第十二章 重积分

例 1 设$K(x,y)$在$[0,1]\times[0,1]$上连续且恒大于0, $f(x)$和$g(x)$在$[0,1]$上连续且恒大于0, 对任意$x\in[0,1]$, 有

$$\int_0^1 f(y)K(x,y)\mathrm{d}y=g(x),\quad \int_0^1 g(y)K(x,y)\mathrm{d}y=f(x),$$

求证：对任意$x\in[0,1]$, 有$f(x)=g(x)$.

证 对任意$x\in[0,1]$, 有

$$f(x)=\int_0^1 g(t)K(x,t)\mathrm{d}t=\int_0^1\int_0^1 f(y)K(t,y)K(x,t)\mathrm{d}y\mathrm{d}t=\int_0^1 f(y)L(x,y)\mathrm{d}y,$$

其中

$$L(x,y)=\int_0^1 K(x,t)K(t,y)\mathrm{d}t,\ \forall(x,y)\in[0,1]^2.$$

类似地，对任意$x\in[0,1]$, 有

$$g(x)=\int_0^1 g(y)L(x,y)\mathrm{d}y.$$

因为对任意$x\in[0,1]$, 有

$$\frac{g(x)}{f(x)}=\int_0^1\frac{g(y)L(x,y)}{f(x)}\mathrm{d}y=\int_0^1\frac{g(y)}{f(y)}\cdot\frac{f(y)L(x,y)}{f(x)}\mathrm{d}y$$

和

$$\int_0^1\frac{f(y)L(x,y)}{f(x)}\mathrm{d}y=\frac{f(x)}{f(x)}=1,$$

所以$\dfrac{g(x)}{f(x)}$恒为常数. 设$\dfrac{g(x)}{f(x)}\equiv C$, 则

$$g(x)=Cf(x)=C\int_0^1 g(y)K(x,y)\mathrm{d}y=C^2\int_0^1 f(y)K(x,y)\mathrm{d}y=C^2g(x),$$

因此$C=1$. 于是对任意$x\in[0,1]$, 有$f(x)=g(x)$. □

例 2 用$\exp(x)$表示e^x, 令

$$F(x)=\frac{x^4}{\exp(x^3)}\int_0^x\int_0^{x-u}\exp(u^3+v^3)\mathrm{d}u\mathrm{d}v,$$

求$\lim\limits_{x\to+\infty}F(x)$或者证明它不存在.

解 计算可得 $\lim\limits_{x\to+\infty}F(x)=\dfrac{2}{9}$. 具体计算如下：作变量替换$u=\dfrac{t+s}{2}$, $v=\dfrac{t-s}{2}$, 则

$$\int_0^x\int_0^{x-u}\exp(u^3+v^3)\mathrm{d}u\mathrm{d}v=\int_0^x\int_{-t}^{t}\exp\left(\frac{1}{4}t^3+\frac{3}{4}ts^2\right)\cdot\frac{1}{2}\mathrm{d}s\mathrm{d}t.$$

由洛必达法则得

$$\begin{aligned}
&\lim_{x\to+\infty}F(x)\\
=&\lim_{x\to+\infty}\frac{\int_0^x\int_{-t}^{t}\exp\left(\frac{1}{4}t^3+\frac{3}{4}ts^2\right)\mathrm{d}s\mathrm{d}t}{2x^{-4}\exp(x^3)}\\
=&\lim_{x\to+\infty}\frac{\int_{-x}^{x}\exp\left(\frac{1}{4}x^3+\frac{3}{4}xs^2\right)\mathrm{d}s}{(6x^{-2}-8x^{-5})\exp(x^3)}\\
=&\lim_{x\to+\infty}\frac{\int_{-x}^{x}\exp\left(\frac{3}{4}xs^2\right)\mathrm{d}s}{(6x^{-2}-8x^{-5})\exp(3x^3/4)}\\
=&\lim_{x\to+\infty}\frac{\frac{1}{\sqrt{x}}\int_{x\sqrt{x}}^{x\sqrt{x}}\exp(3z^2/4)\mathrm{d}z}{(6x^{-2}-8x^{-5})\exp(3x^3/4)}\ (z=s\sqrt{x})\\
=&\lim_{x\to+\infty}\frac{\int_{x\sqrt{x}}^{x\sqrt{x}}\exp(3z^2/4)\mathrm{d}z}{(6x^{-3/2}-8x^{-9/2})\exp(3x^3/4)}\\
=&\lim_{x\to+\infty}\frac{2\cdot\frac{3}{2}\sqrt{x}\cdot\exp(3x^3/4)}{\left[\frac{27}{2}\sqrt{x}+\cdots\right]\exp(3x^3/4)}\\
=&\frac{2}{9}.
\end{aligned}$$

□

例 3 设$f(x,y)$在$D=[0,1]\times[0,1]$上四次连续可微，在∂D上$f(x,y)$恒等于0, 对任意$(x,y)\in D$, 都有$\left|\dfrac{\partial^4 f}{\partial x^2\partial y^2}(x,y)\right|\leqslant 3$, 求证：

$$\left|\iint_D f(x,y)\mathrm{d}x\mathrm{d}y\right|\leqslant\frac{1}{48}.$$

解 首先证明

$$\iint_D f(x,y)\mathrm{d}x\mathrm{d}y=\frac{1}{4}\iint_D x(1-x)y(1-y)\frac{\partial^4 f}{\partial x^2\partial y^2}(x,y)\mathrm{d}x\mathrm{d}y.\tag{$*$}$$

由分部积分法，有

$$\begin{aligned}
&\int_0^1 f(x,y)\mathrm{d}x=\left(x-\frac{1}{2}\right)f(x,y)\Big|_{x=0}^{1}-\int_0^1\left(x-\frac{1}{2}\right)f'_x(x,y)\mathrm{d}x\\
=&-\int_0^1\left(x-\frac{1}{2}\right)f'_x(x,y)\mathrm{d}x=-\frac{1}{2}x(x-1)f'_x(x,y)\Big|_{x=0}^{1}+\frac{1}{2}\int_0^1 x(x-1)f''_{xx}(x,y)\mathrm{d}x\\
=&-\frac{1}{2}\int_0^1 x(1-x)f''_{xx}(x,y)\mathrm{d}x.
\end{aligned}$$

