本章的重点是理解原函数与不定积分的概念，掌握换元积分法和分部积分法，了解一些函数类上的积分方法.

7.1 不定积分的概念

一、基本方法

借助不定积分的线性性质和基本积分表计算不定积分的方法.

例 1 求不定积分$\int \cot^2 x\,\mathrm{d}x$.

解 原式$=\int(\csc^2 x-1)\mathrm{d}x=-x-\cot x+C$. □

二、例题

例 2 已知$f(x)$满足给定条件：$f'(x)=\sqrt{1-\sin 2x} \quad (x\in(-\infty,+\infty))$, 求$f(x)$.

解 $$f(x)=\begin{cases}-(\sin x+\cos x)+4\sqrt{2}k, & x\in\left[2k\pi+\dfrac{\pi}{4},2k\pi+\dfrac{5\pi}{4}\right), \\ (\sin x+\cos x)+4\sqrt{2}k+2\sqrt{2}, & x\in\left[2k\pi+\dfrac{5\pi}{4},2k\pi+\dfrac{9\pi}{4}\right),\end{cases}\quad k\in\mathbb{Z}.$$ □

7.2 换元积分法

一、基本方法

1. 凑微分法

例 1 求不定积分$\int \frac{1}{x}\ln^2 x\,\mathrm{d}x$.

解 原式$=\int \ln^2 x\mathrm{d}(\ln x)=\dfrac{1}{3}\ln^3 x+C$. □

2. 积分第二换元法

例 2 求不定积分$\int x\sqrt{2-5x}\mathrm{d}x$.

解 $t=\sqrt{2-5x}$换元，原式$=-\dfrac{2}{375}(15x+4)(2-5x)^{\frac{3}{2}}+C.$ □

二、例题

例 3 求$\displaystyle\int\frac{2^x\cdot 3^x}{9^x-4^x}\mathrm{d}x.$

解

$$\begin{aligned}\int\frac{2^x\cdot 3^x}{9^x-4^x}\mathrm{d}x&=\int\frac{\left(\frac{2}{3}\right)^x}{1-\left(\frac{2}{3}\right)^{2x}}\mathrm{d}x=\int\frac{\frac{1}{\ln\frac{2}{3}}\mathrm{d}\left(\left(\frac{2}{3}\right)^x\right)}{1-\left(\frac{2}{3}\right)^{2x}}\\&=\frac{1}{\ln\frac{2}{3}}\int\frac{\mathrm{d}u}{1-u^2}\quad\left(u=\left(\frac{2}{3}\right)^x\right)=\frac{1}{2\ln\frac{2}{3}}\ln\left|\frac{1+u}{1-u}\right|+C\\&=\frac{1}{2(\ln 3-\ln 2)}\ln\left|\frac{3^x-2^x}{3^x+2^x}\right|+C.\end{aligned}$$

□

例 4 求$\displaystyle\int\frac{4x^5-1}{(x^5+x+1)^2}\mathrm{d}x.$

解

$$\begin{aligned}\int\frac{4x^5-1}{(x^5+x+1)^2}\mathrm{d}x&=\int\frac{\left(4x^3-\frac{1}{x^2}\right)\mathrm{d}x}{\left(x^4+1+\frac{1}{x}\right)^2}=\int\frac{\mathrm{d}\left(x^4+1+\frac{1}{x}\right)}{\left(x^4+1+\frac{1}{x}\right)^2}\\&=-\frac{1}{x^4+1+\frac{1}{x}}+C=-\frac{x}{x^5+x+1}+C.\end{aligned}$$

□

例 5 求$\displaystyle\int\frac{1+x}{x(1+x\mathrm{e}^x)}\mathrm{d}x.$

解

$$\begin{aligned}\int\frac{1+x}{x(1+x\mathrm{e}^x)}\mathrm{d}x&=\int\frac{(1+x)\mathrm{e}^x\mathrm{d}x}{x\mathrm{e}^x(1+x\mathrm{e}^x)}=\int\frac{\mathrm{d}(x\mathrm{e}^x)}{x\mathrm{e}^x(1+x\mathrm{e}^x)}\\&=\int\frac{\mathrm{d}u}{u(u+1)}\quad(u=x\mathrm{e}^x)=\ln\left|\frac{u}{u+1}\right|+C=\ln\left|\frac{x\mathrm{e}^x}{x\mathrm{e}^x+1}\right|+C.\end{aligned}$$

