数学分析讲义（省身班）

段华贵

数学科学学院

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第11.4节

多元函数的泰勒公式

设函数$f(t)$在t_0处m阶导数存在，则

$$f(t_0+h)=f(t_0)+f'(t_0)h+\cdots+\frac{1}{m!}f^{(m)}(t_0)h^m+R_m(h),$$

其中

$$R_m(h)=\begin{cases} o(|h|^m)\ (h\to 0),\ (\text{佩亚诺余项}) \\ \frac{h^{m+1}}{(m+1)!}f^{(m+1)}(t_0+\theta h),\ (\text{拉格朗日余项}) \\ \frac{h^{m+1}}{m!}f^{(m+1)}(t_0+\theta h)(1-\theta)^m,\ (\text{柯西余项}) \\ \frac{h^{m+1}}{m!}\int_0^1 f^{(m+1)}(t_0+sh)(1-s)^m ds,\ (\text{积分余项}). \end{cases}$$

设函数$f(X)$在点X_0处m阶可微,

令$\varphi(t)=f(X_0+t\Delta x),\forall t\in[0,1]$,则$\varphi(t)$在$t=0$ 处m次可微,且

$$
\begin{aligned}
&\varphi(0)=f(X_0),\\
&\varphi'(0)=\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(X_0)\Delta x_i=\mathrm{d}f(X_0)\big|_{\mathrm{d}x_i=\Delta x_i},\\
&\varphi''(0)=\sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial x_i\partial x_j}(X_0)\Delta x_i\Delta x_j=\mathrm{d}^2 f(X_0)\big|_{\mathrm{d}x_i=\Delta x_i},\\
&\cdots,\\
&\varphi^{(k)}(0)=\sum_{i_1,\cdots,i_k=1}^{n}\frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}(X_0)\Delta x_{i_1}\cdots\Delta x_{i_k}=\mathrm{d}^k f(X_0)\big|_{\mathrm{d}x_i=\Delta x_i}.
\end{aligned}
$$

由一元函数泰勒公式($\varphi(t) = f(X_0 + t\Delta x), \forall t \in [0,1]$)

$$\varphi(1) = \varphi(0)+\varphi'(0)+\frac{1}{2!}\varphi''(0)+\cdots+\frac{1}{m!}\varphi^m(0)+\frac{1}{(m+1)!}\varphi^{m+1}(\theta)$$

即得

$$f(X_0+\Delta X) = f(X_0) + \sum_{k=1}^{m}\frac{1}{k!}\mathrm{d}^k f(X_0)\Big|_{\mathrm{d}x_i=\Delta x_i} + R_m(\Delta X)$$

记

$$\alpha = (\alpha_1, \cdots, \alpha_n), \quad |\alpha| = \sum_{i=1}^{n} \alpha_i,$$
$$\alpha! = \alpha_1! \cdots \alpha_n!, \quad X^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

则

$$\begin{aligned}
\sum_{k=1}^{m} \frac{1}{k!} \mathrm{d}^k f(X_0) &= \sum_{k=1}^{m} \sum_{|\alpha|=k} \frac{D^\alpha f(X_0)}{\alpha!} (dX)^\alpha \\
&= \sum_{k=1}^{m} \sum_{|\alpha|=k} \frac{\partial^{|\alpha|} f(X_0)}{\alpha!\, \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \mathrm{d}x_1^{\alpha_1} \cdots \mathrm{d}x_n^{\alpha_n} \\
&= \sum_{k=1}^{m} \sum_{i_1, \cdots, i_k=1}^{n} \frac{\partial^k f(X_0)}{\partial x_{i_1} \cdots \partial x_{i_k}} \mathrm{d}x_{i_1} \cdots \mathrm{d}x_{i_k}.
\end{aligned}$$

Theorem (Peano余项)

设$D \subseteq \mathbb{R}^n$是一个开区域，$X_0 = (a_1, a_2, \cdots, a_n) \in D$，$f \in C^m(D)$. 则有

$$f(X_0 + \Delta X) = f(X_0) + \sum_{k=1}^{m} \frac{1}{k!} \mathrm{d}^k f(X_0) \Big|_{\mathrm{d}x_i = \Delta x_i} + o(|\Delta X|^m),$$

这里$\lim_{|\Delta X| \to 0} \frac{o(|\Delta X|^m)}{|\Delta X|^m} = 0$.

对$m=1$，由函数的微分定义，知定理的结论成立.

对于$m\geq 2$, 设定理对$m-1$成立. 记$\Delta X=U=(u_1,\cdots,u_n)$,

$$
\begin{aligned}
P_m(U) &\equiv f(X_0)+\sum_{k=1}^{m}\frac{1}{k!}\mathrm{d}^k f(X_0)\Big|_{\mathrm{d}x_i=u_i} \\
&= f(X_0)+\sum_{k=1}^{m}\frac{1}{k!}\sum_{i_1,\cdots,i_k=1}^{n}\frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}(X_0)u_{i_1}\cdots u_{i_k}.
\end{aligned}
$$

从而$R_m(U)=f(X_0+U)-P_m(U)$,且$R_m(O)=0$.

