换元积分法

数学分析I

第28讲

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我们通常假设被积函数都有原函数, 并且也忽略原函数成立的区间. 沿用上一节的记号，设$F(x)$是$f(x)$在区间I上的一个原函数, 于是有

$$\int f(x)\mathrm{d}x = F(x) + C. \tag{1}$$

设函数$G(t) = F(\varphi(t))$是函数$F(x)$与函数$\varphi(t)$的复合函数, 则由复合函数求导法, $G'(t) = \left.\frac{\mathrm{d}F}{\mathrm{d}x}\right|_{x=\varphi(t)} \cdot \varphi'(t) = f(\varphi(t))\varphi'(t)$. 于是,

$$\int f(\varphi(t))\varphi'(t)\mathrm{d}t = G(t) + C = F(\varphi(t)) + C. \tag{2}$$

从而,

$$\int f(x)\mathrm{d}x = \int f(\varphi(t))\varphi'(t)\mathrm{d}t. \tag{3}$$

(3)式左边要用$x = \varphi(t)$代入才能成立. 从此式可以看出在积分表达式中配上“$\mathrm{d}x$”的好处, 它起到了微分的作用.

(3)是换元法的理论基础，从(3)右端出发，往左边变换，这就是**第一换元法**，也称**凑微分法**．它的格式如下

$$\begin{aligned}\int f(\varphi(t))\varphi'(t)\mathrm{d}t &= \int f(\varphi(t))\mathrm{d}\varphi(t) = \int f(x)\mathrm{d}x = F(x)|_{x=\varphi(t)} + C \\ &= F(\varphi(t)) + C.\end{aligned}$$

例 1

求不定积分 $\int \sin 2x\mathrm{d}x$.

$$\int \sin 2x\mathrm{d}x = \frac{1}{2}\int \sin t\mathrm{d}t\ (t = 2x) = -\frac{1}{2}\cos 2x + C.$$

$$\int \sin 2x\mathrm{d}x = 2\int t\mathrm{d}t\ (t = \sin x) = t^2 + C = \sin^2 x + C.$$

积累各种“凑微分”形式

- $\mathrm{d}x = \frac{1}{a}\mathrm{d}(ax+b),\ a \neq 0;$
- $x^\alpha \mathrm{d}x = \frac{1}{\alpha+1}\mathrm{d}\left(x^{\alpha+1}\right),\ \alpha \neq -1;$
- $\dfrac{1}{x}\mathrm{d}x = \mathrm{d}(\ln x);$
- $a^x \mathrm{d}x = \frac{1}{\ln a}\mathrm{d}(a^x),\ a > 0, a \neq 1;$
- $\mathrm{e}^x \mathrm{d}x = \mathrm{d}(\mathrm{e}^x),\ \mathrm{e}^{-x}\mathrm{d}x = -\mathrm{d}(\mathrm{e}^{-x});$
- $\sin x \mathrm{d}x = -\mathrm{d}(\cos x);$
- $\cos x \mathrm{d}x = \mathrm{d}(\sin x);$
- $\sec^2 x \mathrm{d}x = \mathrm{d}(\tan x);$
- $\dfrac{\mathrm{d}x}{\sqrt{1-x^2}} = \mathrm{d}(\arcsin x);$
- $\dfrac{\mathrm{d}x}{1+x^2} = \mathrm{d}(\arctan x);$

积累各种“凑微分”形式

- $\sinh x \mathrm{d}x = \mathrm{d}(\cosh x)$;
- $\cosh x \mathrm{d}x = \mathrm{d}(\sinh x)$;
- $\dfrac{1}{\cosh^2 x}\mathrm{d}x = \mathrm{d}(\tanh x)$;
- $\dfrac{\mathrm{d}x}{x \ln x} = \mathrm{d}(\ln \ln x)$;
- $\left(1 - \dfrac{1}{x^2}\right)\mathrm{d}x = \mathrm{d}\left(x + \dfrac{1}{x}\right)$;
- $\left(1 + \dfrac{1}{x^2}\right)\mathrm{d}x = \mathrm{d}\left(x - \dfrac{1}{x}\right)$;
- $(x+1)\mathrm{e}^x\mathrm{d}x = \mathrm{d}(x\mathrm{e}^x)$;
- $(\ln x + 1)\mathrm{d}x = \mathrm{d}(x \ln x)$;
- $\cdots$

例 2

求不定积分 $\int \frac{x}{1+x^4}\mathrm{d}x$.

