=21528522+31320173525\sqrt[3]{2}+659369214\sqrt[3]{4}\,,
\end{aligned}$$

$$
\begin{aligned}
&s_3^{(7)}(14,17,20,23,26,29)\\
&= 12615313654771886039774918936481484789732904072330249995951189 2\,,\\
&s_4^{(-1/2)}(14,17,20,23,26,29)\\
&= -\frac{252455039549405466513}{147573952589676412928}\,,\\
&s_5^{(4+3\sqrt{-1})}(14,17,20,23,26,29)\\
&= 586049555846415789540309665304848752532973290001015604 80\\
&\quad - 69984733631939902694215153740002368436325991046609895240\sqrt{-1}\,.
\end{aligned}
$$

In fact, the weight power sum of nonrepresentable numbers is given by

$$
\begin{aligned}
&\lambda^1\cdot 1^\mu+\lambda^2\cdot 2^\mu+\lambda^3\cdot 3^\mu+\lambda^4\cdot 4^\mu+\lambda^5\cdot 5^\mu+\lambda^6\cdot 6^\mu+\lambda^7\cdot 7^\mu+\lambda^8\cdot 8^\mu\\
&+\lambda^9\cdot 9^\mu+\lambda^{10}\cdot 10^\mu+\lambda^{11}\cdot 11^\mu+\lambda^{12}\cdot 12^\mu+\lambda^{13}\cdot 13^\mu+\lambda^{15}\cdot 15^\mu\\
&+\lambda^{16}\cdot 16^\mu+\lambda^{18}\cdot 18^\mu+\lambda^{19}\cdot 19^\mu+\lambda^{21}\cdot 21^\mu+\lambda^{22}\cdot 22^\mu+\lambda^{24}\cdot 24^\mu\\
&+\lambda^{25}\cdot 25^\mu+\lambda^{27}\cdot 27^\mu+\lambda^{30}\cdot 30^\mu+\lambda^{32}\cdot 32^\mu+\lambda^{33}\cdot 33^\mu+\lambda^{35}\cdot 35^\mu\\
&+\lambda^{36}\cdot 36^\mu+\lambda^{38}\cdot 38^\mu+\lambda^{39}\cdot 39^\mu+\lambda^{41}\cdot 41^\mu+\lambda^{44}\cdot 44^\mu+\lambda^{47}\cdot 47^\mu\\
&+\lambda^{50}\cdot 50^\mu+\lambda^{53}\cdot 53^\mu+\lambda^{61}\cdot 61^\mu+\lambda^{64}\cdot 64^\mu+\lambda^{67}\cdot 67^\mu\,.
\end{aligned}
$$

When $\mu=1$ in Theorem 2, we have the following.

Theorem 3. *If $\lambda\ne 0$ and $\lambda^{a_1}\ne 1$, then*

$$
\begin{aligned}
&s^{(\lambda)}(a_1,a_2,\dots,a_k;p):=s_1^{(\lambda)}(a_1,a_2,\dots,a_k)\\
&=\frac{1}{\lambda^{a_1}-1}\sum_{i=0}^{a_1-1}m_i^{(p)}\lambda^{m_i^{(p)}}-\frac{a_1\lambda^{a_1}}{(\lambda^{a_1}-1)^2}\sum_{i=0}^{a_1-1}\lambda^{m_i^{(p)}}+\frac{\lambda}{(\lambda-1)^2}\,.
\end{aligned}
$$

When $k=2$, by $m_i=b(pa+i)$, Theorem 3 is reduced to a formula for the weighted sum of positive integers whose number of representations of two coprime positive integers (a,b) are less than or equal to p.

Corollary 3. *Assume that $\lambda\ne 0$, $\lambda^a\ne 1$ and $\lambda^b\ne 1$. For $p\ge 0$, we have*

$$
\begin{aligned}
s^{(\lambda)}(a,b;p):&=\sum_{d(n;a,b)\le p}\lambda^n n\\
&=\frac{\lambda}{(\lambda-1)^2}+\frac{ab\lambda^{pab}\bigl((p+1)\lambda^{ab}-p\bigr)}{(\lambda^a-1)(\lambda^b-1)}
\end{aligned}
$$

$$-\frac{\lambda^{pab}(\lambda^{ab}-1)\big((a+b)\lambda^{a+b}-a\lambda^a-b\lambda^b\big)}{(\lambda^a-1)^2(\lambda^b-1)^2}\,.$$

Remark. When $p=0$ in Theorem 3, it is reduced to Theorem 1 in [16].

