第19讲

Infinite products

An infinite product of cardinals is defined using infinite products of sets. If $\{X_i : i \in I\}$ is a family of sets, then the *product* is defined as follows:

$$(5.14) \qquad \prod_{i \in I} X_i = \{f : f \text{ is a function on } I \text{ and } f(i) \in X_i \text{ for each } i \in I\}.$$

Note that if some X_i is empty, then the product is empty. If all the X_i are nonempty, then AC implies that the product is nonempty.

If $\{\kappa_i : i \in I\}$ is a family of cardinal numbers, we define

$$(5.15) \qquad \prod_{i \in I} \kappa_i = \left| \prod_{i \in I} X_i \right|,$$

where $\{X_i : i \in I\}$ is a family of sets such that $|X_i| = \kappa_i$ for each $i \in I$. (We abuse the notation by using $\prod$ both for the product of sets and for the product of cardinals.)

Again, it follows from AC that the definition does not depend on the choice of the sets X_i (Exercise 5.10).

Some rules of infinite sums and products

If $\kappa_i = \kappa$ for each $i \in I$, and $|I| = \lambda$, then $\prod_{i \in I} \kappa_i = \kappa^\lambda$. Also, infinite sums and products satisfy some of the rules satisfied by finite sums and products. For instance, $\prod_i \kappa_i^\lambda = (\prod_i \kappa_i)^\lambda$, or $\prod_i \kappa^{\lambda_i} = \kappa^{\sum_i \lambda_i}$. Or if I is a disjoint union $I = \bigcup_{j \in J} A_j$, then

$$\prod_{i \in I} \kappa_i = \prod_{j \in J} \Big(\prod_{i \in A_j} \kappa_i \Big). \tag{5.16}$$

An inequality

If $\kappa_i \geq 2$ for each $i \in I$, then

$$\sum_{i \in I} \kappa_i \leq \prod_{i \in I} \kappa_i. \tag{5.17}$$

(The assumption $\kappa_i \geq 2$ is necessary: $1+1 > 1 \cdot 1$.) If I is finite, then (5.17) is certainly true; thus assume that I is infinite. Since $\prod_{i \in I} \kappa_i \geq \prod_{i \in I} 2 = 2^{|I|} > |I|$, it suffices to show that $\sum_i \kappa_i \leq |I| \cdot \prod_i \kappa_i$. If $\{X_i : i \in I\}$ is a disjoint family, we assign to each $x \in \bigcup_i X_i$ a pair (i, f) such that $x \in X_i$, $f \in \prod_i X_i$ and $f(i) = x$. Thus we have (5.17).

A lemma

Lemma 5.9. *If λ is an infinite cardinal and $\langle \kappa_i : i < \lambda \rangle$ is a nondecreasing sequence of nonzero cardinals, then*

$$\prod_{i<\lambda} \kappa_i = (\sup_i \kappa_i)^\lambda.$$

Proof. Let $\kappa = \sup_i \kappa_i$. Since $\kappa_i \leq \kappa$ for each $i < \lambda$, we have

$$\prod_{i<\lambda} \kappa_i \leq \prod_{i<\lambda} \kappa = \kappa^\lambda.$$

To prove that $\kappa^\lambda \leq \prod_{i<\lambda} \kappa_i$, we consider a partition of λ into λ disjoint sets A_j, each of cardinality λ:

$$\lambda = \bigcup_{j<\lambda} A_j. \tag{5.18}$$

(To get a partition (5.18), we can, e.g., use the canonical pairing function $\Gamma : \lambda \times \lambda \to \lambda$ and let $A_j = \Gamma(\lambda \times \{j\})$.) Since a product of nonzero cardinals is greater than or equal to each factor, we have $\prod_{i\in A_j} \kappa_i \geq \sup_{i\in A_j} \kappa_i = \kappa$, for each $j < \lambda$. Thus, by (5.16),

$$\prod_{i<\lambda} \kappa_i = \prod_{j<\lambda} \Big(\prod_{i\in A_j} \kappa_i \Big) \geq \prod_{j<\lambda} \kappa = \kappa^\lambda. \qquad \square$$

