# 第16讲

# Cofinality

Let $\alpha > 0$ be a limit ordinal. We say that an increasing $\beta$-sequence $\langle \alpha_\xi : \xi < \beta \rangle$, $\beta$ a limit ordinal, is *cofinal* in $\alpha$ if $\lim_{\xi \to \beta} \alpha_\xi = \alpha$. Similarly, $A \subset \alpha$ is *cofinal* in $\alpha$ if $\sup A = \alpha$. If $\alpha$ is an infinite limit ordinal, the *cofinality* of $\alpha$ is

$$\begin{aligned} \operatorname{cf} \alpha = {} & \text{the least limit ordinal } \beta \text{ such that there is an increasing} \\ & \beta\text{-sequence } \langle \alpha_\xi : \xi < \beta \rangle \text{ with } \lim_{\xi \to \beta} \alpha_\xi = \alpha. \end{aligned}$$

Obviously, $\operatorname{cf} \alpha$ is a limit ordinal, and $\operatorname{cf} \alpha \leq \alpha$. Examples: $\operatorname{cf}(\omega + \omega) = \operatorname{cf} \aleph_\omega = \omega$.

**Lemma 3.6.** $\operatorname{cf}(\operatorname{cf} \alpha) = \operatorname{cf} \alpha$.

*Proof.* If $\langle \alpha_\xi : \xi < \beta \rangle$ is cofinal in $\alpha$ and $\langle \xi(\nu) : \nu < \gamma \rangle$ is cofinal in $\beta$, then $\langle \alpha_{\xi(\nu)} : \nu < \gamma \rangle$ is cofinal in $\alpha$. $\square$

# Two useful facts about cofinality

**Lemma 3.7.** *Let $\alpha > 0$ be a limit ordinal.*

(i) *If $A \subset \alpha$ and $\sup A = \alpha$, then the order-type of $A$ is at least $\operatorname{cf} \alpha$.*

(ii) *If $\beta_0 \leq \beta_1 \leq \ldots \leq \beta_\xi \leq \ldots$, $\xi < \gamma$, is a nondecreasing $\gamma$-sequence of ordinals in $\alpha$ and $\lim_{\xi \to \gamma} \beta_\xi = \alpha$, then $\operatorname{cf} \gamma = \operatorname{cf} \alpha$.*

*Proof.* (i) The order-type of $A$ is the length of the increasing enumeration of $A$ which is an increasing sequence with limit $\alpha$.

(ii) If $\gamma = \lim_{\nu \to \operatorname{cf} \gamma} \xi(\nu)$, then $\alpha = \lim_{\nu \to \operatorname{cf} \gamma} \beta_{\xi(\nu)}$, and the nondecreasing sequence $\langle \beta_{\xi(\nu)} : \nu < \operatorname{cf} \gamma \rangle$ has an increasing subsequence of length $\leq \operatorname{cf} \gamma$, with the same limit. Thus $\operatorname{cf} \alpha \leq \operatorname{cf} \gamma$.

To show that $\operatorname{cf} \gamma \leq \operatorname{cf} \alpha$, let $\alpha = \lim_{\nu \to \operatorname{cf} \alpha} \alpha_\nu$. For each $\nu < \operatorname{cf} \alpha$, let $\xi(\nu)$ be the least $\xi$ greater than all $\xi(\iota)$, $\iota < \nu$, such that $\beta_\xi > \alpha_\nu$. Since $\lim_{\nu \to \operatorname{cf} \alpha} \beta_{\xi(\nu)} = \alpha$, it follows that $\lim_{\nu \to \operatorname{cf} \alpha} \xi(\nu) = \gamma$, and so $\operatorname{cf} \gamma \leq \operatorname{cf} \alpha$. $\square$

# Regular cadinals and singular cardinals

An infinite cardinal $\aleph_\alpha$ is *regular* if $\operatorname{cf} \omega_\alpha = \omega_\alpha$. It is *singular* if $\operatorname{cf} \omega_\alpha < \omega_\alpha$.

**Lemma 3.8.** *For every limit ordinal $\alpha$, $\operatorname{cf} \alpha$ is a regular cardinal.*

*Proof.* It is easy to see that if $\alpha$ is not a cardinal, then using a mapping of $|\alpha|$ onto $\alpha$, one can construct a cofinal sequence in $\alpha$ of length $\leq |\alpha|$, and therefore $\operatorname{cf} \alpha < \alpha$.