If $\lambda\ne 1$ and $\lambda^{a_1}=\lambda^{a_2}=\cdots=\lambda^{a_k}=1$, then $\gcd(a_1,a_2,\dots,a_k)\ne 1$. So, we can choose a_j such that $\lambda^{a_j}\ne 1$. Nevertheless, if $\lambda\ne 1$ and $\lambda^{a_1}=1$, then we have the following

Theorem 4. *If $\lambda\ne 0,1$ and $\lambda^{a_1}=1$, then for $p\ge 0$*

$$s^{(\lambda)}(a_1,a_2,\dots,a_k;p)=\frac{1}{2a_1}\sum_{i=0}^{a_1-1}\big(m_i^{(p)}\big)^2\lambda^i-\frac{1}{2}\sum_{i=0}^{a_1-1}m_i^{(p)}\lambda^i+\frac{\lambda}{(\lambda-1)^2}\,.$$

Proof. Since $\lambda^{a_1}=1$, the weighted sum of elements in $\mathrm{R}(a_1,a_2,\dots,a_k)$ congruent to i $(1\le i\le a_1-1)$ modulo a_1 is given by

$$\begin{aligned}
&\sum_{j=1}^{\ell}\lambda^{m_i^{(p)}-ja_1}(m_i^{(p)}-ja_1)=m_i^{(p)}\sum_{j=1}^{\ell}\lambda^i-a_1\sum_{j=1}^{\ell}\lambda^i j\\
&=m_i^{(p)}\ell\lambda^i-a_1\frac{\ell(\ell+1)}{2}\lambda^i\\
&=\frac{m_i^{(p)}(m_i^{(p)}-i)\lambda^i}{a_1}-\frac{(m_i^{(p)}-i)^2\lambda^i}{2a_1}-\frac{(m_i^{(p)}-i)\lambda^i}{2}\,.
\end{aligned}$$

For $i=0$ and $p\ge 1$, it is given by

$$\begin{aligned}
&\sum_{j=1}^{\ell}\lambda^{m_0^{(p)}-ja_1}(m_0^{(p)}-ja_1)\\
&=\frac{m_0^{(p)}(m_0^{(p)}-a_1)}{a_1}-\frac{(m_0^{(p)}-a_1)^2}{2a_1}-\frac{(m_0^{(p)}-a_1)}{2}\\
&=\frac{\big(m_0^{(p)}\big)^2}{2a_1}-\frac{m_0^{(p)}}{2}\,.
\end{aligned}$$

This is also valid for $i=0$ and $p=0$. Therefore,

$$\begin{aligned}
&s^{(\lambda)}(a_1,a_2,\dots,a_k;p)\\
&=\frac{1}{a_1}\left(\sum_{i=1}^{a_1-1}\big(m_i^{(p)}\big)^2\lambda^i-\sum_{i=1}^{a_1-1}im_i^{(p)}\lambda^i\right)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2a_1}\left(\sum_{i=1}^{a_1-1}\bigl(m_i^{(p)}\bigr)^2\lambda^i-2\sum_{i=1}^{a_1-1}im_i^{(p)}\lambda^i+\sum_{i=1}^{a_1-1}i^2\lambda^i\right)\\
&-\frac{1}{2}\left(\sum_{i=1}^{a_1-1}m_i^{(p)}\lambda^i-\sum_{i=1}^{a_1-1}i\lambda^i\right)+\frac{\bigl(m_0^{(p)}\bigr)^2}{2a_1}-\frac{m_0^{(p)}}{2}\\
=&\frac{1}{2a_1}\sum_{i=1}^{a_1-1}\bigl(m_i^{(p)}\bigr)^2\lambda^i-\frac{1}{2}\sum_{i=1}^{a_1-1}m_i^{(p)}\lambda^i\\
&-\frac{1}{2a_1}\frac{a_1^2(\lambda-1)-2a_1\lambda}{(\lambda-1)^2}+\frac{1}{2}\frac{a_1}{\lambda-1}+\frac{\bigl(m_0^{(p)}\bigr)^2}{2a_1}-\frac{m_0^{(p)}}{2}\\
=&\frac{1}{2a_1}\sum_{i=0}^{a_1-1}\bigl(m_i^{(p)}\bigr)^2\lambda^i-\frac{1}{2}\sum_{i=0}^{a_1-1}\bigl(m_i^{(p)}\bigr)\lambda^i+\frac{\lambda}{(\lambda-1)^2}\,.
\end{aligned}
$$

□

In particular, when $\lambda=-1$ and a_1 is odd in Theorem 3, we have the formula for alternate sums. When $k=2$ and $A=\{a,b\}$, the formulas are obtained in terms of Bernoulli or Euler numbers in [35].