令 $t=x^2$,

$$\int \frac{x}{1+x^4}\mathrm{d}x = \frac{1}{2}\int \frac{\mathrm{d}(x^2)}{1+(x^2)^2} = \frac{1}{2}\int \frac{\mathrm{d}t}{1+t^2}$$

$$= \frac{1}{2}\arctan t + C = \frac{1}{2}\arctan x^2 + C.$$

求
$$\int \frac{x^2 - x^4}{(x^2+1)^4}\mathrm{d}x.$$

$$\begin{aligned}
\int \frac{x^2 - x^4}{(x^2+1)^4}\mathrm{d}x &= \int \frac{\frac{1}{x^2} - 1}{\left(x + \frac{1}{x}\right)^4}\mathrm{d}x \\
= \quad & -\int \frac{\mathrm{d}\left(x + \frac{1}{x}\right)}{\left(x + \frac{1}{x}\right)^4} = \frac{1}{3\left(x + \frac{1}{x}\right)^3} + C \\
= \quad & \frac{x^3}{3(x^2+1)^3} + C.
\end{aligned}$$

例 3

求不定积分 $\int \tan x \mathrm{d}x$.

$$\int \tan x \mathrm{d}x = \int \frac{\sin x}{\cos x}\mathrm{d}x = -\int \frac{\mathrm{d}(\cos x)}{\cos x} = -\ln|\cos x| + C.$$

例 4

求 $\displaystyle\int \frac{\mathrm{d}x}{a^2-x^2}$, $a>0$.

注意到

$$\frac{1}{a^2-x^2}=\frac{1}{2a}\left(\frac{1}{a-x}+\frac{1}{a+x}\right),$$

而且

$$\int \frac{\mathrm{d}x}{a-x}=-\int \frac{\mathrm{d}(a-x)}{a-x}==-\ln|a-x|+C,$$

$$\int \frac{\mathrm{d}x}{a+x}=\int \frac{\mathrm{d}(a+x)}{a+x}=\ln|a+x|+C,$$

问题就解决了.

例 5

求$\displaystyle\int \sec x\mathrm{d}x$.

参看例4的结果, 我们有

$$\begin{aligned}\int \sec x\mathrm{d}x &= \int \frac{\cos x}{\cos^2 x}\mathrm{d}x = \int \frac{\mathrm{d}\sin x}{1-\sin^2 x}\\ &= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C = \frac{1}{2}\ln\left|\frac{1+\sin x}{\cos x}\right|^2 + C\\ &= \ln|\sec x + \tan x| + C.\end{aligned}$$

用$t=\tan\dfrac{x}{2}$换元，得

$$\int \sec x\mathrm{d}x = \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C.$$

例 6

求 $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}}, a > 0.$

$$\int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = \int \frac{\mathrm{d}\left(\frac{x}{a}\right)}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} = \arcsin \frac{x}{a} + C.$$