由$f(x,y)$在∂D上恒等于0知$f''_{xx}(x,0)\equiv 0$, $f''_{xx}(x,1)\equiv 0$, 于是由分部积分法，有

$$
\begin{aligned}
&\int_0^1 f''_{xx}(x,y)\mathrm{d}y = \left(y-\frac{1}{2}\right) f''_{xx}(x,y)\Big|_{y=0}^{1} - \int_0^1 \left(y-\frac{1}{2}\right)\frac{\partial^3 f}{\partial x^2 \partial y}(x,y)\mathrm{d}y \\
= & -\int_0^1 \left(y-\frac{1}{2}\right)\frac{\partial^3 f}{\partial x^2 \partial y}(x,y)\mathrm{d}y = -\frac{1}{2}y(y-1)\frac{\partial^3 f}{\partial x^2 \partial y}(x,y)\Big|_{y=0}^{1} + \frac{1}{2}\int_0^1 y(y-1)\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y)\mathrm{d}y \\
= & -\frac{1}{2}\int_0^1 y(1-y)\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y)\mathrm{d}y.
\end{aligned}
$$

因此有

$$
\begin{aligned}
&\iint_D f(x,y)\mathrm{d}x\mathrm{d}y = \int_0^1 \mathrm{d}y\int_0^1 f(x,y)\mathrm{d}x \\
= & -\frac{1}{2}\int_0^1 \mathrm{d}y\int_0^1 x(1-x)f''_{xx}(x,y)\mathrm{d}x = -\frac{1}{2}\int_0^1 x(1-x)\mathrm{d}x\int_0^1 f''_{xx}(x,y)\mathrm{d}y \\
= & \frac{1}{4}\int_0^1 x(1-x)\mathrm{d}x\int_0^1 y(1-y)\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y)\mathrm{d}y = \frac{1}{4}\iint_D x(1-x)y(1-y)\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y)\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

这样就证明了$(*)$式，进一步就得到

$$
\begin{aligned}
&\left|\iint_D f(x,y)\mathrm{d}x\mathrm{d}y\right| \\
= & \frac{1}{4}\left|\iint_D x(1-x)y(1-y)\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y)\mathrm{d}x\mathrm{d}y\right| \\
\leqslant & \frac{1}{4}\iint_D x(1-x)y(1-y)\left|\frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y)\right|\mathrm{d}x\mathrm{d}y \\
\leqslant & \frac{3}{4}\iint_D x(1-x)y(1-y)\mathrm{d}x\mathrm{d}y \\
= & \frac{3}{4}\left(\int_0^1 x(1-x)\mathrm{d}x\right)^2 \\
= & \frac{1}{48}.
\end{aligned}
$$

□

例 4 设$f(x,y)$在$\mathbb{R}^2$上连续，对任意面积为1的长方形区域R, 都有$\iint_R f(x,y)\mathrm{d}x\mathrm{d}y = 0$, 问$f(x,y)$是否一定恒等于0？证明你的结论.

解 $f(x,y)$必恒等于0. 实际上，只需对任意面积为1且两边分别平行于坐标轴的长方形区域R, 都有$\iint_R f(x,y)\mathrm{d}x\mathrm{d}y = 0$, 就可以证得$f(x,y)$必恒等于0. 先证明一个引理：“设$f(x,y)$在$\mathbb{R}^2$上连续，对任意面积为1且两边分别平行于坐标轴的长方形区域R, 都有$\iint_R f(x,y)\mathrm{d}x\mathrm{d}y = 0$,

则$f(x,y)$在R的有公共顶点的两条边上有相等的积分均值”．引理的证明如下：设R的顶点坐标分别是(x,y), $(x+c,y)$, $\left(x,y+\frac{1}{c}\right)$, $\left(x+c,y+\frac{1}{c}\right)$, 其中$c>0$, 不失一般性，考虑顶点$\left(x+c,y+\frac{1}{c}\right)$所在的两条边. 令$F(s,t)=\int_0^s\int_0^t f(u,v)\mathrm{d}u\mathrm{d}v$, 则$F_s'(s,t)=\int_0^t f(s,v)\mathrm{d}v$, $F_t'(s,t)=\int_0^s f(u,t)\mathrm{d}u$. 由$\iint_R f(x,y)\mathrm{d}x\mathrm{d}y=0$得

$$F\left(x+c,y+\frac{1}{c}\right)-F(x+c,y)-F\left(x,y+\frac{1}{c}\right)+F(x,y)=0.$$

上式对任意$c>0$都成立，两边对c求导，得

$$F_s'\left(x+c,y+\frac{1}{c}\right)-\frac{1}{c^2}F_t'\left(x+c,y+\frac{1}{c}\right)-F_s'(x+c,y)+\frac{1}{c^2}F_t'\left(x,y+\frac{1}{c}\right)=0,$$

即

$$\int_0^{y+\frac{1}{c}}f(x+c,v)\mathrm{d}v-\frac{1}{c^2}\int_0^{x+c}f\left(u,y+\frac{1}{c}\right)\mathrm{d}u-\int_0^{y}f(x+c,v)\mathrm{d}v+\frac{1}{c^2}\int_0^{x}f\left(u,y+\frac{1}{c}\right)\mathrm{d}u=0,$$

化简整理得

$$c\int_y^{y+\frac{1}{c}}f(x+c,v)\mathrm{d}v=\frac{1}{c}\int_x^{x+c}f\left(u,y+\frac{1}{c}\right)\mathrm{d}u.$$

这就是说，$f(x,y)$在R的顶点$\left(x+c,y+\frac{1}{c}\right)$所在的两条边有相等的积分均值.

