APPENDIX I

DINI DERIVATIVES AND MONOTONIC FUNCTIONS

1. The Dini Derivatives

1.1. Let a, b, $a < b$, be two real numbers and consider a function $f:]a,b[\to \mathscr{R}$, $t \to f(t)$ and a point $t_0 \in]a,b[$. We assume that the reader is familiar with the notions of

$$\limsup_{t \to t_0} f(t) \quad \text{and} \quad \liminf_{t \to t_0} f(t).$$

The same concepts of lim sup and inf, but for $t \to t_0 +$ mean simply that one considers, in the limiting processes, only the values of $t > t_0$. A similar meaning is attached to $t \to t_0 -$. Remember that, without regularity assumptions on f, any lim sup or inf exists if we accept the possible values $+\infty$ or $-\infty$. The extended real line will be designated hereafter by $\overline{\mathscr{R}}$. Thus $\overline{\mathscr{R}} = \mathscr{R} \cup \{-\infty\} \cup \{+\infty\}$.

The four Dini derivatives of f at t_0 are now defined by the following equations:

$$D^+ f(t_0) = \limsup_{t \to t_0+} \frac{f(t) - f(t_0)}{t - t_0},$$

$$D_+ f(t_0) = \liminf_{t \to t_0+} \frac{f(t) - f(t_0)}{t - t_0},$$

$$D^- f(t_0) = \limsup_{t \to t_0-} \frac{f(t) - f(t_0)}{t - t_0},$$

$$D_-f(t_0) = \liminf_{t \to t_0-} \frac{f(t) - f(t_0)}{t - t_0} .$$

They are called respectively the <u>upper right, lower right</u>, <u>upper left</u> and <u>lower left derivatives of</u> f <u>at</u> t_0. Further, the function $t \to D^+f(t)$ on $]a,b[$ into $\bar{\mathscr{R}}$ is called the <u>upper right derivative of</u> f <u>on the interval</u> $]a,b[$, and similarly for D_+, D^- and D_-.

1.2. <u>Remarks</u>. a) It is clear that, in the absence of regularity assumptions on f, any Dini derivative may equal $-\infty$ or $+\infty$.

b) However, if there is a Lipschitz condition for f on some neighborhood of t_0, then all four derivatives are finite.

c) The four Dini derivatives of f at some point $t_0 \in]a,b[$ are equal if and only if f has a derivative at t_0. This derivative is then of course equal to the common value of the Dini derivatives.

d) The well known properties of lim sup and lim inf yield the elementary rules of calculus applicable to the Dini derivatives. For example, if f_1 and f_2 are two real functions defined on $]a,b[$, one gets for any $t \in]a,b[$ that

$$D^+[f_1(t) + f_2(t)] \leq D^+f_1(t) + D^+f_2(t),$$

and

$$D^+[f_1(t) + f_2(t)] \geq D^+f_1(t) + D_+f_2(t),$$

as long as the additions are possible $[(+\infty) + (-\infty)$ is an example of an addition which is not possible$]$.

e) Another important property is that if f is continuous and g is $\mathscr{C}^1$, then

(1) if for some t: $g(t) \geq 0$, one has $D^+(fg)(t) = f(t)g'(t) + g(t)D^+f(t)$

(2) if for some t: $g(t) \leq 0$, one has $D^+(fg)(t) = f(t)g'(t) + g(t)D_+f(t)$,

where $g'(t)$ is the ordinary derivative.

The proof is as follows:

$$D^+(fg)(t) = \limsup_{h \to 0+} \frac{(fg)(t+h) - (fg)(t)}{h}$$

$$= \limsup_{h \to 0+} \left[f(t+h)\frac{g(t+h) - g(t)}{h} + g(t)\frac{f(t+h) - f(t)}{h}\right]$$

$$= \lim_{h \to 0+} f(t+h)\frac{g(t+h) - g(t)}{h} + \limsup_{h \to 0+} g(t)\frac{f(t+h) - f(t)}{h}.$$

Hence the expected result. On the properties of the lim sup and lim inf which enable one to write the above equalities, as well as on other rules of calculus for Dini derivatives, we refer to E. J. McShane [1944].