例 7

求 $\displaystyle\int \frac{\mathrm{d}x}{1 + \mathrm{e}^x}.$

$$\int \frac{\mathrm{d}x}{1 + \mathrm{e}^x} = \int \frac{\mathrm{e}^{-x}\mathrm{d}x}{1 + \mathrm{e}^{-x}} = -\int \frac{\mathrm{d}(1 + \mathrm{e}^{-x})}{1 + \mathrm{e}^{-x}} = -\ln|1 + \mathrm{e}^{-x}| + C.$$

$$\int f(x)\mathrm{d}x = \int f(\varphi(t))\varphi'(t)\mathrm{d}t. \tag{3}$$

如果在上面(3)式中，右端的积分可以算出来，则可以用来计算左端的积分，只是由于右端积分得到的是t的函数，故需在$x = \varphi(t)$的一个单调区间上，用$x = \varphi(t)$的反函数$t = \varphi^{-1}(x)$代入这个t的函数，从而得到左端的积分．格式如下：

$$\int f(x)\mathrm{d}x = \int f(\varphi(t))\varphi'(t)\mathrm{d}t = G(t)|_{t=\varphi^{-1}(x)} + C = G(\varphi^{-1}(x)) + C.$$

这就是积分**第二换元法**.

第一换元法与第二换元法的理论基础是一样的，只是应用的方向相反．第一换元法，之所以称为凑微分法，是把被积函数中的一个函数“凑”到微分符号的后面去，然后换成新变量的积分，从而把积分计算出来．第二换元法是把积分变量x用某一个合适的函数$\varphi(t)$代换，这时要注意$\mathrm{d}x = \varphi'(t)\mathrm{d}t$, 从而化成积分变量是$t$的函数的积分.

下面几个积分是运用第二换元法的典型例子. 其主要想法是把根号去掉，可以用下面的三角公式:

$$\begin{aligned}
\sqrt{1-\sin^2 x} &= \cos x,\ x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right],\\
\sqrt{1+\tan^2 x} &= \sec x,\ x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right),\\
\sqrt{\sec^2 x-1} &= \begin{cases}\tan x, & x\in\left[0,\frac{\pi}{2}\right),\\ -\tan x, & x\in\left(\frac{\pi}{2},\pi\right].\end{cases}
\end{aligned}$$

也可以利用双曲变换:

$$\sqrt{\cosh^2 t-1}=\sinh t,\quad t\geqslant 0;$$

$$\sqrt{1+\sinh^2 t}=\cosh t,\quad t\in\mathbb{R}.$$

例 8

求$\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}}$, $a>0$.

令$x=a\tan t$, $-\frac{\pi}{2}<t<\frac{\pi}{2}$, $\mathrm{d}x=a\sec^2 t\mathrm{d}t$, 于是有

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}}=\int \frac{a\sec^2 t\,\mathrm{d}t}{\sqrt{a^2\tan^2 t+a^2}}=\int \sec t\mathrm{d}t.$$

令$x=a\sinh t$, $t\in\mathbb{R}$, 于是$\mathrm{d}x=a\cosh t\mathrm{d}t$, 从而

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}}=\int \frac{a\cosh t}{a\cosh t}\mathrm{d}t=\int \mathrm{d}t=t+C.$$

例 9

求$\int \sqrt{a^2-x^2}\mathrm{d}x,\ a>0.$

注意，被积函数的定义域为$[-a,a]$. 令$x=a\sin t,\ t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$，于是$\mathrm{d}x=a\cos t\,\mathrm{d}t$，$\sqrt{a^2-x^2}=a\cos t$. 从而有

$$\int \sqrt{a^2-x^2}\mathrm{d}x=a^2\int \cos^2 t\mathrm{d}t=\frac{a^2}{2}\int(1+\cos 2t)\mathrm{d}t$$

$$=\frac{a^2}{2}t+\frac{a^2}{4}\sin 2t+C$$

$$=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{1}{2}x\sqrt{a^2-x^2}+C.$$

例 10

求$\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}, a>0, |x|>a.$

令$x = a\sec t$, $t \in \left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right)$, 于是$\mathrm{d}x = a\sec t\tan t\ \mathrm{d}t$, $\sqrt{x^2-a^2} = a\sqrt{\tan^2 t} = \pm a\tan t$, 其中当$t\in\left(0,\frac{\pi}{2}\right)$时取正号, 当$t\in\left(\frac{\pi}{2},\pi\right)$时取负号. 从而由例5的结果有

$$\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = \int\frac{a\sec t\tan t}{\pm a\tan t}\mathrm{d}t = \pm\int\sec t\,\mathrm{d}t = \pm\ln|\sec t+\tan t|+C.$$