□

例 6 求$\displaystyle\int\frac{\mathrm{d}x}{(x+a)^m(x+b)^n}$ *(m, n是正整数).*

解 令$t=\dfrac{x+a}{x+b}$, 则$x=-b+\dfrac{a-b}{t-1}$, $\mathrm{d}x=\dfrac{b-a}{(t-1)^2}\mathrm{d}t$, 于是有

$$
\begin{aligned}
& \int \frac{\mathrm{d}x}{(x+a)^m(x+b)^n}=\int \frac{\mathrm{d}x}{\left(\frac{x+a}{x+b}\right)^m (x+b)^{m+n}} \\
= & \int \frac{\frac{b-a}{(t-1)^2}\mathrm{d}t}{t^m\left(\frac{a-b}{t-1}\right)^{m+n}}=-\frac{1}{(a-b)^{m+n-1}}\int \frac{(t-1)^{m+n-2}}{t^m}\mathrm{d}t \\
= & \frac{1}{(a-b)^{m+n-1}}\left[\sum_{\substack{i=0\\ i\neq n-1}}^{m+n-2}\frac{(-1)^{i+1}\mathrm{C}_{m+n-2}^{i}}{n-i-1}\left(\frac{x+a}{x+b}\right)^{n-i-1}+(-1)^n\mathrm{C}_{m+n-2}^{n-1}\ln\left|\frac{x+a}{x+b}\right|\right]+C.
\end{aligned}
$$

□

例 7 求$\displaystyle\int \frac{x^3-x+1}{\sqrt{x^2+2x+2}}\mathrm{d}x$.

解

$$
\begin{aligned}
& \int \frac{x^3-x+1}{\sqrt{x^2+2x+2}}\mathrm{d}x=\int \frac{(x^2-x)(x+1)+1}{\sqrt{x^2+2x+2}}\mathrm{d}x \\
= & \int (x^2-x)\mathrm{d}\sqrt{x^2+2x+2}+\int \frac{\mathrm{d}x}{\sqrt{x^2+2x+2}} \\
= & (x^2-x)\sqrt{x^2+2x+2}-\int (2x-1)\sqrt{x^2+2x+2}\mathrm{d}x+\int \frac{\mathrm{d}x}{\sqrt{x^2+2x+2}} \\
= & (x^2-x)\sqrt{x^2+2x+2}-\int \sqrt{x^2+2x+2}\mathrm{d}(x^2+2x+2)+3\int \sqrt{x^2+2x+2}\mathrm{d}x \\
& +\int \frac{\mathrm{d}x}{\sqrt{x^2+2x+2}} \\
= & (x^2-x)\sqrt{x^2+2x+2}-\frac{2}{3}(x^2+2x+2)\sqrt{x^2+2x+2}+\frac{3(x+1)}{2}\sqrt{x^2+2x+2} \\
& +\frac{3}{2}\ln|x+1+\sqrt{x^2+2x+2}|+\ln|x+1+\sqrt{x^2+2x+2}|+C \\
= & \frac{1}{6}(2x^2-5x+1)\sqrt{x^2+2x+2}+\frac{5}{2}\ln|x+1+\sqrt{x^2+2x+2}|+C.
\end{aligned}
$$

□

注 《微积分学教程》中对$\displaystyle\int \frac{p(x)}{\sqrt{ax^2+bx+c}}\mathrm{d}x$ (其中$p(x)$是实系数多项式，$a>0$, $b^2-4ac<0$)进行了更一般的讨论.

例 8 求$\displaystyle\int \frac{\sqrt{1+x^4}}{x}\mathrm{d}x$.