Since $\operatorname{cf}(\operatorname{cf} \alpha) = \operatorname{cf} \alpha$, it follows that $\operatorname{cf} \alpha$ is a cardinal and is regular. $\square$

# Bound subsets and unbound subsets of a limit ordinal

Let $\kappa$ be a limit ordinal. A subset $X \subset \kappa$ is *bounded* if $\sup X < \kappa$, and *unbounded* if $\sup X = \kappa$.

**Lemma 3.9.** *Let $\kappa$ be an aleph.*

(i) *If $X \subset \kappa$ and $|X| < \operatorname{cf} \kappa$ then $X$ is bounded.*
(ii) *If $\lambda < \operatorname{cf} \kappa$ and $f : \lambda \to \kappa$ then the range of $f$ is bounded.*

It follows from (i) that every unbounded subset of a regular cardinal has cardinality $\kappa$.

*Proof.* (i) Lemma 3.7(i).
(ii) If $X = \operatorname{ran}(f)$ then $|X| \leq \lambda$, and use (i). $\square$

# Examples of regular cadinals and singular cardinals

There are arbitrarily large singular cardinals. For each $\alpha$, $\aleph_{\alpha+\omega}$ is a singular cardinal of cofinality $\omega$.

Using the Axiom of Choice, we shall show in Chapter 5 that every $\aleph_{\alpha+1}$ is regular. (The Axiom of Choice is necessary.)

Therefore, the smallest three singular infinite cardinals are $\aleph_\omega$, $\aleph_{\omega+\omega}$ and $\aleph_{\omega+\omega+\omega}$.

# $\mathrm{cf}(\aleph_\alpha)$

$\mathrm{cf}(\aleph_\alpha) = \aleph_\alpha$ if $\alpha$ is a successor ordinal; $\mathrm{cf}(\aleph_\alpha) = \mathrm{cf}\,\alpha$ if $\alpha > 0$ is a limit ordinal.

**Proof**

If $\alpha = \beta + 1$ is a successor ordinal, then $\aleph_\alpha = \aleph_{\beta+1}$ is a regular cardinal. Hence $\mathrm{cf}(\aleph_\alpha) = \aleph_\alpha$. If $\alpha > 0$ is a limit ordinal, then $\langle \aleph_\beta : \beta < \alpha \rangle$ is cofinal in $\aleph_\alpha$. Hence $\mathrm{cf}(\aleph_\alpha) = \mathrm{cf}\,\alpha$ by Lemma 3.7 (ii).

# The condition that an infinite cardinal is singular

**Lemma 3.10.** *An infinite cardinal $\kappa$ is singular if and only if there exists a cardinal $\lambda < \kappa$ and a family $\{S_\xi : \xi < \lambda\}$ of subsets of $\kappa$ such that $|S_\xi| < \kappa$ for each $\xi < \lambda$, and $\kappa = \bigcup_{\xi<\lambda} S_\xi$. The least cardinal $\lambda$ that satisfies the condition is* $\operatorname{cf}\kappa$.

*Proof.* If $\kappa$ is singular, then there is an increasing sequence $\langle \alpha_\xi : \xi < \operatorname{cf}\kappa\rangle$ with $\lim_\xi \alpha_\xi = \kappa$. Let $\lambda = \operatorname{cf}\kappa$, and $S_\xi = \alpha_\xi$ for all $\xi < \lambda$.

If the condition holds, let $\lambda < \kappa$ be the least cardinal for which there is a family $\{S_\xi : \xi < \lambda\}$ such that $\kappa = \bigcup_{\xi<\lambda} S_\xi$ and $|S_\xi| < \kappa$ for each $\xi < \lambda$. For every $\xi < \lambda$, let $\beta_\xi$ be the order-type of $\bigcup_{\nu<\xi} S_\nu$. The sequence $\langle \beta_\xi : \xi < \lambda\rangle$ is nondecreasing, and by the minimality of $\lambda$, $\beta_\xi < \kappa$ for all $\xi < \lambda$. We shall show that $\lim_\xi \beta_\xi = \kappa$, thus proving that $\operatorname{cf}\kappa \le \lambda$.