$R_m(U)$在原点O的邻域内直到$m-1$阶的偏导数存在, 在原点m阶可微, 并且直到m阶所有偏导数全为零.

$$
\begin{aligned}
R_m(U) &= R_m(U) - R_m(O) \quad (\text{插入一些项}) \\
&= \sum_{i=1}^{n} \left[R_m(u_1, \cdots, u_i, 0, \cdots, 0) - R_m(u_1, \cdots, u_{i-1}, 0, \cdots, 0) \right].
\end{aligned}
$$

由一元函数的拉格朗日中值定理, 存在$\theta_i \in (0,1)$, 对于任意$i = 1, \cdots, n$, 有

$$
\begin{aligned}
& R_m(u_1, \cdots, u_i, 0, \cdots, 0) - R_m(u_1, \cdots, u_{i-1}, 0, \cdots, 0) \\
= \ & \frac{\partial R_m}{\partial u_i}(u_1, u_2, \cdots, \theta_i u_i, 0, \cdots, 0) u_i.
\end{aligned}
$$

而$\frac{\partial R_m}{\partial u_i}(u_1, \cdots, u_n)$在原点处$m-1$次可微.

由归纳假设，并注意到所有偏导数满足$\frac{\partial^k R_m}{\partial u_{i_1}\cdots\partial u_{i_k}}(O)=0$，则有

$$\begin{aligned} & \frac{\partial R_m}{\partial u_i}(u_1,u_2,\cdots,\theta_i u_i,0,\cdots,0) \\ = \ & o(|(u_1,u_2,\cdots,\theta_i u_i,0,\cdots,0)|^{m-1}) \\ = \ & o(|U|^{m-1}), \end{aligned}$$

这里$\frac{o(|U_i|^{m-1})}{|U|^{m-1}}=\frac{o(|U_i|^{m-1})}{|U_i|^{m-1}}\frac{|U_i|^{m-1}}{|U|^{m-1}}\le\frac{o(|U_i|^{m-1})}{|U_i|^{m-1}}\to 0.$
故

$$R_m(U)=\sum_{i=1}^{n}o(|U|^{m-1})u_i=o(|U|^m).$$

Theorem (Lagrange余项)

设$X_0 = (a_1, a_2, \cdots, a_n)$, $f(X)$在某邻域$B_r(X_0)$内$m+1$次可微. 则$\forall X \in B_r(X_0)$, 存在点$\xi = X_0 + \theta(X - X_0)$, $0 < \theta < 1$, 使得

$$\begin{aligned} & f(X) - f(X_0) \\ = \; & \sum_{k=1}^{m} \frac{1}{k!} \mathrm{d}^k f(X_0)\Big|_{\mathrm{d}x_i = \Delta x_i} + \frac{1}{(m+1)!} \mathrm{d}^{m+1} f(\xi)\Big|_{\mathrm{d}x_i = \Delta x_i}. \end{aligned}$$

记$D \equiv \Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}$,则

$$f(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0) + Df(x_0, y_0) + \frac{1}{2!} D^2 f(x_0, y_0)$$
$$+ \cdots + \frac{1}{m!} D^m f(x_0, y_0) + + \frac{1}{(m+1)!} D^{m+1} f(x_0 + \theta \Delta x, y_0 + \theta \Delta y),$$

$$D^k f(x_0, y_0) = \sum_{i=0}^{k} C_k^i \frac{\partial^k}{\partial x^{k-i} \partial y^i} (\Delta x)^{p-i} (\Delta y)^i.$$

对一元函数$\varphi(t)$用泰勒公式，有

$$f(a_1+t\Delta x_1,\cdots,a_n+t\Delta x_n)-f(a_1,\cdots,a_n)$$
$$=\sum_{k=1}^{m}\frac{t^k}{k!}\sum_{i_1,\cdots,i_k=1}^{n}\frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}(X_0)\Delta x_{i_1}\cdots\Delta x_{i_k}+R_m(t\Delta X).$$

若取$t=1$, 并且$f(X)$在D内$m+1$次可微时,

$$f(a_1+\Delta x_1,\cdots,a_n+\Delta x_n)-f(a_1,\cdots,a_n)$$
$$=\sum_{k=1}^{m}\frac{1}{k!}\sum_{i_1,\cdots,i_k=1}^{n}\frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}(X_0)\Delta x_{i_1}\cdots\Delta x_{i_k}+R_m(\Delta X),$$

取拉格朗日余项, 存在$0<\theta<1$，使

$$R_m(\Delta X)$$
$$=\frac{1}{(m+1)!}\sum_{i_1,\cdots,i_{m+1}=1}^{n}\frac{\partial^{m+1}f}{\partial x_{i_1}\cdots\partial x_{i_{m+1}}}(X_0+\theta\Delta X)\Delta x_{i_1}\cdots\Delta x_{i_{m+1}}.$$

$$f(X)-f(X_0)=\langle\nabla f(X_0),\Delta X\rangle+\frac{1}{2}\Delta X\cdot H_f(X_0)\cdot\Delta X^T+o(|\Delta X|^2),$$

$$f(X)-f(X_0)=\langle\nabla f(X_0),\Delta X\rangle+\frac{1}{2}\Delta X\cdot H_f(X_0+\theta\Delta X)\cdot\Delta X^T.$$

(i) 凸区域D内函数$f \in C^1(D)$是凸函数 $\Leftrightarrow$

$$f(X) - f(X_0) \geqslant \langle \nabla f(X_0), X - X_0 \rangle, \quad \forall\, X, X_0 \in D.$$