回到原问题，任意固定$c>0$, 用顶点属于集合$\left\{\left(mc,\frac{n}{c}\right)|m,n\in\mathbb{Z}\right\}$且面积为1的长方形平铺整个平面. 反复使用上面的引理，得

$$\int_0^c f(u,0)\mathrm{d}u=\int_{mc}^{(m+1)c}f(u,0)\mathrm{d}u,\ \forall m\in\mathbb{N}^*.$$

用$\frac{c}{m}$代替c, 得

$$\int_0^{\frac{c}{m}}f(u,0)\mathrm{d}u=\int_c^{c+\frac{c}{m}}f(u,0)\mathrm{d}u,$$

从而得到

$$f(0,0)=\lim_{m\to\infty}\frac{m}{c}\int_0^{\frac{c}{m}}f(u,0)\mathrm{d}u=\lim_{m\to\infty}\frac{m}{c}\int_c^{c+\frac{c}{m}}f(u,0)\mathrm{d}u=f(c,0).$$

同理可证$f(0,0)=f(-c,0)$, 由c的任意性知$f(x,y)$在直线$y=0$上恒为常数. 类似地可以证明$f(x,y)$在任何平行于坐标轴的直线上恒为常数，由此可知$f(x,y)$是$\mathbb{R}^2$上的常数函数，再由$\iint_R f(x,y)\mathrm{d}x\mathrm{d}y=0$即知$f(x,y)$必恒等于0. □

例 5 设$f(x,y)$在$[0,1]\times[0,1]$上两次连续可微，求极限

$$\lim_{n\to\infty} n^2\left(\int_0^1\int_0^1 f(x,y)\mathrm{d}x\mathrm{d}y-\frac{1}{n^2}\sum_{j=1}^n\sum_{k=1}^n f\left(\frac{j-\frac{1}{2}}{n},\frac{k-\frac{1}{2}}{n}\right)\right).$$

证 记$x_i=y_i=\dfrac{i}{n}, i=0,1,\cdots,n, x_i^*=y_i^*=\dfrac{i-\frac{1}{2}}{n}, i=1,2,\cdots,n, D_{jk}=[x_{j-1},x_j]\times[y_{k-1},y_k]$, $j,k=1,2,\cdots,n$, 则

$$\begin{aligned}&\int_0^1\int_0^1 f(x,y)\mathrm{d}x\mathrm{d}y-\frac{1}{n^2}\sum_{j=1}^n\sum_{k=1}^n f\left(\frac{j-\frac{1}{2}}{n},\frac{k-\frac{1}{2}}{n}\right)\\ =\ &\sum_{j=1}^n\sum_{k=1}^n\iint_{D_{jk}}[f(x,y)-f(x_j^*,y_k^*)]\mathrm{d}x\mathrm{d}y.\end{aligned}$$

由泰勒公式得

$$\begin{aligned}&f(x,y)-f(x_j^*,y_k^*)\\ =\ &\frac{\partial f}{\partial x}(x_j^*,y_k^*)(x-x_j^*)+\frac{\partial f}{\partial y}(x_j^*,y_k^*)(y-y_k^*)+\frac{1}{2}\cdot\frac{\partial^2 f}{\partial x^2}(\xi_j(x),\eta_k(y))(x-x_j^*)^2\\ &+\frac{\partial^2 f}{\partial x\partial y}(\xi_j(x),\eta_k(y))(x-x_j^*)(y-y_k^*)+\frac{1}{2}\cdot\frac{\partial^2 f}{\partial y^2}(\xi_j(x),\eta_k(y))(y-y_k^*)^2,\end{aligned}$$

其中$(\xi_j(x),\eta_k(y))$在连结(x,y)与(x_j^*,y_k^*)的线段上. 注意到$\iint_{D_{jk}}(x-x_j^*)\mathrm{d}x\mathrm{d}y=\iint_{D_{jk}}(y-y_k^*)\mathrm{d}x\mathrm{d}y=0$, 就有

$$\begin{aligned}&\iint_{D_{jk}}[f(x,y)-f(x_j^*,y_k^*)]\mathrm{d}x\mathrm{d}y\\ =\ &\iint_{D_{jk}}\Big[\frac{1}{2}\cdot\frac{\partial^2 f}{\partial x^2}(\xi_j(x),\eta_k(y))(x-x_j^*)^2+\frac{\partial^2 f}{\partial x\partial y}(\xi_j(x),\eta_k(y))(x-x_j^*)(y-y_k^*)\\ &\qquad+\frac{1}{2}\cdot\frac{\partial^2 f}{\partial y^2}(\xi_j(x),\eta_k(y))(y-y_k^*)^2\Big]\mathrm{d}x\mathrm{d}y.\end{aligned}$$

因为$f(x,y)$在$[0,1]\times[0,1]$上两次连续可微，所以$f(x,y)$的所有二阶偏导数在$[0,1]\times[0,1]$上一

致连续. 由此可以证明（请自证）

$$\begin{aligned}
&\lim_{n\to\infty} n^2 \sum_{j=1}^{n}\sum_{k=1}^{n} \iint_{D_{jk}} \Big[\frac{1}{2}\cdot\frac{\partial^2 f}{\partial x^2}(\xi_j(x),\eta_k(y))(x-x_j^*)^2 + \frac{\partial^2 f}{\partial x\partial y}(\xi_j(x),\eta_k(y))(x-x_j^*)(y-y_k^*)\\
&\qquad + \frac{1}{2}\cdot\frac{\partial^2 f}{\partial y^2}(\xi_j(x),\eta_k(y))(y-y_k^*)^2\Big]\mathrm{d}x\mathrm{d}y\\
=\ &\lim_{n\to\infty} n^2 \sum_{j=1}^{n}\sum_{k=1}^{n} \iint_{D_{jk}} \Big[\frac{1}{2}\cdot\frac{\partial^2 f}{\partial x^2}(x_j^*,y_k^*)(x-x_j^*)^2 + \frac{\partial^2 f}{\partial x\partial y}(x_j^*,y_k^*)(x-x_j^*)(y-y_k^*)\\
&\qquad + \frac{1}{2}\cdot\frac{\partial^2 f}{\partial y^2}(x_j^*,y_k^*)(y-y_k^*)^2\Big]\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

