第15讲

Some facts about cardinal numbers

(3.4) $+$ and $\cdot$ are associative, commutative and distributive.

(3.5) $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$.

(3.6) $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$.

(3.7) $(\kappa^\lambda)^\mu = \kappa^{\lambda\cdot\mu}$.

(3.8) If $\kappa \leq \lambda$, then $\kappa^\mu \leq \lambda^\mu$.

(3.9) If $0 < \lambda \leq \mu$, then $\kappa^\lambda \leq \kappa^\mu$.

(3.10) $\kappa^0 = 1$; $1^\kappa = 1$; $0^\kappa = 0$ if $\kappa > 0$.

To prove (3.4)–(3.10), one has only to find the appropriate one-to-one functions.

Proof of (3.6)

1.7 Theorem

(a) $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$.

(b) $(\kappa^\lambda)^\mu = \kappa^{\lambda\cdot\mu}$.

(c) $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$.

Proof. Let $\kappa = |K|$, $\lambda = |L|$, $\mu = |M|$. To show (a), assume that L and M are disjoint. We construct a one-to-one mapping F of $K^L \times K^M$ onto $K^{L\cup M}$. If $(f, g) \in K^L \times K^M$, we let $F(f, g) = f \cup g$. We note that $f \cup g$ is a function, in fact a member of $K^{L\cup M}$, and every $h \in K^{L\cup M}$ is equal to $F(f, g)$ for some $(f, g) \in K^L \times K^M$ (namely, $f = h \restriction L$, $g = h \restriction M$). It is easily seen that F is one-to-one.

Proof of (3.5) and (3.7)

To prove (b), we look for a one-to-one map F of $K^{L\times M}$ onto $(K^L)^M$. A typical element of $K^{L\times M}$ is a function $f : L \times M \to K$. We let F assign to f the function $g : M \to K^L$ defined as follows: for all $m \in M$, $g(m) = h \in K^L$ where $h(l) = f(l, m)$ (for all $l \in L$). We leave it to the reader to verify that F is one-to-one and onto.

For a proof of (c) we need a one-to-one mapping F of $K^M \times L^M$ onto $(K \times L)^M$. For each $(f_1, f_2) \in K^M \times L^M$, let $F(f_1, f_2) = g : M \to (K \times L)$ where $g(m) = (f_1(m), f_2(m))$, for all $m \in M$. It is routine to check that F is one-to-one and onto. $\square$

Cardinal numbers and Alephs

An ordinal α is called a *cardinal number* (a cardinal) if $|\alpha| \neq |\beta|$ for all $\beta < \alpha$. We shall use κ, λ, μ, ... to denote cardinal numbers.

If W is a well-ordered set, then there exists an ordinal α such that $|W| = |\alpha|$. Thus we let

$$|W| = \text{the least ordinal such that } |W| = |\alpha|.$$

Clearly, $|W|$ is a cardinal number.

Every natural number is a cardinal (a *finite cardinal*); and if S is a finite set, then $|S| = n$ for some n.

The ordinal ω is the least infinite cardinal. Note that all infinite cardinals are limit ordinals. The infinite ordinal numbers that are cardinals are called *alephs*.

A lemma

Lemma 3.4.

(i) *For every α there is a cardinal number greater than α.*
(ii) *If X is a set of cardinals, then* $\sup X$ *is a cardinal.*

For every α, let α^+ be the least cardinal number greater than α, the *cardinal successor* of α.

Proof. (i) For any set X, let

$$(3.11) \qquad h(X) = \text{the least } \alpha \text{ such that there is no one-to-one function of } \alpha \text{ into } X.$$

There is only a set of possible well-orderings of subsets of X. Hence there is only a set of ordinals for which a one-to-one function of α into X exists. Thus $h(X)$ exists.

If α is an ordinal, then $|\alpha| < |h(\alpha)|$ by (3.11). That proves (i).