Corollary 4. *When a_1 is odd, we have*

$$
s^{(-1)}(a_1,a_2,\dots,a_k;p)=-\frac{1}{2}\sum_{i=0}^{a_1-1}(-1)^{m_i^{(p)}}m_i^{(p)}+\frac{a_1}{4}\sum_{i=0}^{a_1-1}(-1)^{m_i^{(p)}}+\frac{a_1-1}{4}\,.
$$

When $k=2$ in Theorem 3, by $\{m_i^{(p)}|1\le i\le a-1\}=\{b(pa+i)|1\le i\le a-1\}$, if $\lambda^b\ne 1$, we have the formula in Corollary 3. If $\lambda\ne 1$ and $\lambda^b=1$, we have

$$
\begin{aligned}
&s^{(\lambda)}(a,b;p)\\
&=\frac{1}{\lambda^a-1}\sum_{i=0}^{a-1}b(pa+i)\lambda^{b(pa+i)}-\frac{a\lambda^a}{(\lambda^a-1)^2}\sum_{i=0}^{a-1}\lambda^{b(pa+i)}+\frac{\lambda}{(\lambda-1)^2}\\
&=\frac{1}{\lambda^a-1}\sum_{i=0}^{a-1}b(pa+i)-\frac{a^2\lambda^a}{(\lambda^a-1)^2}+\frac{\lambda}{(\lambda-1)^2}\\
&=\frac{ab\bigl((2p+1)a-1\bigr)}{2(\lambda^a-1)}-\frac{a^2\lambda^a}{(\lambda^a-1)^2}+\frac{\lambda}{(\lambda-1)^2}\,. \qquad (12)
\end{aligned}
$$

For example, for the primitive 5th root of unity ζ_5, from (12) we have

$$
\begin{aligned}
s^{(\zeta_5)}(7,5;0) &= 34\zeta_5 + 2\zeta_5^2 + 65\zeta_5^3 + 13\zeta_5^4\,,\\
s^{(\zeta_5)}(7,5;1) &= 286\zeta_5 + 156\zeta_5^2 + 366\zeta_5^3 + 216\zeta_5^4 + 105\,,\\
s^{(\zeta_5)}(7,5;2) &= 784\zeta_5 + 555\zeta_5^2 + 912\zeta_5^3 + 664\zeta_5^4 + 455\,,\\
s^{(\zeta_5)}(7,5;3) &= 1525\zeta_5 + 1199\zeta_5^2 + 1703\zeta_5^3 + 1357\zeta_5^4 + 1050\,,\\
s^{(\zeta_5)}(7,5;4) &= 2512\zeta_5 + 2088\zeta_5^2 + 2739\zeta_5^3 + 2295\zeta_5^4 + 1890\,,\\
s^{(\zeta_5)}(7,5;5) &= 3744\zeta_5 + 3222\zeta_5^2 + 4020\zeta_5^3 + 3478\zeta_5^4 + 2975\,.
\end{aligned}
$$

4 Three variables

When the number of variables $k \ge 3$, it is not easy to find an explicit form of Frobenius number, Sylvester number or Sylvester sum. Nevertheless, for $p = 0$, explicit froms have been discovered in some paricular cases, including arithmetic, geometric-like, Fibonacci, Mersenne, and triangular (see, e.g., [23] and references therein) and so on. However, for $p \ge 1$, any explicit form has been found even in such particular cases because it is not easy to find $m_i^{(p)}$ in order to use Theorem 1 or Corollary 1.

Here, we give only one result.

Proposition 1. *For $p \ge 1$,*

$$
\begin{aligned}
g(t_3,t_4,t_5;t_{p-1}) &= 30p - 1\,,\\
n(t_3,t_4,t_5;t_{p-1}) &= 30p - 16\,,\\
s(t_3,t_4,t_5;t_{p-1}) &= 15(30p^2 - 31p + 12)\,,
\end{aligned}
$$

where $t_n = n(n+1)/2$ is the n-th triangular number.

In fact, there does not exist a positive integer n such that for some nonnegative integer m $d(n;6,10,15) \ne t_m$.