Proof. 充分性：

$\forall X_1$, $X_2 \in D$, 记$X_0 = \lambda X_1 + (1-\lambda) X_2, \lambda \in (0,1)$, 则

$$f(X_1) - f(X_0) \geqslant \langle \nabla f(X_0), X_1 - X_0 \rangle = (1-\lambda)\langle \nabla f(X_0), X_1 - X_2 \rangle,$$

$$f(X_2) - f(X_0) \geqslant \langle \nabla f(X_0), X_2 - X_0 \rangle = \lambda \langle \nabla f(X_0), X_2 - X_1 \rangle,$$

由上面两式子可得

$$\lambda f(X_1) + (1-\lambda) f(X_2) - f(\lambda X_1 + (1-\lambda) X_2) \geqslant 0.$$

(ii) $f \in C^2(D)$是凸函数的充要条件$H_f(X) \geqslant 0.$

Proof. **必要性**: 由(i),$f(X)$是凸函数$\Rightarrow$

$$f(X+\Delta X) - f(X) \geqslant \langle \nabla f(X), \Delta X \rangle$$

上式左边−右边$= \frac{1}{2}\Delta X \cdot H_f(X) \cdot \Delta X^T + o(|\Delta X|^2) \geqslant 0.$ 从而

$$V \cdot H_f(X) \cdot V^T \geqslant 0,\ \forall\, V \in \mathbb{R}^n.$$

即$H_f(X) \geqslant 0$. 事实上, 若存在$V_0 \neq 0$使$V_0 \cdot H_f(X_0) \cdot V_0^T = \lambda < 0$, 则令$\Delta X = tV_0$, 则当$|t|$很小时,有

$$\Delta X \cdot H_f(X_0) \cdot \Delta X^T + o(|\Delta X|^2) = \lambda t^2 + o(t^2|V_0|) < 0,$$

矛盾.

若$\Delta X \cdot H_f(X) \cdot \Delta X^T \geqslant 0$ 在$X \in D$成立, 则

$$
\begin{aligned}
f(X) - f(X_0) &= \langle \nabla f(X_0), \Delta X \rangle + \frac{1}{2} \Delta X \cdot H_f(X_0 + \theta \Delta X) \cdot \Delta X^T \\
&\geq \langle \nabla f(X_0), \Delta X \rangle.
\end{aligned}
$$

从而$f(X)$在D内是凸函数.

设$f:\mathbb{R}^n\to\mathbb{R}$为二阶连续可微的函数,若$H_f(x)\geq I_n,\ \forall x\in\mathbb{R}^n$. 则$f$存在唯一的最小值.

第11.5节

隐函数存在定理

Kepler方程: $F(x,y) = y - x - e\sin y = 0$, 其中$0 < e < 1$.

隐函数 由一个方程或一个方程组所确定的函数.

平面上圆的方程: $x^2 + y^2 = 1$.

Theorem (隐函数定理)

设二元函数$f(x,y)$满足

(i) $f(x_0,y_0)=0$,

(ii) $f(x,y)$, $f'_y(x,y)$在(x_0,y_0)的一个邻域内连续,

(iii) $f'_y(x_0,y_0)\neq 0$.

则存在$\delta,\eta>0$和唯一的函数$y=y(x):(x_0-\delta,x_0+\delta)\to(y_0-\eta,y_0+\eta)$, 使得

$$y_0=y(x_0),\quad f(x,y(x))\equiv 0,\quad \forall x\in(x_0-\delta,x_0+\delta),$$

且$y(x)$在$(x_0-\delta,x_0+\delta)$内连续.

Theorem (续)

[注] 把$f(x,y)=0$当作y的一元方程，在(x_0,y_0)的一个邻域$(x_0-\delta,x_0+\delta)\times(y_0-\eta,y_0+\eta)$有唯一连续解$y=y(x)$，且$y(x_0)=y_0$.

(iv) 若$f'_x(x,y)$在(x_0,y_0)的一个邻域内连续，则上述的$y=y(x)$在x_0的一个邻域内一阶导数连续，且

$$y'(x)=-\frac{f'_x(x,y(x))}{f'_y(x,y(x))}.$$

不妨设$f'_y(x_0,y_0)>0$,由假设(ii)可知，存在$\eta>0$，使得$f(x,y)$, $f'_y(x,y)$在

$$D=\{(x,y)\in\mathbb{R}^2|\ |x-x_0|\leqslant\eta,|y-y_0|\leqslant\eta\}$$

上连续，而且

$$\begin{aligned}&f'_y(x,y)>0,\quad \forall(x,y)\in D,\\&f(x_0,y_0-\eta)<0,\\&f(x_0,y_0+\eta)>0.\end{aligned}$$

从而存在$0<\delta\leqslant\eta$，使得当$x\in(x_0-\delta,x_0+\delta)$时

$$f(x,y_0-\eta)<0,\quad f(x,y_0+\eta)>0.$$

再由$f(x,y)$关于y连续且严格单增，从而(由介值定理)存在唯一的$y(x)\in(y_0-\eta,y_0+\eta)$，使得$f(x,y(x))=0$.