化简计算的结果

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}$$

$$= \begin{cases} \ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+C=\ln|x+\sqrt{x^2-a^2}|+C, & x>a, \\ -\ln\left|\frac{x}{a}-\frac{\sqrt{x^2-a^2}}{a}\right|+C=-\ln|x-\sqrt{x^2-a^2}|+C, & x<-a. \end{cases}$$

又因为当$x<-a$时，有

$$-\ln|x-\sqrt{x^2-a^2}| = -\ln(\sqrt{x^2-a^2}-x) = -\ln\frac{(x^2-a^2)-x^2}{x+\sqrt{x^2-a^2}}$$

$$= \quad -\ln\frac{a^2}{|x+\sqrt{x^2-a^2}|} = \ln|x+\sqrt{x^2-a^2}| - \ln a^2.$$

所以对所有$|x|>a$，均有$\displaystyle\int\frac{\mathrm{d}x}{\sqrt{x^2-a^2}}=\ln|x+\sqrt{x^2-a^2}|+C.$

例 11

求$\displaystyle\int \frac{\mathrm{d}x}{x\sqrt{x^2-1}}, x>1.$

令$x=\csc t,\ t\in\left(0,\dfrac{\pi}{2}\right)$，于是$\mathrm{d}x=-\csc t\cdot\cot t\,\mathrm{d}t$，
$\sqrt{x^2-1}=\cot t$. 从而有

$$\int \frac{\mathrm{d}x}{x\sqrt{x^2-1}}=-\int \frac{\csc t\cot t}{\csc t\cot t}\mathrm{d}t=-\int \mathrm{d}t=-t+C.$$

利用凑微分法，$\displaystyle\int \frac{\mathrm{d}x}{x\sqrt{x^2-1}}=-\int \frac{\mathrm{d}\left(\frac{1}{x}\right)}{\sqrt{1-\left(\frac{1}{x}\right)^2}}=-\arcsin\frac{1}{x}+C.$

应用倒数代换解决问题

求 $\displaystyle\int \frac{\mathrm{d}x}{x^2\sqrt{4-x^2}}$.

$$
\begin{aligned}
&\int \frac{\mathrm{d}x}{x^2\sqrt{4-x^2}} = \int \frac{-\frac{2}{t^2}\mathrm{d}t}{\frac{4}{t^2}\cdot\sqrt{4-\frac{4}{t^2}}} \quad \left(x = \frac{2}{t}\right) \\
= \quad & \mp\frac{1}{4}\int \frac{t}{\sqrt{t^2-1}}\mathrm{d}t \ (x \in (0,2)\text{取}-\text{号，}\ x \in (-2,0)\text{取}+\text{号}) \\
= \quad & \mp\frac{1}{8}\int \frac{\mathrm{d}(t^2-1)}{\sqrt{t^2-1}} = \mp\frac{1}{4}\sqrt{t^2-1} + C \\
= \quad & \mp\frac{1}{4}\sqrt{\frac{4}{x^2}-1} + C \\
= \quad & -\frac{\sqrt{4-x^2}}{4x} + C.
\end{aligned}
$$

特定被积函数的特殊换元方法在课外学习中积累

求$\displaystyle\int \frac{\mathrm{d}x}{\sqrt{(x-a)(b-x)}}$, $a < x < b$.

换元方法

令$x = a\cos^2\theta + b\sin^2\theta$, $0 < \theta < \dfrac{\pi}{2}$换元.

习题7第5题

求$\displaystyle\int \frac{\mathrm{d}x}{(x+a)^m(x+b)^n}$ (m, n是正整数).

换元方法

令$t = \dfrac{x+a}{x+b}$换元.