计算得到

$$\begin{aligned}
&\iint_{D_{jk}} (x-x_j^*)^2\mathrm{d}x\mathrm{d}y = \frac{1}{12n^4},\\
&\iint_{D_{jk}} (x-x_j^*)(y-y_k^*)\mathrm{d}x\mathrm{d}y = 0,\\
&\iint_{D_{jk}} (y-y_k^*)^2\mathrm{d}x\mathrm{d}y = \frac{1}{12n^4},
\end{aligned}$$

于是

$$\begin{aligned}
&\lim_{n\to\infty} n^2 \sum_{j=1}^{n}\sum_{k=1}^{n} \iint_{D_{jk}} \Big[\frac{1}{2}\cdot\frac{\partial^2 f}{\partial x^2}(x_j^*,y_k^*)(x-x_j^*)^2 + \frac{\partial^2 f}{\partial x\partial y}(x_j^*,y_k^*)(x-x_j^*)(y-y_k^*)\\
&\qquad + \frac{1}{2}\cdot\frac{\partial^2 f}{\partial y^2}(x_j^*,y_k^*)(y-y_k^*)^2\Big]\mathrm{d}x\mathrm{d}y\\
=\ &\lim_{n\to\infty} \sum_{j=1}^{n}\sum_{k=1}^{n} \left[\frac{1}{24}\cdot\frac{\partial^2 f}{\partial x^2}(x_j^*,y_k^*)\cdot\frac{1}{n^2} + \frac{1}{24}\cdot\frac{\partial^2 f}{\partial y^2}(x_j^*,y_k^*)\cdot\frac{1}{n^2}\right]\\
=\ &\frac{1}{24}\iint_{[0,1]^2}\left[\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y)\right]\mathrm{d}x\mathrm{d}y,
\end{aligned}$$

即

$$\begin{aligned}
&\lim_{n\to\infty} n^2\left(\int_0^1\int_0^1 f(x,y)\mathrm{d}x\mathrm{d}y - \frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n} f\left(\frac{j-\frac{1}{2}}{n},\frac{k-\frac{1}{2}}{n}\right)\right)\\
=\ &\frac{1}{24}\iint_{[0,1]^2}\left[\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y)\right]\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

□

例 6 设D_1, D_2, $\cdots$, D_n是平面上n个闭圆盘，用a_{ij}来记$D_i\cap D_j$的面积，$i,j=1,2\cdots,n$, 求证：对任意给定的实数$x_1,x_2,\cdots,x_n$, 有

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j \geqslant 0.$$

证 更一般地，下面证明当D_1, D_2, $\cdots$, D_n都是平面上若尔当可测的有界集时命题成立. 取定长方形H使得$D_i \subseteq H$, $i=1,2,\cdots,n$, 对于H的子集E, 用

$$\chi_E(x,y)=\begin{cases}1, & (x,y)\in E,\\ 0, & (x,y)\in H\setminus E,\end{cases}$$

来记E的特征函数在H上的限制，则有

$$a_{ij}=V_J(D_i\cap D_j)=\iint_H \chi_{D_i\cap D_j}(x,y)\mathrm{d}x\mathrm{d}y=\iint_H \chi_{D_i}(x,y)\chi_{D_j}(x,y)\mathrm{d}x\mathrm{d}y,$$

其中$i,j=1,2\cdots,n$. 因此

$$\begin{aligned}
&\sum_{i=1}^n\sum_{j=1}^n a_{ij}x_ix_j\\
=\ &\sum_{i=1}^n\sum_{j=1}^n x_ix_j\iint_H \chi_{D_i}(x,y)\chi_{D_j}(x,y)\mathrm{d}x\mathrm{d}y\\
=\ &\iint_H\left(\sum_{i=1}^n\sum_{j=1}^n x_i\chi_{D_i}(x,y)x_j\chi_{D_j}(x,y)\right)\mathrm{d}x\mathrm{d}y\\
=\ &\iint_H\left(\sum_{i=1}^n x_i\chi_{D_i}(x,y)\right)^2\mathrm{d}x\mathrm{d}y\\
\geqslant\ &0.
\end{aligned}$$

□

例 7 一个矩形被分割成若干个更小的矩形, 每个小矩形的长和宽中至少有一个是正整数. 证明: 该矩形的长和宽中也至少有一个是正整数.

证 将矩形T放入直角坐标系，使得矩形T的各边平行于坐标轴且左下顶点是坐标原点，设$T=[0,b]\times[0,d]$. 用T_i记分割成的小矩形，设$T_i=[a_i,b_i]\times[c_i,d_i]$, $i=1,2,\cdots,n$. 因为每个小矩形T_i的长和宽中至少有一个是正整数，即b_i-a_i或d_i-c_i是正整数，所以

$$\begin{aligned}
&\iint_{T_i}\sin 2\pi x\cdot\sin 2\pi y\mathrm{d}x\mathrm{d}y\\
=\ &\int_{a_i}^{b_i}\sin 2\pi x\mathrm{d}x\int_{c_i}^{d_i}\sin 2\pi y\mathrm{d}y\\
=\ &\frac{(\cos 2\pi a_i-\cos 2\pi b_i)(\cos 2\pi c_i-\cos 2\pi d_i)}{4\pi^2}\\
=\ &0,
\end{aligned}$$

其中$i=1,2,\cdots,n$. 于是由重积分的区域可加性，有

$$\iint_T \sin 2\pi x\cdot\sin 2\pi y\mathrm{d}x\mathrm{d}y=\sum_{i=1}^{n}\iint_{T_i}\sin 2\pi x\cdot\sin 2\pi y\mathrm{d}x\mathrm{d}y=0.$$

因为

$$\iint_T \sin 2\pi x\cdot\sin 2\pi y\mathrm{d}x\mathrm{d}y=\frac{(1-\cos 2\pi b)(1-\cos 2\pi d)}{4\pi^2},$$