$\forall\ \bar{x} \in (x_0-\delta, x_0+\delta)$，记$\bar{y}=y(\bar{x})$，则$|\bar{y}-y_0|<\eta$. 任给$0<\varepsilon<\min\{y_0+\eta-\bar{y}, \bar{y}-y_0+\eta\}$. 由于$f(\bar{x},y)$是$y$的严格单增函数，从而由$f(\bar{x},\bar{y})=0$可得

$$f(\bar{x},\bar{y}-\varepsilon)<0, \quad f(\bar{x},\bar{y}+\varepsilon)>0.$$

再由f的连续性可知，存在$0<r<\min\{x_0+\delta-\bar{x}, \bar{x}-x_0+\delta\}$，使得当$|x-\bar{x}|<r$时有

$$f(x,\bar{y}-\varepsilon)<0, f(x,\bar{y}+\varepsilon)>0. \quad \text{注:}f(x,y(x))=0.$$

于是当$|x-\bar{x}|<r$时 (因为$f(x,y)$关于y连续且严格单增)

$$|y(x)-\bar{y}|=|y(x)-y(\bar{x})|<\varepsilon,$$

即$y(x)$在$\bar{x}$连续，由$\bar{x}$的任意性可知$y=y(x)$在$(x_0-\delta, x_0+\delta)$连续.

Claim: 在(iv)下，$y = y(x)$在x_0的某邻域内一阶导数连续.

事实上，任取$x \in (x_0 - \delta, x_0 + \delta)$,

当$0 < |\Delta x| < \min\{x_0 + \delta - x, x - x_0 + \delta\}$时,

$$\begin{aligned} 0 &= f(x+\Delta x, y(x+\Delta x)) - f(x, y(x)) \\ &= f(x+\Delta x, y(x+\Delta x)) - f(x, y(x+\Delta x)) \\ &\quad + f(x, y(x+\Delta x)) - f(x, y(x)). \end{aligned}$$

由一元函数的中值定理可得

$$0 = f'_x(x + \theta_1 \Delta x, y(x+\Delta x))\Delta x + f'_y(x, y(x) + \theta_2 \Delta y)\Delta y,$$

其中$\Delta y = y(x+\Delta x) - y(x)$, $0 < \theta_1, \theta_2 < 1$.

$$0 = f'_x(x+\theta_1\Delta x, y(x+\Delta x))\Delta x + f'_y(x, y(x)+\theta_2\Delta y)\Delta y.$$

由于在D上$f'_y(x,y)\neq 0$, 从而

$$\frac{\Delta y}{\Delta x} = -\frac{f'_x(x+\theta_1\Delta x, y(x+\Delta x))}{f'_y(x, y(x)+\theta_2\Delta y)}.$$

由$f'_x(x,y)$和$f'_y(x,y)$的连续性可得

$$y'(x) = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = -\frac{f'_x(x, y(x))}{f'_y(x, y(x))}.$$

Corollary

在定理的条件下，进一步如果$f(x,y)$在(x_0,y_0)的某邻域k阶偏导数连续，则$y=y(x)$在x_0的某邻域k阶导数连续.

证明 (归纳法) 当$k=1$时，结论成立. 设当$k=m$时，结论成立. 当$k=m+1$时，由归纳假设$y(x)$在x_0的某邻域m阶导数连续. 又由定理可知

$$y'(x)=-\frac{f'_x(x,y(x))}{f'_y(x,y(x))}.$$

由于$f(x,y)$在(x_0,y_0)的某邻域$m+1$阶偏导数连续，从而$y'(x)$在x_0的某邻域m阶导数连续，即$y(x)$的$m+1$阶导数连续.

Theorem (隐函数定理)

假设

(i) $f(x_{01},\cdots,x_{0n},y_0)=0$;

(ii) $f(x_1,\cdots,x_n,y)$, $f'_y(x_1,\cdots,x_n,y)$在$(x_{01},\cdots,x_{0n},y_0)$的一个邻域内连续;

(iii) $f'_y(x_{01},\cdots,x_{0n},y_0)\neq 0$.

则存在$(x_{01},\cdots,x_{0n})$的邻域U及y_0的邻域V以及唯一的n元函数$y:U\to V$使得

$$f(x_1,\cdots,x_n,y(x_1,\cdots,x_n))=0,\quad \forall(x_1,\cdots,x_n)\in U.$$

进一步$y(x_1,\cdots,x_n)$连续且$y_0=y(x_{01},\cdots,x_{0n})$.

Theorem (续)

(iv) 进一步, 若 $f'_{x_i}(x_1,\cdots,x_n,y)$连续, 则$\dfrac{\partial y}{\partial x_i}$连续, 且

$$\frac{\partial y}{\partial x_i}(x_1,\cdots,x_n)=-\frac{f'_{x_i}(x_1,\cdots,x_n,y(x_1,\cdots,x_n))}{f'_y(x_1,\cdots,x_n,y(x_1,\cdots,x_n))}.$$

*(v)*若$f(x_1,\cdots,x_n,y)$为k阶偏导数连续, 则$y(x_1,\cdots,x_n)$是k阶偏导数连续.

Example

设$f(x,y)\in C^2$且$f'_x\neq 0$, 从$z=f(x,y)$中解出函数$x=g(y,z)$, 如果$f''_{xx}f''_{yy}-f''^2_{xy}=0$, 求证

$$\frac{\partial^2 g}{\partial y^2}\frac{\partial^2 g}{\partial z^2}-\left(\frac{\partial^2 g}{\partial y\partial z}\right)^2=0.$$