Let $\beta = \lim_{\xi\to\lambda} \beta_\xi$. There is a one-to-one mapping $f$ of $\kappa = \bigcup_{\xi<\lambda} S_\xi$ into $\lambda \times \beta$: If $\alpha \in \kappa$, let $f(\alpha) = (\xi, \gamma)$, where $\xi$ is the least $\xi$ such that $\alpha \in S_\xi$ and $\gamma$ is the order-type of $S_\xi \cap \alpha$. Since $\lambda < \kappa$ and $|\lambda \times \beta| = \lambda \cdot |\beta|$, it follows that $\beta = \kappa$. $\square$

# Some propositions

One cannot prove without the Axiom of Choice that $\omega_1$ is not a countable union of countable sets. Compare this with Exercise 3.13

**3.13 (ZF).** Show that $\omega_2$ is not a countable union of countable sets.

[Assume that $\omega_2 = \bigcup_{n<\omega} S_n$ with $S_n$ countable and let $\alpha_n$ be the order-type of $S_n$. Then $\alpha = \sup_n \alpha_n \leq \omega_1$ and there is a mapping of $\omega \times \alpha$ onto $\omega_2$.]

The only cardinal inequality we have proved so far is Cantor's Theorem $\kappa < 2^\kappa$. It follows that $\kappa < \lambda^\kappa$ for every $\lambda > 1$, and in particular $\kappa < \kappa^\kappa$ (for $\kappa \neq 1$). The following theorem gives a better inequality. This and other cardinal inequalities will also follow from König's Theorem 5.10, to be proved in Chapter 5.

# An inequality

**Theorem 3.11.** *If $\kappa$ is an infinite cardinal, then $\kappa < \kappa^{\operatorname{cf}\kappa}$.*

*Proof.* Let $F$ be a collection of $\kappa$ functions from $\operatorname{cf}\kappa$ to $\kappa$: $F = \{f_\alpha : \alpha < \kappa\}$. It is enough to find $f : \operatorname{cf}\kappa \to \kappa$ that is different from all the $f_\alpha$. Let $\kappa = \lim_{\xi \to \operatorname{cf}\kappa} \alpha_\xi$. For $\xi < \operatorname{cf}\kappa$, let

$$f(\xi) = \text{least } \gamma \text{ such that } \gamma \neq f_\alpha(\xi) \text{ for all } \alpha < \alpha_\xi.$$

Such $\gamma$ exists since $|\{f_\alpha(\xi) : \alpha < \alpha_\xi\}| \leq |\alpha_\xi| < \kappa$. Obviously, $f \neq f_\alpha$ for all $\alpha < \kappa$. $\square$

Consequently, $\kappa^\lambda > \kappa$ whenever $\lambda \geq \operatorname{cf}\kappa$.

## Another inequality

$$\text{If } \lambda \geq 2 \text{ and } \kappa \text{ is infinite, then } \mathrm{cf}(\lambda^\kappa) > \kappa.$$

### Proof

If $\mathrm{cf}\left(\lambda^\kappa\right) \leq \kappa$, then

$$\left(\lambda^\kappa\right)^{\mathrm{cf}\left(\lambda^\kappa\right)} \leq \left(\lambda^\kappa\right)^\kappa = \lambda^{\kappa \cdot \kappa} = \lambda^\kappa,$$

and this contradicts Theorem 3.11.

# Weakly inaccessible cardinals

An uncountable cardinal $\kappa$ is *weakly inaccessible* if it is a limit cardinal and is regular. There will be more about inaccessible cardinals later, but let me mention at this point that existence of (weakly) inaccessible cardinals is not provable in ZFC.

To get an idea of the size of an inaccessible cardinal, note that if $\aleph_\alpha > \aleph_0$ is limit and regular, then $\aleph_\alpha = \operatorname{cf} \aleph_\alpha = \operatorname{cf} \alpha \le \alpha$, and so $\aleph_\alpha = \alpha$.

Since the sequence of alephs is a normal sequence, it has arbitrarily large fixed points; the problem is whether some of them are regular cardinals. For instance, the least fixed point $\aleph_\alpha = \alpha$ has cofinality $\omega$:

$$\kappa = \lim \langle \omega, \omega_\omega, \omega_{\omega_\omega}, \ldots \rangle = \lim_{n\to\omega} \kappa_n$$

$$\text{where } \kappa_0 = \omega,\ \kappa_{n+1} = \omega_{\kappa_n}.$$