Sketch of the proof of Proposition 1. Since $(t_3,t_4,t_5) = (6,10,15)$, we consider the representation $10x_2 + 15x_3 = 5(2x_2 + 3x_3)$ $(x_2,x_3 \ge 0)$. As the generating function

$$
\begin{aligned}
&\frac{1}{(1-z^2)(1-z^3)}\\
&= \frac{1}{2(1-z^2)} + \frac{1}{3(1-z^3)} - \frac{z}{3(1-z^3)} + \frac{1}{6(1-z)^2}
\end{aligned}
$$

$$= \frac{1}{2}\sum s_1(n)z^n + \frac{1}{3}\sum s_2(n)z^n - \frac{1}{3}\sum s_3(n)z^n + \frac{1}{6}\sum (n+1)z^n\,,$$

where

$$s_1(n) = \begin{cases} 1 & n \equiv 0 \pmod 2; \\ 0 & \text{otherwise}, \end{cases} \qquad s_2(n) = \begin{cases} 1 & n \equiv 0 \pmod 3; \\ 0 & \text{otherwise}, \end{cases}$$
$$s_3(n) = \begin{cases} 1 & n \equiv 1 \pmod 3; \\ 0 & \text{otherwise}\,, \end{cases}$$

the sequence is given by

$$\begin{aligned} & 1,0,1,1,1,1,2,1,2,2,2,2,3,2,3,3,3,3,4,3,4,4,4,4,5,4,5,5,5,5,6,5,\dots \\ & = 1,0,\overline{m,m,m,m,m+1,m}_{m=1}^{\infty} \qquad (13) \\ & = \left\{ \left\lfloor \frac{n+2}{2} \right\rfloor - \left\lfloor \frac{n+2}{3} \right\rfloor \right\}_{n\ge 0} \end{aligned}$$

(*Cf.* [27, A008615]). Hence, for $n \ge 0$

$$d(n;2,3) = \left\lfloor \frac{n+2}{2} \right\rfloor - \left\lfloor \frac{n+2}{3} \right\rfloor = d(5n;10,15)\,,$$

and for $p \ge 0$, $g_p(2,3) = 6p+1$. Thus,

$$m_5^{(p')} \pmod 6 = \max_{0\le i\le 5} m_i^{(p')} = 5(6p+1)\,.$$

Here, because the number of representation of $30p+5$ in terms of $(6,10,15)$ is just p more than that of $30p-1$, by applying Corollary 1 (7), we have

$$g\bigl(t_3,t_4,t_5;p'\bigr) = 30p-1\,,$$

where $p' = 1+2+\cdots+(p-1) = t_{p-1}$. Similarly, from the sequence (13), we see that for $p' = t_{p-1}$

$$\begin{aligned} & m_1^{(p')} = 5(6p-1), \quad m_2^{(p')} = 5(6p-2), \quad m_3^{(p')} = 5(6p-3), \\ & m_4^{(p')} = 5(6p-4), \quad m_0^{(p')} = 5(6p-6)\,. \end{aligned}$$

By applying Corollary 1 (8) and (9), we have the second and the third identities. □

For example, since

$$\{(x_2, x_3) \mid 2x_2 + 3x_3 = 19,\ x_2, x_3 \geq 0\} = \{(2,5), (5,3), (8,1)\}\,,$$

we see that

$$\begin{aligned}&\{(x_1, x_2, x_3) \mid 6x_1 + 5(2x_2 + 3x_3) = 89,\ x_1, x_2, x_3 \geq 0\}\\&= \{(4,2,3), (4,5,1), (9,2,1)\}\,,\\&\{(x_1, x_2, x_3) \mid 6x_1 + 5(2x_2 + 3x_3) = 95,\ x_1, x_2, x_3 \geq 0\}\\&= \{(5,2,3), (5,5,1), (10,2,1), (0,2,5), (0,5,3), (0,8,1)\}\,.\end{aligned}$$

We can also obtain expressions of weighted sums for this triple (t_3, t_4, t_5). Finally, we leave one of our expectations:

$$g(t_n, t_{n+1}, t_{n+2}; 1) = n^3 + 3n^2 + 2n - 1 \quad (n \geq 2)\,.$$

But we have not expected anything for general (t_n, t_{n+1}, t_{n+2}).

References

[1] L. Bardomero and M. Beck, *Frobenius coin-exchange generating functions*, Amer. Math. Monthly **127** (2020), no. 4, 308–315.

[2] M. Beck, I. M. Gessel and T. Komatsu, *The polynomial part of a restricted partition function related to the Frobenius problem*, Electron. J. Combin. **8** (No.1) (2001), #N7.

[3] M. Beck and C. Kifer, *An extreme family of generalized Frobenius numbers*, Integers **11** (2011), A24, 639–645.

[4] M. Beck and S. Robins, *A formula related to the Frobenius problem in two dimensions*, Number theory(New York, 2003), Springer, New York, 2004, pp. 17–23, arXiv:math.NT/0204037.