证 由$z=f(x,y)$, 将x视为y,z的函数, 在这个恒等式中分别对y,z求偏导, 有

$$0=f'_x\frac{\partial g}{\partial y}+f'_y,$$

$$1=f'_x\frac{\partial g}{\partial z}.$$

前式对y,z求偏导，后式对z求偏导，可得

$$\begin{aligned}
0 &= f''_{xx}\left(\frac{\partial g}{\partial y}\right)^2 + 2f''_{xy}\frac{\partial g}{\partial y} + f''_{yy} + f'_x\frac{\partial^2 g}{\partial y^2},\\
0 &= f''_{xx}\frac{\partial g}{\partial y}\frac{\partial g}{\partial z} + f'_x\frac{\partial^2 g}{\partial y\partial z} + f''_{xy}\frac{\partial g}{\partial z},\\
0 &= f''_{xx}\left(\frac{\partial g}{\partial z}\right)^2 + f'_x\frac{\partial^2 g}{\partial z^2}.
\end{aligned}$$

从而

$$\begin{aligned}
& {f'_x}^2\frac{\partial^2 g}{\partial y^2}\frac{\partial^2 g}{\partial z^2} - {f'_x}^2\left(\frac{\partial^2 g}{\partial y\partial z}\right)^2\\
= \ & f''_{xx}\left(\frac{\partial g}{\partial z}\right)^2\left(f''_{xx}\left(\frac{\partial g}{\partial y}\right)^2 + 2f''_{xy}\frac{\partial g}{\partial y} + f''_{yy}\right)\\
& - \left(f''_{xx}\frac{\partial g}{\partial z}\frac{\partial g}{\partial y} + f''_{xy}\frac{\partial g}{\partial z}\right)^2 = \left(\frac{\partial g}{\partial z}\right)^2\left(f''_{xx}f''_{yy} - {f''_{xy}}^2\right) = 0.
\end{aligned}$$

于是

$$\frac{\partial^2 g}{\partial y^2}\frac{\partial^2 g}{\partial z^2} - \left(\frac{\partial^2 g}{\partial y\partial z}\right)^2 = 0.$$

对于方程组

$$\begin{cases} f_1(x_1,\cdots,x_n,y_1,\cdots,y_{m-1},y_m)=0, \\ \cdots\quad\cdots \\ f_{m-1}(x_1,\cdots,x_n,y_1,\cdots,y_{m-1},y_m)=0, \\ f_m(x_1,\cdots,x_n,y_1,\cdots,y_{m-1},y_m)=0, \end{cases} \tag{4}$$

记

$$F(X,Y)=(f_1(X,Y),\cdots,f_m(X,Y))^T=0.$$

Theorem (隐函数定理)

假设

(i) $(X_0, Y_0) = (x_{01}, \cdots, x_{0n}, y_{01}, \cdots, y_{0m})$满足方程组;

(ii) 映射F及$\dfrac{\partial F}{\partial y_k}, 1 \leqslant k \leqslant m$, 均在$(X_0, Y_0)$的一个邻域内连续;

(iii) F关于$y_1, \cdots, y_m$的雅可比行列式

$$\frac{D(f_1, \cdots, f_m)}{D(y_1, \cdots, y_m)}(X_0, Y_0) = \begin{vmatrix} \dfrac{\partial f_1}{\partial y_1} & \cdots & \dfrac{\partial f_1}{\partial y_m} \\ & \cdots & \\ \dfrac{\partial f_m}{\partial y_1} & \cdots & \dfrac{\partial f_m}{\partial y_m} \end{vmatrix}(X_0, Y_0) \neq 0,$$

则在(X_0, Y_0)的一个邻域内, 方程组唯一确定映射$Y = Y(X)$, 即函数组

$$y_i = y_i(x_1, \cdots, x_n), i = 1, 2, \cdots, m,$$

而且$Y(X)$在X_0的一个邻域内连续, 同时满足$Y_0 = Y(X_0)$.

Theorem (续)

进一步若F在(X_0, Y_0)的一个邻域内关于$x_i(1 \leqslant i \leqslant n)$的偏导数连续，则$y_1, \cdots, y_m$ 在$(x_{01}, \cdots, x_{0n})$的一个邻域内关于x_i的偏导数存在，连续且

$$\begin{pmatrix} \dfrac{\partial y_1}{\partial x_i} \\ \cdots \\ \dfrac{\partial y_m}{\partial x_i} \end{pmatrix} = - \begin{pmatrix} \dfrac{\partial f_1}{\partial y_1} & \cdots & \dfrac{\partial f_1}{\partial y_m} \\ & \cdots & \\ \dfrac{\partial f_m}{\partial y_1} & \cdots & \dfrac{\partial f_m}{\partial y_m} \end{pmatrix}^{-1} \begin{pmatrix} \dfrac{\partial f_1}{\partial x_i} \\ \cdots \\ \dfrac{\partial f_m}{\partial x_i} \end{pmatrix}. \tag{5}$$

这时如果上式对$i = 1, \cdots, n$均成立，写成紧凑的矩阵形式有

$$\frac{\partial(y_1, y_2, \cdots, y_m)}{\partial(x_1, x_2, \cdots, x_n)} = -\left(\frac{\partial(f_1, f_2, \cdots, f_m)}{\partial(y_1, y_2, \cdots, y_m)}\right)^{-1} \frac{\partial(f_1, f_2, \cdots, f_m)}{\partial(x_1, x_2, \cdots, x_n)}.$$