解

$$\begin{aligned}
&\int \frac{\sqrt{1+x^4}}{x}\mathrm{d}x = \int \frac{\sqrt{1+x^4}\cdot x^3\mathrm{d}x}{x^4} = \frac{1}{4}\int \frac{\sqrt{1+x^4}\mathrm{d}(x^4)}{x^4}\\
= &\ \frac{1}{4}\int \frac{\sqrt{1+u}\mathrm{d}u}{u} \quad (u=x^4) = \frac{1}{4}\int \frac{t\mathrm{d}(t^2-1)}{t^2-1} \quad (u=t^2-1,\ t>1)\\
= &\ \frac{1}{2}t+\frac{1}{4}\ln\left|\frac{t-1}{t+1}\right|+C = \frac{1}{2}\sqrt{x^4+1}+\frac{1}{4}\ln\left(\frac{\sqrt{x^4+1}-1}{\sqrt{x^4+1}+1}\right)+C.
\end{aligned}$$

□

7.3 分部积分法

一、基本方法

1. 分部积分法

例 1 求$\int \mathrm{e}^{2x}\sin^2 x\,\mathrm{d}x$.

解 原式$=\int \mathrm{e}^{2x}\cdot\frac{1-\cos 2x}{2}\mathrm{d}x=\frac{\mathrm{e}^{2x}}{8}(2-\sin 2x-\cos 2x)+C$. □

2. 推导递推关系计算不定积分

例 2 导出不定积分$I_n=\int \sin^n x\,\mathrm{d}x$的递推公式.

解 当$n>1$时，有

$$\begin{aligned}
I_n &= \int \sin^{n-1} x\mathrm{d}(-\cos x) = -\sin^{n-1} x\cos x+(n-1)\int \sin^{n-2} x\cos^2 x\mathrm{d}x\\
&= -\sin^{n-1} x\cos x+(n-1)I_{n-2}-(n-1)I_n,
\end{aligned}$$

故递推公式为$I_n=\frac{n-1}{n}I_{n-2}-\frac{\sin^{n-1} x\cos x}{n}$. □

二、例题

例 3 求$\int\left(1+x-\frac{1}{x}\right)\mathrm{e}^{x+\frac{1}{x}}\,\mathrm{d}x$.

解

$$\begin{aligned}&\int\left(1+x-\frac{1}{x}\right)\mathrm{e}^{x+\frac{1}{x}}\,\mathrm{d}x=\int\mathrm{e}^{x+\frac{1}{x}}\,\mathrm{d}x+\int\left(x-\frac{1}{x}\right)\mathrm{e}^{x+\frac{1}{x}}\,\mathrm{d}x\\=\;&x\mathrm{e}^{x+\frac{1}{x}}-\int x\mathrm{e}^{x+\frac{1}{x}}\cdot\left(1-\frac{1}{x^2}\right)\mathrm{d}x+\int\left(x-\frac{1}{x}\right)\mathrm{e}^{x+\frac{1}{x}}\,\mathrm{d}x=x\mathrm{e}^{x+\frac{1}{x}}+C.\end{aligned}$$

□

例 4 求$\int\sin\,x\,\ln(\tan x)\mathrm{d}x$.

解

$$\begin{aligned}&\int\sin\,x\,\ln(\tan x)\mathrm{d}x=-\cos\,x\,\ln(\tan x)+\int\cos\,x\cdot\frac{1}{\tan x}\cdot\sec^2x\\=\;&-\cos\,x\,\ln(\tan x)+\int\frac{1}{\csc x}\mathrm{d}x=-\cos\,x\,\ln(\tan x)+\ln\left|\tan\frac{x}{2}\right|+C.\end{aligned}$$

□

例 5 求$\int\sin(\ln\,x)\mathrm{d}x$.