(ii) Let $\alpha = \sup X$. If f is a one-to-one mapping of α onto some $\beta < \alpha$, let $\kappa \in X$ be such that $\beta < \kappa \leq \alpha$. Then $|\kappa| = |\{f(\xi) : \xi < \kappa\}| \leq \beta$, a contradiction. Thus α is a cardinal. $\square$

The increasing enumeration of all Alephs

用超限归纳定义ω_α如下.

$$
\begin{aligned}
\omega_0 &= \omega, \\
\omega_{\alpha+1} &= h(\omega_\alpha), \\
\omega_\alpha &= \sup\{\omega_\beta : \beta < \alpha\},\ \text{当}\alpha\text{为非0极限序数}.
\end{aligned}
$$

Using Lemma 3.4, we define the increasing enumeration of all alephs. We usually use $\aleph_\alpha$ when referring to the cardinal number, and ω_α to denote the order-type:

$$
\aleph_0 = \omega_0 = \omega, \qquad \aleph_{\alpha+1} = \omega_{\alpha+1} = \aleph_\alpha^+,
$$

$$
\aleph_\alpha = \omega_\alpha = \sup\{\omega_\beta : \beta < \alpha\} \quad \text{if } \alpha \text{ is a limit ordinal.}
$$

Successor cardinals and limit cardinals

Sets whose cardinality is $\aleph_0$ are called *countable*; a set is *at most countable* if it is either finite or countable. Infinite sets that are not countable are *uncountable*.

A cardinal $\aleph_{\alpha+1}$ is a *successor cardinal*. A cardinal $\aleph_\alpha$ whose index is a limit ordinal is a *limit cardinal*.

Multiplication of alephs

Addition and multiplication of alephs is a trivial matter, due to the following fact:

Theorem 3.5. $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$.

As a corollary we have

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \max\{\aleph_\alpha, \aleph_\beta\}. \tag{3.14}$$

Exponentiation of cardinals will be dealt with in Chapter 5. Without the Axiom of Choice, one cannot prove that $2^{\aleph_\alpha}$ is an aleph (or that $P(\omega_\alpha)$ can be well-ordered), and there is very little one can prove about $2^{\aleph_\alpha}$ or $\aleph_\alpha^{\aleph_\beta}$.

The canonical well-ordering of $\alpha \times \alpha$

We define a well-ordering of the class $Ord \times Ord$ of ordinal pairs. Under this well-ordering, each $\alpha \times \alpha$ is an initial segment of Ord^2; the induced well-ordering of α^2 is called the *canonical well-ordering* of α^2. Moreover, the well-ordered class Ord^2 is isomorphic to the class Ord, and we have a one-to-one function Γ of Ord^2 onto Ord. For many α's the order-type of $\alpha \times \alpha$ is α; in particular for those α that are alephs.

$$
\begin{aligned}
(\alpha, \beta) < (\gamma, \delta) \leftrightarrow \ & \text{either } \max\{\alpha, \beta\} < \max\{\gamma, \delta\}, \\
& \text{or } \max\{\alpha, \beta\} = \max\{\gamma, \delta\} \text{ and } \alpha < \gamma, \\
& \text{or } \max\{\alpha, \beta\} = \max\{\gamma, \delta\},\ \alpha = \gamma \text{ and } \beta < \delta.
\end{aligned} \tag{3.12}
$$

Some properties of the canonical well-ordering of $\alpha \times \alpha$

The relation $<$ defined in (3.12) is a linear ordering of the class $Ord \times Ord$. Moreover, if $X \subset Ord \times Ord$ is nonempty, then X has a least element. Also, for each α, $\alpha \times \alpha$ is the initial segment given by $(0, \alpha)$. If we let

$$\Gamma(\alpha, \beta) = \text{the order-type of the set } \{(\xi, \eta) : (\xi, \eta) < (\alpha, \beta)\},$$

then Γ is a one-to-one mapping of Ord^2 onto Ord, and

$$(\alpha, \beta) < (\gamma, \delta) \quad \text{if and only if} \quad \Gamma(\alpha, \beta) < \Gamma(\gamma, \delta). \tag{3.13}$$

Note that $\Gamma(\omega\times\omega) = \omega$ and since $\gamma(\alpha) = \Gamma(\alpha\times\alpha)$ is an increasing function of α, we have $\gamma(\alpha) \geq \alpha$ for every α. However, $\gamma(\alpha)$ is also continuous, and so $\Gamma(\alpha \times \alpha) = \alpha$ for arbitrarily large α.