证 $(m \geqslant 2)$,设定理对$m-1$个方程成立.
记$X=(x_1,\cdots,x_n), Y=(y_1,\cdots,y_m)$. 由假设$\dfrac{D(f_1,\cdots,f_m)}{D(y_1,\cdots,y_m)}(X_0,Y_0)\neq 0$, 不妨设$\dfrac{D(f_1,\cdots,f_{m-1})}{D(y_1,\cdots,y_{m-1})}(X_0,Y_0)\neq 0$. 用归纳假设在(4)中从前$m-1$个方程可唯一解出$y_i=\bar{y}_i(X,y_m), \quad i=1,2,\cdots,m-1$, 满足

$$y_{0i}=\bar{y}_i(X_0,y_{0m}), \quad i=1,2,\cdots,m-1, \tag{6}$$

且

$$\begin{pmatrix} \dfrac{\partial \bar{y}_1}{\partial y_m} \\ \cdots \\ \dfrac{\partial \bar{y}_{m-1}}{\partial y_m} \end{pmatrix} = -\begin{pmatrix} \dfrac{\partial f_1}{\partial y_1} & \cdots & \dfrac{\partial f_1}{\partial y_{m-1}} \\ & \cdots & \\ \dfrac{\partial f_{m-1}}{\partial y_1} & \cdots & \dfrac{\partial f_{m-1}}{\partial y_{m-1}} \end{pmatrix}^{-1} \begin{pmatrix} \dfrac{\partial f_1}{\partial y_m} \\ \cdots \\ \dfrac{\partial f_{m-1}}{\partial y_m} \end{pmatrix}. \tag{7}$$

把$y_i=\bar{y}_i(X,y_m)$, $i=1,2,\cdots,m-1$ 代入方程组(4)的最后一个方程, 得

$$g(X,y_m)\equiv f_m(X,\bar{y}_1(X,y_m),\cdots,\bar{y}_{m-1}(X,y_m),y_m)=0.$$

若$g'_{y_m}(X_0,y_{0m})\neq 0$, 则在$(X_0,y_{0m})$的一个更小的邻域内唯一确定函数$y_m=y_m(X)$, 代入$y_i=\bar{y}_i(X,y_m)$, $i=1,\cdots,m-1$中即得

$$y_i=y_i(X)=\bar{y}_i(X,y_m(X)),\ i=1,\cdots,m.$$

事实上, 由链式法则, 可得

$$\begin{aligned} g'_{y_m}(X_0,y_{0m}) &= \sum_{i=1}^{m-1}\frac{\partial f_m}{\partial y_i}(X_0,Y_0)\frac{\partial \bar{y}_i}{\partial y_m}(X_0,Y_0)+\frac{\partial f_m}{\partial y_m}(X_0,Y_0) \\ &= -\beta B^{-1}\alpha^T+\frac{\partial f_m}{\partial y_m}(X_0,Y_0), \quad \text{由(7)} \end{aligned}$$

这里

$$
\begin{aligned}
\beta &= \left(\frac{\partial f_m}{\partial y_1}, \cdots, \frac{\partial f_m}{\partial y_{m-1}}\right)(X_0, Y_0), \\
\alpha &= \left(\frac{\partial f_1}{\partial y_m}, \cdots, \frac{\partial f_{m-1}}{\partial y_m}\right)(X_0, Y_0), \\
B &= \begin{pmatrix} \dfrac{\partial f_1}{\partial y_1} & \cdots & \dfrac{\partial f_1}{\partial y_{m-1}} \\ & \cdots & \\ \dfrac{\partial f_{m-1}}{\partial y_1} & \cdots & \dfrac{\partial f_{m-1}}{\partial y_{m-1}} \end{pmatrix}(X_0, Y_0),
\end{aligned}
$$

显然

$$
\frac{\partial(f_1, \cdots, f_m)}{\partial(y_1, \cdots, y_m)}(X_0, Y_0) = \begin{pmatrix} B & \alpha^T \\ \beta & \dfrac{\partial f_m}{\partial y_m}(X_0, Y_0) \end{pmatrix}. \tag{8}
$$

注意到

$$\begin{pmatrix} I_{n-1} & O \\ -\beta\cdot B^{-1} & 1 \end{pmatrix}\begin{pmatrix} B & \alpha^T \\ \beta & \dfrac{\partial f_m}{\partial y_m}(X_0,Y_0) \end{pmatrix}$$

$$= \begin{pmatrix} B & \alpha^T \\ O & \dfrac{\partial f_m}{\partial y_m}(X_0,Y_0)-\beta\cdot B^{-1}\cdot\alpha^T \end{pmatrix},$$

由条件, 雅可比行列式

$$\frac{D(f_1,\cdots,f_m)}{D(y_1,\cdots,y_m)}(X_0,Y_0) = \det\begin{pmatrix} B & \alpha^T \\ O & \dfrac{\partial f_m}{\partial y_m}(X_0,Y_0)-\beta\cdot B^{-1}\alpha^T \end{pmatrix}\neq 0 \tag{9}$$

可得 $\dfrac{\partial f_m}{\partial y_m}(X_0,Y_0)-\beta\cdot B^{-1}\cdot\alpha^T\neq 0$.