König theorem

Theorem 5.10 (König). *If* $\kappa_i < \lambda_i$ *for every* $i \in I$, *then*

$$\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i.$$

Proof. We shall show that $\sum_i \kappa_i \not\geq \prod_i \lambda_i$. Let T_i, $i \in I$, be such that $|T_i| = \lambda_i$ for each $i \in I$. It suffices to show that if Z_i, $i \in I$, are subsets of $T = \prod_{i\in I} T_i$, and $|Z_i| \leq \kappa_i$ for each $i \in I$, then $\bigcup_{i\in I} Z_i \neq T$.

For every $i \in I$, let S_i be the projection of Z_i into the ith coordinate:

$$S_i = \{f(i) : f \in Z_i\}.$$

Since $|Z_i| < |T_i|$, we have $S_i \subset T_i$ and $S_i \neq T_i$. Now let $f \in T$ be a function such that $f(i) \notin S_i$ for every $i \in I$. Obviously, f does not belong to any Z_i, $i \in I$, and so $\bigcup_{i\in I} Z_i \neq T$. $\square$

Corollaries of König theorem

Corollary 5.11. $\kappa < 2^\kappa$ *for every* κ.

Proof. $\underbrace{1+1+\dots}_{\kappa \text{ times}} < \underbrace{2\cdot 2\cdot\dots}_{\kappa \text{ times}}$. ◻

Corollary 5.12. $\mathrm{cf}(2^{\aleph_\alpha}) > \aleph_\alpha$.

Proof. It suffices to show that if $\kappa_i < 2^{\aleph_\alpha}$ for $i < \omega_\alpha$, then $\sum_{i<\omega_\alpha} \kappa_i < 2^{\aleph_\alpha}$.
Let $\lambda_i = 2^{\aleph_\alpha}$.

$$\sum_{i<\omega_\alpha} \kappa_i < \prod_{i<\omega_\alpha} \lambda_i = (2^{\aleph_\alpha})^{\aleph_\alpha} = 2^{\aleph_\alpha}.$$ ◻

Corollaries of König theorem

Corollary 5.13. $\mathrm{cf}(\aleph_\alpha^{\aleph_\beta}) > \aleph_\beta$.

Proof. We show that if $\kappa_i < \aleph_\alpha^{\aleph_\beta}$ for $i < \omega_\beta$, then $\sum_{i<\omega_\beta} \kappa_i < \aleph_\alpha^{\aleph_\beta}$. Let $\lambda_i = \aleph_\alpha^{\aleph_\beta}$.

$$\sum_{i<\omega_\beta} \kappa_i < \prod_{i<\omega_\beta} \lambda_i = (\aleph_\alpha^{\aleph_\beta})^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta}. \qquad \square$$

Corollary 5.14. $\kappa^{\mathrm{cf}\,\kappa} > \kappa$ *for every infinite cardinal* κ.

Proof. Let $\kappa_i < \kappa$, $i < \mathrm{cf}\,\kappa$, be such that $\kappa = \sum_{i<\mathrm{cf}\,\kappa} \kappa_i$. Then

$$\kappa = \sum_{i<\mathrm{cf}\,\kappa} \kappa_i < \prod_{i<\mathrm{cf}\,\kappa} \kappa = \kappa^{\mathrm{cf}\,\kappa}. \qquad \square$$

GCH

Cantor's Theorem 3.1 states that $2^{\aleph_\alpha} > \aleph_\alpha$, and therefore $2^{\aleph_\alpha} \geq \aleph_{\alpha+1}$, for all α. The *Generalized Continuum Hypothesis* (GCH) is the statement

$$2^{\aleph_\alpha} = \aleph_{\alpha+1}$$

for all α. GCH is independent of the axioms of ZFC. Under the assumption of GCH, cardinal exponentiation is evaluated as follows:

Theorem 5.15. *If* GCH *holds and κ and λ are infinite cardinals then:*

(i) *If $\kappa \leq \lambda$, then $\kappa^\lambda = \lambda^+$.*
(ii) *If* $\operatorname{cf}\kappa \leq \lambda < \kappa$, *then $\kappa^\lambda = \kappa^+$.*
(iii) *If* $\lambda < \operatorname{cf}\kappa$, *then $\kappa^\lambda = \kappa$.*

Proof. (i) Lemma 5.6.