所以由$\iint_T \sin 2\pi x\cdot\sin 2\pi y\mathrm{d}x\mathrm{d}y=0$知$\cos 2\pi b=1$或$\cos 2\pi d=1$, 从而$b$或$d$是正整数. 因此，矩形$T$的长和宽中至少有一个是正整数. □

例 8 设$f(x,y)$在$[0,1]\times[0,1]$上连续，求证：

$$\begin{aligned}&\int_0^1\left(\int_0^1 f(x,y)\mathrm{d}x\right)^2\mathrm{d}y+\int_0^1\left(\int_0^1 f(x,y)\mathrm{d}y\right)^2\mathrm{d}x\\ \leqslant\ &\left(\int_0^1\int_0^1 f(x,y)\mathrm{d}x\mathrm{d}y\right)^2+\int_0^1\int_0^1[f(x,y)]^2\mathrm{d}x\mathrm{d}y.\end{aligned}$$

证 要证的不等式等价于

$$\int_0^1\int_0^1\int_0^1\int_0^1 F(x,y,z,w)^2\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}w\geqslant 0,$$

其中

$$F(x,y,z,w)=f(x,y)+f(z,w)-f(x,w)-f(z,y).$$

由被积函数的非负性即证. □

另证 因为定积分和重积分都是积分和数的极限, 所以只需证明相应的离散型不等式: 对任何实数x_{ij}, $i,j=1,\cdots,n$, 有

$$n\sum_{i=1}^{n}\left(\sum_{j=1}^{n}x_{ij}\right)^2+n\sum_{j=1}^{n}\left(\sum_{i=1}^{n}x_{ij}\right)^2\leqslant\left(\sum_{i=1}^{n}\sum_{j=1}^{n}x_{ij}\right)^2+n^2\sum_{i=1}^{n}\sum_{j=1}^{n}x_{ij}^2.$$

注意到上式中右边与左边的差为

$$\frac{1}{4}\sum_{i,j,k,l=1}^{n}(x_{ij}+x_{kl}-x_{il}-x_{kj})^2,$$

显然这是非负的，故上式成立. 这就完成了证明. □

例 9 设$R=\left\{(x,y,z)\middle|x\geqslant 0,y\geqslant 0,z\geqslant 0,x+y+z\leqslant 1\right\}$, $w=1-x-y-z$, 将三重积分$\iiint_R x^1y^9z^8w^4\mathrm{d}x\mathrm{d}y\mathrm{d}z$表示为$\dfrac{a!b!c!d!}{n!}$的形式，其中$a$, b, c, d和n都是正整数.

解 作变量替换$x=u(1-v)$, $y=uv(1-t)$, $z=uvt$, 则积分区域R化为

$$R'=\left\{(u,v,t)\middle|0\leqslant u\leqslant 1,0\leqslant v\leqslant 1,0\leqslant t\leqslant 1\right\},$$

直接计算得到$\dfrac{D(x,y,z)}{D(u,v,t)}=u^2v$, 于是

$$\begin{aligned}&\iiint_R x^1y^9z^8w^4\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ =&\iiint_{R'} u(1-v)\cdot u^9v^9(1-t)^9\cdot u^8v^8t^8\cdot(1-u)^4\cdot u^2v\mathrm{d}u\mathrm{d}v\mathrm{d}t\\ =&\int_0^1 u^{20}(1-u)^4\mathrm{d}u\int_0^1 v^{18}(1-v)\mathrm{d}v\int_0^1 t^8(1-t)^9\mathrm{d}t.\end{aligned}$$

由分部积分法推导递推关系，不难得到$\displaystyle\int_0^1 x^m(1-x)^n\mathrm{d}x=\frac{m!n!}{(m+n+1)!}$, 其中$m$和$n$是自然数. 因此

$$\iiint_R x^1y^9z^8w^4\mathrm{d}x\mathrm{d}y\mathrm{d}z=\frac{20!4!}{25!}\cdot\frac{18!1!}{20!}\cdot\frac{8!9!}{18!}=\frac{1!9!8!4!}{25!}.$$

□

注 学习了贝塔函数与伽玛函数后，按定义知$\displaystyle\int_0^1 x^m(1-x)^n\mathrm{d}x$就是$B(m+1,n+1)$, 由贝塔函数与伽玛函数的性质得$B(m+1,n+1)=\dfrac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}=\dfrac{m!n!}{(m+n+1)!}$.

例 10 计算积分

$$\int_0^1\int_0^1\int_0^1\frac{1}{(1+x^2+y^2+z^2)^2}\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

解 用I来记要计算的积分，令$D=\left\{(x,y,z)\middle|0\leqslant y\leqslant x\leqslant 1,0\leqslant z\leqslant 1\right\}$, 则由对称性得

$$I=2\iiint_D\frac{1}{(1+x^2+y^2+z^2)^2}\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

作柱坐标变换$x=r\cos\theta$, $y=r\sin\theta$, $z=z$, 则区域D对应于

$$D'=\left\{(r,\theta,z)\middle|0\leqslant\theta\leqslant\frac{\pi}{4},0\leqslant r\leqslant\sec\theta,0\leqslant z\leqslant 1\right\},$$

从而有

$$
\begin{aligned}
I &= 2\iiint_D \frac{1}{(1+x^2+y^2+z^2)^2}\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= 2\iiint_{D'} \frac{1}{(1+r^2+z^2)^2} r\mathrm{d}r\mathrm{d}\theta\mathrm{d}z \\
&= 2\int_0^{\frac{\pi}{4}}\mathrm{d}\theta\int_0^1\mathrm{d}z\int_0^{\sec\theta}\frac{r}{(1+r^2+z^2)^2}\mathrm{d}r \\
&= 2\int_0^{\frac{\pi}{4}}\mathrm{d}\theta\int_0^1 \frac{-1}{2(1+r^2+z^2)}\Big|_{r=0}^{r=\sec\theta}\mathrm{d}z \\
&= \int_0^{\frac{\pi}{4}}\mathrm{d}\theta\int_0^1\frac{\sec^2\theta}{(1+z^2)(1+z^2+\sec^2\theta)}\mathrm{d}z.
\end{aligned}
$$