由以上计算和讨论可知, $y_1, \cdots, y_m$都在X_0的某邻域内有连续的偏导数, 且满足

$$\begin{cases} f_1(x_1,\cdots,x_n,y_1(x_1,\cdots,x_n),\cdots,y_m(x_1,\cdots,x_n))=0, \\ \cdots\quad\cdots \\ f_m(x_1,\cdots,x_n,y_1(x_1,\cdots,x_n),\cdots,y_m(x_1,\cdots,x_n))=0, \end{cases} \tag{10}$$

两边分别对变量x_i求偏导数, 得到

$$\begin{cases} \dfrac{\partial f_1}{\partial x_i}+\displaystyle\sum_{j=1}^m \dfrac{\partial f_1}{\partial y_j}\dfrac{\partial y_j}{\partial x_i}=0, \\ \cdots\quad\cdots \\ \dfrac{\partial f_m}{\partial x_i}+\displaystyle\sum_{j=1}^m \dfrac{\partial f_m}{\partial y_j}\dfrac{\partial y_j}{\partial x_i}=0, \end{cases} \tag{11}$$

用矩阵写成一个紧凑公式为

$$\frac{\partial(f_1,\cdots,f_m)}{\partial(x_1,\cdots,x_n)}+\frac{\partial(f_1,\cdots,f_m)}{\partial(y_1,\cdots,y_m)}\frac{\partial(y_1,\cdots,y_m)}{\partial(x_1,\cdots,x_n)}=0.$$

解此线性方程组可得(3)式, 即

$$\frac{\partial(y_1,\cdots,y_m)}{\partial(x_1,\cdots,x_n)}=-\left(\frac{\partial(f_1,\cdots,f_m)}{\partial(y_1,\cdots,y_m)}\right)^{-1}\frac{\partial(f_1,\cdots,f_m)}{\partial(x_1,\cdots,x_n)}.$$

考虑方程组

$$\begin{cases} y_1 = f_1(x_1, \cdots, x_n), \\ \qquad \cdots \\ y_n = f_n(x_1, \cdots, x_n), \end{cases} \tag{12}$$

(12)可写成$Y = F(X)$, 这里$X = (x_1, \cdots, x_n)$, $Y = (y_1, \cdots, y_n)$, $F = (f_1, \cdots, f_n)$.

Theorem (逆映射定理)

设

(i) $y_{0i}=f_i(x_{01},\cdots,x_{0n}), i=1,2,\cdots,n;$

(ii) $f_1,\cdots,f_n$都在$(x_{01},\cdots,x_{0n})$的一个邻域D内一阶偏导数连续;

(iii) $f_1,\cdots,f_n$关于$x_1,\cdots,x_n$的雅可比行列式

$$\frac{D(f_1,\cdots,f_n)}{D(x_1,\cdots,x_n)}(X_0)=\det\left(\frac{\partial(f_1,\cdots,f_n)}{\partial(x_1,\cdots,x_n)}(X_0)\right)\neq 0.$$

则存在X_0的邻域$U\subseteq D$和Y_0的邻域V, 以及唯一$G:V\to U$, 即

$$\begin{cases} x_1=g_1(y_1,\cdots,y_n), \\ \quad\cdots \\ x_n=g_n(y_1,\cdots,y_n), \end{cases} \tag{13}$$

$$U \xrightarrow{F} V \xrightarrow{G} U \xrightarrow{F} V.$$

Theorem (逆映射定理(续))

使得$V = F(U)$, $U = G(V)$, 且对于任意$i = 1, 2, \cdots, n$有

$$f_i(g_1(y_1, \cdots, y_n), \cdots, g_n(y_1, \cdots, y_n)) = y_i,$$

$$g_i(f_1(x_1, \cdots, x_n), \cdots, f_n(x_1, \cdots, x_n)) = x_i,$$

其中$\forall\ (x_1, \cdots, x_n) \in U,\ (y_1, \cdots, y_n) \in V$.

进一步，$g_1, \cdots, g_n$在V内一阶偏导数连续, 且

$$\frac{\partial(g_1, \cdots, g_n)}{\partial(y_1, \cdots, y_n)}(Y) = \left(\frac{\partial(f_1, \cdots, f_n)}{\partial(x_1, \cdots, x_n)}\right)^{-1}(X).$$

证明 定义$F=(f_1,\cdots,f_n):D\longrightarrow\mathbb{R}^n$, 其中$D$是$X_0$的某邻域.

由条件(i)和(ii)即知$Y_0=F(X_0)$且$F\in C^1(D,\mathbb{R}^n)$.

条件(iii)等价于$J_F(X_0)$非奇异. 对于$i=1,\cdots,n$,令

$$\tilde{f}_i(x_1,\cdots,x_n,y_1,\cdots,y_n)=f_i(x_1,\cdots,x_n)-y_i.$$

由隐函数定理(注意到$\tilde{f}_1,\cdots,\tilde{f}_n$关于$x_1,\cdots,x_n$) 知,
存在X_0的邻域$\widetilde{U}\subseteq D$和Y_0的邻域$\widetilde{V}$, 以及唯一的映射$G:\widetilde{V}\to\widetilde{U}$,其分量表示为函数组(13), 使得

$$F(G(Y))=Y,\quad \forall\ Y\in\widetilde{V},$$

$$G(F(X))=X,\quad \forall\ X\in\widetilde{U},\quad F(X)\in\widetilde{V}.$$

且$G \in C^1(\widetilde{V}, \widetilde{U})$,

$$\frac{\partial(g_1, \cdots, g_n)}{\partial(y_1, \cdots, y_n)}(Y) = \left(\frac{\partial(f_1, \cdots, f_n)}{\partial(x_1, \cdots, x_n)}\right)^{-1}(X) \cdot I_n$$

这里$Y \in \widetilde{V}$, $X = G(Y)$与$Y = F(X)$是对应点. 由于$F \in C^1(\widetilde{U}, \mathbb{R}^n)$, 从而存在$X_0$的开邻域$U \subseteq \widetilde{U}$, 使得$U$ 在映射F下的像集$F(U) \subseteq \widetilde{V}$, 记$V = F(U)$, 显然$Y_0 = F(X_0) \in V$, 并且

$$F(G(Y)) = Y, \quad \forall\ Y \in V,$$

$$G(F(X)) = X, \quad \forall\ X \in U.$$

于是$U = G(F(U)) = G(V)$. 因为U是X_0的开邻域，所以$V = G^{-1}(U)$也是开集，即为Y_0的开邻域.