(ii) This follows from (5.7) and (5.8).

(iii) By Lemma 3.9(ii), the set κ^λ is the union of the sets α^λ, $\alpha < \kappa$, and $|\alpha^\lambda| \leq 2^{|\alpha|\cdot\lambda} = (|\alpha| \cdot \lambda)^+ \leq \kappa$. $\square$

Axiom of Regularity

The Axiom of Regularity states that the relation $\in$ on any family of sets is well-founded:

Axiom of Regularity. *Every nonempty set has an $\in$-minimal element*:

$$\forall S\,(S \neq \emptyset \to (\exists x \in S)\, S \cap x = \emptyset).$$

As a consequence, there is no infinite sequence

$$x_0 \ni x_1 \ni x_2 \ni \ldots .$$

(Consider the set $S = \{x_0, x_1, x_2, \ldots\}$ and apply the axiom.) In particular, there is no set x such that

$$x \in x$$

and there are no "cycles"

$$x_0 \in x_1 \in \ldots \in x_n \in x_0.$$

The cumulative hierarchy

Thus the Axiom of Regularity postulates that sets of certain type do no exist. This restriction on the universe of sets is not contradictory (i.e., the axiom is consistent with the other axioms) and is irrelevant for the development of ordinal and cardinal numbers, natural and real numbers, and in fact of all ordinary mathematics. However, it is extremely useful in the metamathematics of set theory, in construction of models. In particular, all sets can be assigned ranks and can be arranged in a cumulative hierarchy.

We define, by transfinite induction,

$$V_0 = \emptyset, \qquad V_{\alpha+1} = P(V_\alpha),$$
$$V_\alpha = \bigcup_{\beta<\alpha} V_\beta \quad \text{if } \alpha \text{ is a limit ordinal.}$$

The sets V_α have the following properties (by induction):

(i) Each V_α is transitive.
(ii) If $\alpha < \beta$, then $V_\alpha \subset V_\beta$.
(iii) $\alpha \subset V_\alpha$.

Transitive closure

Lemma 6.1. *For every set S there exists a transitive set $T \supset S$.*

Proof. We define by induction

$$S_0 = S, \qquad S_{n+1} = \bigcup S_n$$

and

$$T = \bigcup_{n=0}^{\infty} S_n. \tag{6.1}$$

Clearly, T is transitive and $T \supset S$. □

Since every transitive set must satisfy $\bigcup T \subset T$, it follows that the set in (6.1) is the smallest transitive $T \supset S$; it is called *transitive closure* of S:

$$\mathrm{TC}(S) = \bigcap\{T : T \supset S \text{ and } T \text{ is transitive}\}.$$

Every nonempty class has an $\in$ –minimal element

Lemma 6.2. *Every nonempty class C has an $\in$-minimal element.*

Proof. Let $S \in C$ be arbitrary. If $S \cap C = \emptyset$, then S is a minimal element of C; if $S \cap C \neq \emptyset$, we let $X = T \cap C$ where $T = \mathrm{TC}(S)$. X is a nonempty set and by the Axiom of Regularity, there is $x \in X$ such that $x \cap X = \emptyset$. It follows that $x \cap C = \emptyset$; otherwise, if $y \in x$ and $y \in C$, then $y \in T$ since T is transitive, and so $y \in x \cap T \cap C = x \cap X$. Hence x is a minimal element of C. $\square$