[5] A. Brauer, *On a problem of partitions*, Amer. J. Math. **64** (1942), 299–312.

[6] A. Brauer and B. M. Shockley, *On a problem of Frobenius*, J. Reine. Angew. Math. **211** (1962), 215–220.

[7] T. C. Brown and P. J. Shiue, *A remark related to the Frobenius problem*, Fibonacci Quart. **31** (1993), 32–36.

[8] L. Comtet, *Advanced Combinatorics*, Rev. and enlarged ed., Dordrecht, D. Reidel Publishing Company, (1974).

[9] F. Curtis, *On formulas for the Frobenius number of a numerical semigroup*, Math. Scand. **67** (1990), 190–192.

[10] L. G. Fel, T. Komatsu and A. I. Suriajaya, *A sum of negative degrees of the gaps values in* 2 *and* 3*-generated numerical semigroup*, Ann. Math. Inform. **52** (2020), 85–95. DOI: 10.33039/ami.2020.08.001

[11] L. G. Fel and B. Y. Rubinstein, *Power sums related to semigroups* $S(d_1, d_2, d_3)$, Semigroup Forum **74** (2007), 93–98.

[12] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, Reading MA, 1988.

[13] D. D. Grant, *On linear forms whose coefficients are in arithmetic progression*, Israel J. Math. **15** (1973), 204–209.

[14] S. M. Johnson, *A Linear Diophantine problem*, Canad. J. Math. **12** (1960), 390–398.

[15] T. Komatsu, *On the number of solutions of the Diophantine equation of Frobenius–General case*, Math. Commun. **8** (2003), 195–206.

[16] T. Komatsu and Y. Zhang, *Weighted Sylvester sums on the Frobenius set*, Irish Math. Soc. Bull. **87** (2021), 21–29.

[17] T. Komatsu and Y. Zhang, *Weighted Sylvester sums on the Frobenius set in more variables*, Kyushu J. Math. **76** (2022), no.1 (to appear). arXiv:2101.04298

[18] P. Moree, *Numerical semigroups, cyclotomic polynomials, and Bernoulli numbers*, Am. Math. Mon. **121** (2014), 890–902.

[19] D. C. Ong and V. Ponomarenko, *The Frobenius number of geometric sequences*, Integers **8** (2008), Article A33, 3 p.

[20] P. Punyani and A. Tripathi, *On changes in the Frobenius and Sylvester numbers*, Integers **18B** (2018), #A8, 12 p.

[21] J. L. Ramìrez Alfonsìn, *The Diophantine Frobenius Problem*, Oxford University Press, Oxford, 2005.

[22] J. B. Roberts, *Notes on linear forms*, Proc. Amer. Math. Soc. **7** (1956), 465–469.

[23] A. M. Robles-Pérez and J. C. Rosales, *The Frobenius number for sequences of triangular and tetrahedral numbers*, J. Number Theory **186** (2018), 473–492.

[24] Ø. J. Rødseth, *A note on Brown and Shiue's paper on a remark related to the Frobenius problem*, Fibonacci Quart. **32** (1994), 407–408.

[25] E. S. Selmer, *On the linear diophantine problem of Frobenius*, J. Reine Angew. Math. **293/294** (1977), 1–17.

[26] J. Shallit and J. Stankewicz, *Unbounded discrepancy in Frobenius numbers*, Integers **11** (2011), A2, 27-34.

[27] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, available at oeis.org. (2021).

[28] J. J. Sylvester, *On subinvariants, i.e. semi-invariants to binary quantics of an unlimited order*, Am. J. Math. **5** (1882), 119–136.

[29] J. J. Sylvester, *Mathematical questions with their solutions*, Educational Times **41** (1884), 21.

[30] A. Tripathi, *The number of solutions to $ax + by = n$*, Fibonacci Quart. **38** (2000), 290–293.

[31] A. Tripathi, *On sums of positive integers that are not of the form $ax+by$*, Amer. Math. Monthly **115** (2008), 363–364.

[32] A. Tripathi, *Formulae for the Frobenius number in three variables*, J. Number Theory **170** (2017), 368–389.

[33] A. Tripathi, *On a special case of the Frobenius problem*, J. Integer Seq. **20** (2017), Article 17.7.2, 12 pp.

[34] H. J. H. Tuenter, *The Frobenius problem, sums of powers of integers, and recurrences for the Bernoulli numbers*, J. Number Theory **117** (2006), 376–386.

[35] W. Wang and T. Wang, *Alternate Sylvester sums on the Frobenius set*, Comput. Math. Appl. **56** (2008), 1328–1334.