解 因为

$$\int\sin(\ln\,x)\mathrm{d}x=x\sin(\ln x)-\int\cos(\ln x)\mathrm{d}x=x\sin(\ln x)-x\cos(\ln x)-\int\sin(\ln x)\mathrm{d}x,$$

所以

$$\int\sin(\ln\,x)\mathrm{d}x=\frac{x}{2}[\sin(\ln x)-\cos(\ln x)]+C.$$

□

注 也可以先换元，再用分部积分计算$\int\mathrm{e}^t\sin t\mathrm{d}t$.

$$\begin{aligned}&\int\sin(\ln\,x)\mathrm{d}x=\int\sin t\cdot\mathrm{e}^t\mathrm{d}t\quad(x=\mathrm{e}^t)\\=\;&\frac{\mathrm{e}^t}{2}(\sin t-\cos t)=\frac{x}{2}[\sin(\ln x)-\cos(\ln x)]+C.\end{aligned}$$

例 6 求$\int\frac{x\sin x}{\cos^3x}\mathrm{d}x$.

解 因为$\int\frac{\sin x}{\cos^3x}\mathrm{d}x=-\int\frac{\mathrm{d}\cos x}{\cos^3x}=\frac{1}{2\cos^2x}$, 所以

$$\int\frac{x\sin x}{\cos^3x}\mathrm{d}x=\frac{x}{2\cos^2x}-\int\frac{1}{2\cos^2x}\mathrm{d}x=\frac{x}{2\cos^2x}-\frac{1}{2}\int\sec^2x\mathrm{d}x=\frac{x}{2\cos^2x}-\frac{1}{2}\tan x+C.$$

□

例 7 设n是自然数，$I_n=\int\frac{\mathrm{d}x}{\sin^n x}$, 求$I_n$的递推公式.

解 当$n>1$时，有

$$
\begin{aligned}
I_n=\int\frac{\mathrm{d}x}{\sin^n x}&=\int\frac{\sin^2 x+\cos^2 x}{\sin^n x}\mathrm{d}x=I_{n-2}+\int\frac{\cos^2 x}{\sin^n x}\mathrm{d}x\\
&= I_{n-2}-\frac{1}{n-1}\int\cos x\ \mathrm{d}\left(\frac{1}{\sin^{n-1}x}\right)=I_{n-2}-\left[\frac{\cos x}{(n-1)\sin^{n-1}x}+\frac{1}{n-1}\int\frac{\mathrm{d}x}{\sin^{n-2}x}\right],\\
&= I_{n-2}-\frac{\cos x}{(n-1)\sin^{n-1}x}-\frac{I_{n-2}}{n-1}=\frac{n-2}{n-1}I_{n-2}-\frac{\cos x}{(n-1)\sin^{n-1}x}.
\end{aligned}
$$

因此当$n>1$时，有

$$I_n=\frac{n-2}{n-1}I_{n-2}-\frac{\cos x}{(n-1)\sin^{n-1}x}.$$

□

7.4 有理函数的积分

一、基本方法

赋值法与待定系数法相结合.

例 1 求$\int\frac{x^2-4x-2}{x(x^2+1)}\mathrm{d}x$.

解 设有分解式

$$\frac{x^2-4x-2}{x(x^2+1)}=\frac{a}{x}+\frac{bx+c}{x^2+1}.$$

去分母得

$$x^2-4x-2=a(x^2+1)+(bx+c)x$$

分别令$x=0,\ \mathrm{i}$, 这里$\mathrm{i}=\sqrt{-1}$,

$$\begin{cases}a=-2,\\ -b+c\mathrm{i}=-4\mathrm{i}-3.\end{cases}$$

从而有$a=-2,\ b=3,\ c=-4$.

$$
\begin{aligned}
\int\frac{x^2-4x-2}{x(x^2+1)}\mathrm{d}x&=-2\int\frac{\mathrm{d}x}{x}+\int\frac{3x-4}{x^2+1}\mathrm{d}x\\
&=-2\ln|x|+\tfrac{3}{2}\ln(x^2+1)-4\arctan x+C.
\end{aligned}
$$

□

二、例题

例 2 求$\displaystyle\int\frac{\mathrm{d}x}{x(x^3+1)^2}$.