2. Continuous Monotonic Functions

2.1. Theorem. Suppose f is continuous on $]a,b[$. Then f is increasing on $]a,b[$ if and only if $D^+f(t) \geq 0$ for every $t \in]a,b[$.

Remember that, in this book, f is called increasing on $]a,b[$ if, for any $t_1, t_2 \in]a,b[$, $t_1 < t_2$, one has $f(t_1) \leq f(t_2)$.

Proof. The condition is obviously necessary. Let us prove that it is sufficient.

a) Assume first that $D^+f(t) > 0$ on $]a,b[$. If there exist two points $\alpha, \beta \in]a,b[$, $\alpha < \beta$, with $f(\alpha) > f(\beta)$, then there exist a μ with $f(\alpha) > \mu > f(\beta)$ and some points $t \in [\alpha,\beta]$ such that $f(t) > \mu$. Let ξ be the sup of these points. Of course, ξ is an interior point of $[\alpha,\beta]$, and, due to the continuity of f: $f(\xi) = \mu$. Therefore, for every $t \in]\xi,\beta[$:

$$\frac{f(t) - f(\xi)}{t - \xi} < 0$$

and $D^+f(\xi) \leq 0$, which is absurd.

b) Assume now, as in the statement of the theorem, that $D^+f(t) \geq 0$ on $]a,b[$. For any $\varepsilon > 0$, one gets

$$D^+[f(t) + \varepsilon t] = D^+f(t) + \varepsilon \geq \varepsilon > 0.$$

Hence $f(t) + \varepsilon t$ is increasing on $]a,b[$. And since this is true for any ε, $f(t)$ is also increasing on $]a,b[$.

Q.E.D.

2.2. Remarks. a) This theorem remains true if one replaces the inequality $D^+f(t) \geq 0$ by $D_+f(t) \geq 0$, because the latter implies the former.

b) One proves similarly that $D^+f(t) \geq 0$ can also be replaced by $D^-f(t) \geq 0$: it suffices to substitute the inf of the points t where $f(t) < \mu$ to the sup of the points t where $f(t) > \mu$.

c) In the new theorem thus obtained, D^- may be replaced by D_-. As a consequence, we get the following statement.

2.3. Theorem. Suppose f is continuous on $]a,b[$. Then f is increasing on $]a,b[$ if and only if any of the four Dini derivatives of f is ≥ 0 on $]a,b[$.

2.4. Corollary. If any Dini derivative of the continuous function f is ≥ 0 on $]a,b[$, the same is true of the other three.

2.5. Remark. Analogous monotonicity properties can be established using less than the continuity of f (cf. E. J. McShane [1944]).

2.6. Functions with a bounded Dini derivative. The following theorem is used to estimate the average rate of decrease of a function possessing a Dini derivative bounded from below. It is a straightforward consequence of Theorem 2.3. Hereafter, the symbol D^*f represents any of the four Dini derivatives of f.

Theorem. Let $f: [a,b] \to \mathcal{R}$ be a continuous function such that for any $t \in]a,b[$ and some $A > 0$:

$$D^*f(t) \geq -A. \tag{2.1}$$

Then

$$\frac{f(a) - f(b)}{b - a} \leq A.$$

Proof. One deduces from (2.1) that $D^*(f(t) + At) \geq 0$, and therefore, using Theorem 2.3, that $f(t) + At$ is increasing on $]a,b[$. Therefore $f(b) + Ab \geq f(a) + Aa$. Q.E.D.

2.7. Dini derivative of the maximum of two functions. Concerning three functions f, g, h such that $h(t) = \max (f(t), g(t))$, the following theorem gives an estimation of a Dini derivative of $h(t)$ in terms of the corresponding derivatives of f and g.