A fixed point lemma

Let $f : Ord \to Ord$ be normal, that is, f is increasing and continuous, then for any ordinal α, there exists an ordinal β such that $\beta \geq \alpha$ and $f(\beta) = \beta$.

This lemma shows that the class of fixed points of any normal function is nonempty and unbound.

Hint

Let $\alpha_0 = \alpha$, $\alpha_{n+1} = f(\alpha_n)$, $n = 0, 1, 2, \cdots$, $\beta = \sup\{\alpha_n : n \in \omega\}$.

$\Gamma(\alpha \times \alpha)$

除了$\Gamma(\alpha, \beta)$之外，教材中也用了记号$\Gamma(\alpha \times \alpha)$, 但没有给出$\Gamma(\alpha \times \alpha)$的定义.

这里，$\Gamma(\alpha \times \alpha)$实际上是集合$\alpha \times \alpha$的序型. 对任意$\alpha > 0$, 都有

$$\Gamma(\alpha \times \alpha) = \Gamma(0, \alpha).$$

例如，

$$\begin{aligned}
\Gamma(\omega \times \omega) &= \Gamma(0, \omega) = \omega, \\
\Gamma((\omega + 1) \times (\omega + 1)) &= \Gamma(0, \omega + 1) = \omega \cdot 3 + 1.
\end{aligned}$$

Proof of theorem 3.5

Addition and multiplication of alephs is a trivial matter, due to the following fact:

Theorem 3.5. $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$.

Proof of Theorem 3.5. Consider the canonical one-to-one mapping Γ of $Ord \times Ord$ onto Ord. We shall show that $\Gamma(\omega_\alpha \times \omega_\alpha) = \omega_\alpha$. This is true for $\alpha = 0$. Thus let α be the least ordinal such that $\Gamma(\omega_\alpha \times \omega_\alpha) \neq \omega_\alpha$. Let $\beta, \gamma < \omega_\alpha$ be such that $\Gamma(\beta, \gamma) = \omega_\alpha$. Pick $\delta < \omega_\alpha$ such that $\delta > \beta$ and $\delta > \gamma$. Since $\delta \times \delta$ is an initial segment of $Ord \times Ord$ in the canonical well-ordering and contains (β, γ), we have $\Gamma(\delta \times \delta) \supset \omega_\alpha$, and so $|\delta \times \delta| \geq \aleph_\alpha$. However, $|\delta \times \delta| = |\delta| \cdot |\delta|$, and by the minimality of α, $|\delta| \cdot |\delta| = |\delta| < \aleph_\alpha$. A contradiction. $\square$

Cofinality

Let $\alpha > 0$ be a limit ordinal. We say that an increasing β-sequence $\langle \alpha_\xi : \xi < \beta \rangle$, β a limit ordinal, is *cofinal* in α if $\lim_{\xi \to \beta} \alpha_\xi = \alpha$. Similarly, $A \subset \alpha$ is *cofinal* in α if $\sup A = \alpha$. If α is an infinite limit ordinal, the *cofinality* of α is

$$\begin{aligned} \operatorname{cf} \alpha = {} & \text{the least limit ordinal } \beta \text{ such that there is an increasing} \\ & \beta\text{-sequence } \langle \alpha_\xi : \xi < \beta \rangle \text{ with } \lim_{\xi \to \beta} \alpha_\xi = \alpha. \end{aligned}$$

Obviously, $\operatorname{cf} \alpha$ is a limit ordinal, and $\operatorname{cf} \alpha \le \alpha$. Examples: $\operatorname{cf}(\omega + \omega) = \operatorname{cf} \aleph_\omega = \omega$.

Lemma 3.6. $\operatorname{cf}(\operatorname{cf} \alpha) = \operatorname{cf} \alpha$.

Proof. If $\langle \alpha_\xi : \xi < \beta \rangle$ is cofinal in α and $\langle \xi(\nu) : \nu < \gamma \rangle$ is cofinal in β, then $\langle \alpha_{\xi(\nu)} : \nu < \gamma \rangle$ is cofinal in α. $\square$