Theorem

设D是$\mathbb{R}^n$中的开集，$F \in C^1(D, \mathbb{R}^n)$. 如果$J_F(X)$在$D$内处处非奇异，则$F$把$D$的任何开子集映成$\mathbb{R}^n$中的开集.

Example

考虑$\mathbb{R}^n$中的球坐标

$$\begin{cases} x_1 = r\ \cos\theta_1, \\ x_2 = r\ \sin\theta_1\cos\theta_2, \\ x_3 = r\ \sin\theta_1\sin\theta_2\cos\theta_3, \\ \qquad \cdots \\ x_{n-1} = r\ \sin\theta_1\cdots\sin\theta_{n-2}\cos\theta_{n-1}, \\ x_n = r\ \sin\theta_1\cdots\sin\theta_{n-2}\sin\theta_{n-1}, \end{cases} \tag{14}$$

其中$r \geqslant 0, 0 \leqslant \theta_i \leqslant \pi, i = 1, \cdots, n-2, 0 \leqslant \theta_{n-1} < 2\pi$. 求雅可比行列式$\dfrac{D(x_1, \cdots, x_n)}{D(r, \theta_1, \cdots, \theta_{n-1})}$.

设

$$\begin{aligned} f_1(r,\theta_1,\cdots,\theta_{n-1},x_1,\cdots,x_n) &= r^2-(x_1^2+\cdots+x_n^2),\\ f_2(r,\theta_1,\cdots,\theta_{n-1},x_1,\cdots,x_n) &= r^2\sin^2\theta_1-(x_2^2+\cdots+x_n^2),\\ \cdots &\quad \cdots\\ f_n(r,\theta_1,\cdots,\theta_{n-1},x_1,\cdots,x_n) &= r^2\sin^2\theta_1\cdots\sin^2\theta_{n-1}-x_n^2. \end{aligned}$$

显然(14)满足

$$f_i(r,\theta_1,\cdots,\theta_{n-1},x_1,\cdots,x_n)=0,\quad i=1,\cdots,n.$$

从而

$$\frac{\partial(f_1,\cdots,f_n)}{\partial(r,\theta_1,\cdots,\theta_{n-1})}+\frac{\partial(f_1,\cdots,f_n)}{\partial(x_1,\cdots,x_n)}\frac{\partial(x_1,\cdots,x_n)}{\partial(r,\theta_1,\cdots,\theta_{n-1})}=0$$

解答(续1)

由此可得

$$\frac{D(x_1,\cdots,x_n)}{D(r,\theta_1,\cdots,\theta_{n-1})}=(-1)^n\left[\frac{D(f_1,\cdots,f_n)}{D(x_1,\cdots,x_n)}\right]^{-1}\frac{D(f_1,\cdots,f_n)}{D(r,\theta_1,\cdots,\theta_{n-1})}. \quad (15)$$

由于

$$\frac{D(f_1,\cdots,f_n)}{D(x_1,\cdots,x_n)}=\begin{vmatrix}-2x_1 & -2x_2 & \cdots & -2x_n\\ 0 & -2x_2 & \cdots & -2x_n\\ \cdots & \cdots & \cdots & \cdots\\ 0 & \cdots & 0 & -2x_n\end{vmatrix}=(-1)^n2^nx_1\cdots x_n$$

$$=\quad(-1)^n2^nr^n\sin^{n-1}\theta_1\sin^{n-2}\theta_2\cdots\sin\theta_{n-1}\cos\theta_1\cdots\cos\theta_{n-1},$$

又容易算出

$$\begin{aligned}&\frac{D(f_1,\cdots,f_n)}{D(r,\theta_1,\cdots,\theta_{n-1})}\\=\ &2^n r^{2n-1}\sin^{2n-3}\theta_1\sin^{2n-5}\theta_2\cdots\sin\theta_{n-1}\cos\theta_1\cdots\cos\theta_{n-1}.\end{aligned}$$

由(15)可得

$$\frac{D(x_1,\cdots,x_n)}{D(r,\theta_1,\cdots,\theta_{n-1})}=r^{n-1}\sin^{n-2}\theta_1\cdots\sin\theta_{n-2}.$$

当$x_1x_2\cdots x_n=0$时, (15)式没有意义, 但是所有这样的点在坐标面上, 可用满足条件$x_1x_2\cdots x_n\neq 0$的点任意逼近, 由(15)式左边的连续性, 可知结论对任意点均成立.

当$n=2$时为极坐标

$$\begin{cases} x = r\ \cos\theta \\ y = r\ \sin\theta, \end{cases}$$

则$\dfrac{D(x,y)}{D(r,\theta)} = r.$

当$n=3$时为空间球坐标

$$\begin{cases} x = r\ \sin\varphi\cos\theta, \\ y = r\ \sin\varphi\sin\theta, \\ z = r\ \cos\varphi, \end{cases}$$

其中$r \geqslant 0,\ 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \varphi \leqslant \pi,$ 则$\dfrac{D(x,y,z)}{D(r,\varphi,\theta)} = r^2\sin\varphi.$