令$z=\tan\varphi$换元，得

$$\int_0^1\frac{\sec^2\theta}{(1+z^2)(1+z^2+\sec^2\theta)}\mathrm{d}z=\int_0^{\frac{\pi}{4}}\frac{\sec^2\theta}{\sec^2\varphi+\sec^2\theta}\mathrm{d}\varphi.$$

于是记$\Omega=\left\{(x,y)\middle|x,y\in\left[0,\frac{\pi}{4}\right]\right\}$, 就有

$$I=\int_0^{\frac{\pi}{4}}\mathrm{d}\theta\int_0^{\frac{\pi}{4}}\frac{\sec^2\theta}{\sec^2\varphi+\sec^2\theta}\mathrm{d}\varphi=\iint_\Omega\frac{\sec^2\theta}{\sec^2\varphi+\sec^2\theta}\mathrm{d}\theta\mathrm{d}\varphi.$$

由对称性可见

$$I=\iint_\Omega\frac{\sec^2\varphi}{\sec^2\theta+\sec^2\varphi}\mathrm{d}\theta\mathrm{d}\varphi.$$

相加，得

$$2I=\iint_\Omega\left(\frac{\sec^2\theta}{\sec^2\varphi+\sec^2\theta}+\frac{\sec^2\varphi}{\sec^2\varphi+\sec^2\theta}\right)\mathrm{d}\theta\mathrm{d}\varphi=\iint_\Omega\mathrm{d}\theta\mathrm{d}\varphi=\frac{\pi^2}{16}.$$

故

$$I=\frac{\pi^2}{32}.$$

□

例 11 设A是4阶正定对称方阵，记$X=(x_1,x_2,x_3,x_4)$, $h(X)=XAX^T$, 令$D=\left\{X\in\mathbb{R}^4\middle|h(X)\leqslant 1\right\}$, 计算重积分$\displaystyle\int_D \mathrm{e}^{h(X)}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4$.

解　存在四阶正交矩阵P, 使得$PAP^T=\begin{pmatrix}\lambda_1&0&0&0\\0&\lambda_2&0&0\\0&0&\lambda_3&0\\0&0&0&\lambda_4\end{pmatrix}$, 其中$\lambda_i>0$, $i=1,2,3,4$, 记$Y=(y_1,y_2,y_3,y_4)$, 作变量替换$X=YP$, 就得到

$$\int_D \mathrm{e}^{h(X)}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4=\int_{D_1}\mathrm{e}^{\lambda_1y_1^2+\lambda_2y_2^2+\lambda_3y_3^2+\lambda_4y_4^2}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3\mathrm{d}y_4,$$

其中$D_1=\left\{Y\in\mathbb{R}^4\middle|\lambda_1y_1^2+\lambda_2y_2^2+\lambda_3y_3^2+\lambda_4y_4^2\leqslant 1\right\}$. 再作变量替换$t_i=\sqrt{\lambda_i}y_i$, $i=1,2,3,4$, 得

$$\int_{D_1}\mathrm{e}^{\lambda_1y_1^2+\lambda_2y_2^2+\lambda_3y_3^2+\lambda_4y_4^2}\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3\mathrm{d}y_4=\frac{1}{\sqrt{\lambda_1\lambda_2\lambda_3\lambda_4}}\int_{D_2}\mathrm{e}^{t_1^2+t_2^2+t_3^2+t_4^2}\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4,$$

其中$D_2=\left\{(t_1,t_2,t_3,t_4)\in\mathbb{R}^4\middle|t_1^2+t_2^2+t_3^2+t_4^2\leqslant 1\right\}$. 利用极坐标变换（解每日一题106的方法）可以计算得到

$$\int_{D_2}\mathrm{e}^{t_1^2+t_2^2+t_3^2+t_4^2}\mathrm{d}t_1\mathrm{d}t_2\mathrm{d}t_3\mathrm{d}t_4=\pi^2.$$

结合$\lambda_1\lambda_2\lambda_3\lambda_4=\det A$就得到

$$\int_D \mathrm{e}^{h(X)}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4=\frac{\pi^2}{\sqrt{\det A}}.$$ □

例 12 设$0<r\leqslant 1$, 求$\mathbb{R}^n$中有界闭区域$D_n(r)$的n维体积，这里

$$D_n(r)=\left\{(x_1,x_2,\cdots,x_n)\in[0,1]^n\middle|x_1x_2\cdots x_n\leqslant r\right\}.$$

解　用$V_n(r)$来记$D_n(r)$的n维体积，则$V_1(r)=\int_0^r\mathrm{d}x_1=r$,

$$V_2(r)=\int_0^r\mathrm{d}x_1\int_0^1\mathrm{d}x_2+\int_r^1\mathrm{d}x_1\int_0^{\frac{r}{x_1}}\mathrm{d}x_2=r-r\ln r.$$

一般地，当$n>1$时，有

$$\begin{aligned}V_n(r)&=\int_0^r\mathrm{d}x_1\int_{[0,1]^{n-1}}\mathrm{d}x_2\cdots\mathrm{d}x_{n-1}+\int_r^1\mathrm{d}x_1\int_{D_{n-1}\left(\frac{r}{x_1}\right)}\mathrm{d}x_2\cdots\mathrm{d}x_{n-1}\\&=r+\int_r^1V_{n-1}\left(\frac{r}{x_1}\right)\mathrm{d}x_1\\&=r+r\int_r^1\frac{V_{n-1}(t)}{t^2}\mathrm{d}t\ \ \left(t=\frac{r}{x_1}\right).\end{aligned}$$