Theorem. Let f, g, and h be three continuous functions on $[a,b]$ into $\mathscr{R}$, such that $h(t) = \max\,(f(t),g(t))$. If $D^+f(t) \le 0$ and $D^+g(t) \le 0$ for $t \in \,]a,b[$, then $D^+h(t) \le 0$ for $t \in \,]a,b[$.

Proof. Otherwise, one would have, by Theorem 2.1, for two points a', b' with $a \le a' < b' \le b$, that $\max\,(f(b'), g(b')) > \max\,(f(a'),g(a'))$, and therefore either $f(b') > f(a')$ or $g(b') > g(a')$. But, using Theorem 2.1 again, this contradicts either $D^+f \le 0$ or $D^+g \le 0$. Q.E.D.

3. The Derivative of a Monotonic Function

3.1. The theorem stated (without proof) in this section is a key theorem for Liapunov's direct method. It mentions the Lebesgue integral of a function f on an interval $[a,b]$, which will be written

$$\int_a^b f(\tau)d\tau$$

On this concept, we refer to E. J. McShane [1944] or to A. N. Kolmogorov and S. V. Fomin [1961]. Only one of its elementary properties will be recalled below, in order to clear the statement of the theorem.

3.2. A subset E of the real line $\mathscr{R}$ is said to have measure zero if there exists, for every $\varepsilon > 0$, a finite or countable collection $I_1, I_2, \ldots$ of open intervals such that $\cup I_i \supset E$ and $\Sigma \Delta I_i < \varepsilon$, where ΔI_i is the length of I_i. When a property is verified at each point of some interval $[a,b] \in \mathscr{R}$, except at the points of a set of measure zero, one says that the property is true almost everywhere on $[a,b]$ or for almost all $t \in [a,b]$.

If a function $f: [a,b] \to \mathcal{R}$ is Lebesgue integrable over $[a,b]$, then any function $g: [a,b] \to \mathcal{R}$ which is equal to f almost everywhere on $[a,b]$ is also Lebesgue integrable on $[a,b]$, and the integral of g equals the integral of f. Therefore, it makes sense to speak of the integral over $[a,b]$ of a function which is defined only almost everywhere on $[a,b]$: it can be extended to the whole of $[a,b]$ by choosing arbitrary values at the points where it was originally undefined.

3.3. Theorem. If $f: [a,b] \to \mathcal{R}$ is an increasing function, f has a finite derivative $f'(t)$ almost everywhere on $[a,b]$; this derivative is Lebesgue integrable and one has, for any $t \in [a,b]$,

$$f(t) = \int_a^t f'(\tau)d\tau + h(t)$$

where h is an increasing function and $h'(t)$ vanishes almost everywhere on $[a,b]$.

For a proof, see E. J. McShane [1944] or H. L. Royden [1963].

3.4. Corollary. In the hypotheses of Theorem 3.3,

$$f(b) - f(a) \geq \int_a^b f'(\tau)d\tau. \tag{3.1}$$

3.5. Remarks. a) This inequality becomes an equality if one adds the hypothesis that f is absolutely continuous on $[a,b]$. On this point, cf. the reference books already mentioned. It does exist an example of a function f on $[a,b]$ into $\mathcal{R}$, which is increasing, uniformly continuous, whose derivative vanishes almost everywhere, and such that

$f(b) > f(a)$. Of course, for this function, which is not absolutely continuous

$$f(b) - f(a) > \int_a^b f'(\tau)d\tau = 0.$$

Cf. K. Kuratowski [1961], p. 187.

b) Since the derivative of f, when it exists, equals all four Dini derivatives, the inequality (3.1) can also be written under the form

$$\int_a^b D^+f(\tau)d\tau \leq f(b) - f(a),$$

or similarly while replacing D^+ by D_+, D^- or D_-. Remember that it is valid when f is increasing.