用数学归纳法可以证明

$$V_n(r)=r\sum_{k=0}^{n-1}\frac{(-1)^k(\ln r)^k}{k!}.$$

(请自证) $\square$

例 13 设$f(x)$在$[0,1]$上连续，求证：

$$\lim_{n\to\infty}\int_0^1\int_0^1\cdots\int_0^1 f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n=f\left(\frac{1}{2}\right).$$

解 不妨设$f\left(\frac{1}{2}\right)=0$, 否则以$f(x)-f\left(\frac{1}{2}\right)$代替$f(x)$来讨论. 由$f(x)$的连续性知对任意$\varepsilon>0$, 存在$\delta\in\left(0,\frac{1}{2}\right)$, 使得当$\left|x-\frac{1}{2}\right|\leqslant\delta$时，有$|f(x)|<\dfrac{\varepsilon}{2}$. 由$f(x)$在$[0,1]$上连续知$f(x)$在$[0,1]$上有界，故有$M>0$使得$|f(x)|\leqslant M$. 记$X=(x_1,\cdots,x_n)$, $\sigma(X)=\dfrac{x_1+x_2+\cdots+x_n}{n}$, $D=\left\{X\in[0,1]^n\middle|\left|\sigma(X)-\frac{1}{2}\right|\leqslant\delta\right\}$, $E=\left\{X\in[0,1]^n\middle|\left|\sigma(X)-\frac{1}{2}\right|\geqslant\delta\right\}$, 就有

$$\begin{aligned}
&\left|\int_{[0,1]^n}f(\sigma(X))\mathrm{d}x_1\cdots\mathrm{d}x_n\right|\\
=&\left|\int_D f(\sigma(X))\mathrm{d}x_1\cdots\mathrm{d}x_n+\int_E f(\sigma(X))\mathrm{d}x_1\cdots\mathrm{d}x_n\right|\\
\leqslant&\int_D|f(\sigma(X))|\mathrm{d}x_1\cdots\mathrm{d}x_n+\int_E|f(\sigma(X))|\mathrm{d}x_1\cdots\mathrm{d}x_n\\
<&\int_D\frac{\varepsilon}{2}\mathrm{d}x_1\cdots\mathrm{d}x_n+\int_E M\mathrm{d}x_1\cdots\mathrm{d}x_n\\
\leqslant&\frac{\varepsilon}{2}+M\int_E\mathrm{d}x_1\cdots\mathrm{d}x_n.
\end{aligned}$$

适当放大$\displaystyle\int_E\mathrm{d}x_1\cdots\mathrm{d}x_n$, 得

$$\int_E\mathrm{d}x_1\cdots\mathrm{d}x_n\leqslant\int_E\frac{\left[\sigma(X)-\frac{1}{2}\right]^2}{\delta^2}\mathrm{d}x_1\cdots\mathrm{d}x_n<\frac{1}{\delta^2}\int_{[0,1]^n}\left[\sigma(X)-\frac{1}{2}\right]^2\mathrm{d}x_1\cdots\mathrm{d}x_n.$$

下面来计算$\displaystyle\int_{[0,1]^n}\left[\sigma(X)-\frac{1}{2}\right]^2\mathrm{d}x_1\cdots\mathrm{d}x_n$. 由对称性，有

$$\int_{[0,1]^n}\sigma(X)\mathrm{d}x_1\cdots\mathrm{d}x_n=\frac{1}{n}\cdot n\int_{[0,1]^n}x_1\mathrm{d}x_1\cdots\mathrm{d}x_n=\frac{1}{2},$$

$$\begin{aligned}
&\int_{[0,1]^n}[\sigma(X)]^2\mathrm{d}x_1\cdots\mathrm{d}x_n\\
=\ &\frac{1}{n^2}\cdot\left(n\int_{[0,1]^n}x_1^2\mathrm{d}x_1\cdots\mathrm{d}x_n+(n^2-n)\int_{[0,1]^n}x_1x_2\mathrm{d}x_1\cdots\mathrm{d}x_n\right)\\
=\ &\frac{1}{n^2}\left(\frac{n}{3}+\frac{n^2-n}{4}\right)\\
=\ &\frac{1}{4}+\frac{1}{12n},
\end{aligned}$$

从而

$$\begin{aligned}
&\int_{[0,1]^n}\left[\sigma(X)-\frac{1}{2}\right]^2\mathrm{d}x_1\cdots\mathrm{d}x_n\\
=\ &\int_{[0,1]^n}[\sigma(X)]^2\mathrm{d}x_1\cdots\mathrm{d}x_n-\int_{[0,1]^n}\sigma(X)\mathrm{d}x_1\cdots\mathrm{d}x_n+\int_{[0,1]^n}\frac{1}{4}\mathrm{d}x_1\cdots\mathrm{d}x_n\\
=\ &\frac{1}{4}+\frac{1}{12n}-\frac{1}{2}+\frac{1}{4}\\
=\ &\frac{1}{12n}.
\end{aligned}$$

由此得到$\displaystyle\int_E\mathrm{d}x_1\cdots\mathrm{d}x_n<\frac{1}{12n\delta^2}$. 对上述的$\varepsilon>0$和$\delta>0$, 存在正整数$N$, 使得当$n>N$时，有$\dfrac{M}{12n\delta^2}<\dfrac{\varepsilon}{2}$. 因此，当$n>N$时，有

$$\left|\int_{[0,1]^n}f(\sigma(X))\mathrm{d}x_1\cdots\mathrm{d}x_n\right|<\frac{\varepsilon}{2}+M\int_E\mathrm{d}x_1\cdots\mathrm{d}x_n<\frac{\varepsilon}{2}+\frac{M}{12n\delta^2}<\varepsilon.$$

按极限定义知

$$\lim_{n\to\infty}\int_{[0,1]^n}f(\sigma(X))\mathrm{d}x_1\cdots\mathrm{d}x_n=0.$$

这就完成了证明. □

补充题12

(A)

1. 计算二重积分$\displaystyle\iint_D y\mathrm{d}x\mathrm{d}y$, 其中$D=\big\{(x,y)\big|0\leqslant x\leqslant 1,\arcsin x\leqslant y\leqslant\pi-\arcsin x\big\}$.

2. 设$D=\big\{(x,y)\big|0\leqslant x\leqslant y,2y\leqslant x^2+y^2\leqslant 4y\big\}$, 将积分$\displaystyle\iint_D f(x,y)\mathrm{d}x\mathrm{d}y$化为对极坐标的累次积分.

3. 计算积分$\displaystyle\iint_D(x^2-xy-y^2)\mathrm{d}x\mathrm{d}y$, 其中$D=\{(x,y)\big||x|+|y|\leqslant 1\}$.

4. 计算积分$\int_0^1 \mathrm{d}x \int_0^{1-x} \mathrm{e}^{\frac{y}{x+y}} \mathrm{d}y$.

5. 计算积分$\iint_D \mathrm{e}^{-(x^2+xy+y^2)} \mathrm{d}x\mathrm{d}y$, 其中$D = \{(x,y) \big| x^2+xy+y^2 \leqslant 1\}$.

6. 计算二重积分$\iint_D \dfrac{x+2}{x+y+4} \mathrm{d}x\mathrm{d}y$, 其中$D = \big\{(x,y) \big| |x|+|y| \leqslant 1\big\}$.

7. 计算二重积分$\iint_D (x+y)(x+1) \mathrm{d}x\mathrm{d}y$, 其中$D = \big\{(x,y) \big| x^2+2y^2 \leqslant 1, x \leqslant 0\big\}$.

8. 设V是由平面$x = 0$, $y = 1$, $x - z = 0$, $y - z = 0$所围成的四面体区域，计算三重积分$\iiint_V x \mathrm{d}x\mathrm{d}y\mathrm{d}z$.

9. 设$V = \big\{(x,y,z) \big| x \geqslant 0, y \geqslant 0, x^2+y^2+z^2 \leqslant 1\big\}$, 计算三重积分$\iiint_V x \mathrm{d}x\mathrm{d}y\mathrm{d}z$.

10. 计算积分$\iiint_V \dfrac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{(x+y+z+1)^3}$, 其中$V = \{(x,y,z) \big| x+y+z \leqslant 1, x \geqslant 0, y \geqslant 0, z \geqslant 0\}$.

11. 设$R > 0$, $V = \big\{(x,y,z) \big| x^2+y^2+z^2 \leqslant R^2\big\}$, 计算三重积分$\iiint_V |xyz| \mathrm{d}x\mathrm{d}y\mathrm{d}z$.

12. 设A是3阶实正定对称矩阵，A的3个特征根分别是λ_1, λ_2和λ_3, 记$X = (x_1, x_2, x_3)$, 令$V = \big\{X \in \mathbb{R}^3 \big| XAX^T \leqslant 1\big\}$, 计算三重积分$\iiint_V XX^T \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$.

13. 求曲面$z = x^2+2y^2$, $z = 2-x^2$所围立体的体积.

14. 求曲面$\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\right)^2 = z^3$ $(a > 0,\ b > 0,\ c > 0)$所围成的立体的体积.

15. 求三个圆柱面$x^2+y^2 = a^2$, $x^2+z^2 = a^2$, $y^2+z^2 = a^2$所围立体的体积, 其中$a > 0$.

16. 求曲面$z = \dfrac{x^2}{2a}+\dfrac{y^2}{2b}$被柱面$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = 1$截下部分的面积，其中$a, b > 0$.

(B)

1. 设$f(x,y)$在$H = [0,1] \times [0,1]$上连续，对任意$(a,b) \in H$, 用$S(a,b)$来记以(a,b)为中心、包含于H且各边与H的边平行的最大正方形，如果总有$\iint_{S(a,b)} f(x,y) \mathrm{d}x\mathrm{d}y = 0$, 问$f(x,y)$在$H$上恒等于0吗？证明你的结论.

2. 设$f(x)$在$[a,b]$连续，在$(-\infty,a) \cup (b,+\infty)$上恒等于0, $h > 0$是一个常数，令$\varphi(x) = \dfrac{1}{2h}\int_{x-h}^{x+h} f(t) \mathrm{d}t$, 求证：

$$\int_a^b |\varphi(x)| \, \mathrm{d}x \leqslant \int_a^b |f(x)| \, \mathrm{d}x.$$

3. 设$a > 0$, $b > 0$, 计算积分$\int_0^a \mathrm{d}x \int_0^b \mathrm{e}^{\max\{b^2x^2, a^2y^2\}} \mathrm{d}y$.

4. 设$V = \big\{(x,y,z) \big| 0 \leqslant z \leqslant 1, x^2+y^2 \leqslant z^2\big\}$.

(1) 求证：

$$\lim_{n\to\infty} n^3 \iiint_V \mathrm{e}^{-nz} \mathrm{d}x\mathrm{d}y\mathrm{d}z = 2\pi;$$

(2) 设函数$f(x,y,z)$在V上可积，在点$(0,0,0)$处连续，求证：

$$\lim_{n\to\infty} n^3 \iiint_V \mathrm{e}^{-nz} f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z = 2\pi f(0,0,0).$$

5. 设$V=\big\{(x,y,u,v)\big|x^2+y^2+u^2+v^2\leqslant 1\big\}$, 计算四重积分$\displaystyle\int_V \mathrm{e}^{x^2+y^2-u^2-v^2}\mathrm{d}x\mathrm{d}y\mathrm{d}u\mathrm{d}v$.

6. 设$D=\left\{(x_1,x_2,\cdots,x_n)\,\middle|\,x_1,x_2,\cdots,x_n\text{都非负，}\sum\limits_{k=1}^{n}\dfrac{x_k}{k}\leqslant 1\right\}$, 对任何实数$t$, 计算$n$重积分

$$\int_D \mathrm{e}^{t(x_1+x_2+\cdots+x_n)}\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n.$$

7. n维空间内几何体各点坐标满足$\big\{(x_1,x_2,\cdots,x_n)\big|x_i^2+x_j^2\leqslant 1, i\neq j\big\}$, 求其体积$V_n$.