4. Dini Derivative of a Function along the Solutions of a Differential Equation

4.1. For some τ, $-\infty \leq \tau < \infty$ and some open subset $\Omega \subset \mathscr{R}^n$, consider a continuous function

$$f\colon]\tau,\infty[\times \Omega \to \mathscr{R}^n, \ (t,x) \to f(t,x)$$

and the associated differential equation $\dot{x} = f(t,x)$. Further, let $V\colon]\tau,\infty[\times \Omega \to \mathscr{R}$ be a continuous function, satisfying a local Lipschitz condition for x, uniformly with respect to t.

4.2. One has often to verify that a function like $V(t,x)$ is, so to say, <u>decreasing along the solutions of the differential equation</u>. This means that for any solution $x\colon J \to \mathscr{R}^n$, J an open interval, of the equation $\dot{x} = f(t,x)$, the function $\tilde{V}\colon J \to \mathscr{R}^n$, $t \to \tilde{V}(t) = V(t,x(t))$ is decreasing. The following theorem is crucial, for it enables one to check this property <u>without any knowledge of the solutions</u>.

4.3. <u>Theorem</u> (T. Yoshizawa [1966]). In these general hypotheses, let $x: J \to \mathscr{R}^n$ be any solution and let $t^* \in J$. Putting $x(t^*) = x^*$, one gets

$$D^+\tilde{V}(t^*) = \limsup_{h \to 0+} \frac{V(t^* + h, x^* + hf(t^*,x^*)) - V(t^*,x^*)}{h}. \quad (4.1)$$

<u>Proof</u>. One has, for $h > 0$ small,

$$\begin{aligned} &V(t^* + h, x(t^* + h)) - V(t^*, x(t^*)) = \\ &V[t^* + h, x^* + hf(t^*,x^*) + h\varepsilon(t^*,x^*,h)] - V(t^*,x^*) \\ &\leq V(t^* + h, x^* + hf(t^*,x^*)) + kh||\varepsilon(t^*,x^*,h)|| - V(t^*,x^*), \end{aligned}$$

where $\varepsilon \to 0$ with h and k is a Lipschitz constant on some neighborhood of x^*. Therefore

$$\begin{aligned} D^+\tilde{V}(t^*) &= \limsup_{h \to 0+} \frac{V(t^* + h, x(t^* + h)) - V(t^*, x(t^*))}{h} \\ &\leq \limsup_{h \to 0+} \frac{V(t^* + h, x^* + hf(t^*,x^*)) - V(t^*,x^*)}{h}. \end{aligned}$$

One obtains similarly for $h > 0$ small, that

$$V(t^* + h, x(t^* + h)) - V(t^*, x(t^*)) \geq$$

$$V(t^* + h, x^* + hf(t^*,x^*)) - kh||\varepsilon(t^*,x^*,h)|| - V(t^*,x^*),$$

whence

$$D^+\tilde{V}(t^*) \geq \limsup_{h \to 0+} \frac{V(t^* + h, x^* + hf(t^*,x^*)) - V(t^*,x^*)}{h}.$$

Q.E.D.

4.4. <u>Remarks</u>. a) We shall admit the symbol $D^+V(t^*,x^*)$ to represent the second member of (4.1), and this quantity will be called occasionally the upper right Dini derivative of

$V(t,x)$ (along the solutions of the differential equation).

b) It is a simple consequence of Theorems 2.1 and 4.3 that if $D^+V(t,x) \geq 0$ on $]\tau,\infty[\times \Omega$, then $V(t,x)$ is increasing along the solutions of the differential equation. The analogous statement for $V(t,x)$ decreasing is obvious.

c) There is a theorem similar to 4.3 for any other Dini derivative D_+, D^- and D_-.

d) It is noticeable that no uniqueness property has been assumed for the solutions of the differential equation.

e) If, for some $\varepsilon > 0$ and every $(t,x) \in]\tau,\infty[\times \Omega$, one has $D^+V(t,x) \geq -\varepsilon$, then for any solution $x: J \to \mathcal{R}^n$ and any points $a, b \in J$, $a < b$,

$$\frac{\tilde{V}(a) - \tilde{V}(b)}{b - a} \leq \varepsilon.$$

This is a consequence of Theorems 2.6 and 4.3.