解

$$\begin{aligned}&\int\frac{\mathrm{d}x}{x(x^3+1)^2}=\frac{1}{3}\int\frac{\mathrm{d}t}{(t-1)t^2}\ (t=x^3+1)=\frac{1}{3}\int\left(\frac{1}{t-1}-\frac{1}{t}-\frac{1}{t^2}\right)\mathrm{d}t\\=&\ \frac{1}{3}\left(\ln|t-1|-\ln|t|+\frac{1}{t}\right)+C=\frac{1}{3}\left(\ln\left|\frac{x^3}{x^3+1}\right|+\frac{1}{x^3+1}\right)+C.\end{aligned}$$

□

例 3 求$\displaystyle\int\frac{x^2}{(x^2+2x+2)^2}\mathrm{d}x$.

解

$$\begin{aligned}&\int\frac{x^2}{(x^2+2x+2)^2}\mathrm{d}x=\int\frac{\mathrm{d}x}{x^2+2x+2}-\int\frac{2x+2}{(x^2+2x+2)^2}\mathrm{d}x\\=&\ \int\frac{\mathrm{d}(x+1)}{(x+1)^2+1}-\int\frac{\mathrm{d}(x^2+2x+2)}{(x^2+2x+2)^2}=\arctan(x+1)+\frac{1}{x^2+2x+2}+C.\end{aligned}$$

□

7.5 三角函数有理式的积分

一、基本方法

1. 万能代换

例 1 求$\displaystyle\int\frac{\mathrm{d}x}{2\sin x-\cos x+5}$.

解 由万能代换，得原式 $=\dfrac{1}{\sqrt{5}}\arctan\left(\dfrac{3\tan\frac{x}{2}+1}{\sqrt{5}}\right)+C$. □

2. 特殊情形下的换元

例 2 求$\displaystyle\int\frac{\sin x\ \cos x}{1+\sin^4 x}\mathrm{d}x$.

解 原式 $=\dfrac{1}{2}\displaystyle\int\frac{\mathrm{d}t}{1+t^2}\ (t=\sin^2 x)=\frac{1}{2}\arctan(\sin^2 x)+C$. □

二、例题

例 3 求$\displaystyle\int\frac{\mathrm{d}x}{1+\varepsilon\cos\,x}$，其中$0<\varepsilon<1$.

解

$$\int\frac{\mathrm{d}x}{1+\varepsilon\cos\,x}=\int\frac{\frac{2\mathrm{d}t}{1+t^2}}{1+\varepsilon\cdot\frac{1-t^2}{1+t^2}}\ \left(t=\tan\frac{x}{2}\right)=\frac{2}{1-\varepsilon}\int\frac{\mathrm{d}t}{t^2+\frac{1+\varepsilon}{1-\varepsilon}}$$
$$=\ \frac{2}{1-\varepsilon}\cdot\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\arctan\left(\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\ t\right)+C=\frac{2}{\sqrt{1-\varepsilon^2}}\arctan\left(\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\tan\frac{x}{2}\right)+C.$$
□

例 4 求$\displaystyle\int\frac{\mathrm{d}x}{\sin\,x-\sin\,a},\cos a\neq0$.

解

$$\int\frac{\mathrm{d}x}{\sin\,x-\sin\,a}=\frac{1}{\cos a}\int\frac{\cos\left(\frac{x+a}{2}-\frac{x-a}{2}\right)\mathrm{d}x}{\sin x-\sin a}=\frac{1}{\cos a}\int\frac{\left(\cos\frac{x+a}{2}\cos\frac{x-a}{2}+\sin\frac{x+a}{2}\sin\frac{x-a}{2}\right)\mathrm{d}x}{2\cos\frac{x+a}{2}\sin\frac{x-a}{2}}$$
$$=\ \frac{1}{\cos a}\left(\int\frac{\mathrm{d}\sin\frac{x-a}{2}}{\sin\frac{x-a}{2}}+\int\frac{-\mathrm{d}\cos\frac{x+a}{2}}{\cos\frac{x+a}{2}}\right)=\frac{1}{\cos a}\ln\left|\frac{\sin\frac{x-a}{2}}{\cos\frac{x+a}{2}}\right|+C.$$
□

例 5 求$\displaystyle\int\frac{\sin^2x}{\cos^6x}\mathrm{d}x$.

解

$$\int\frac{\sin^2x}{\cos^6x}\mathrm{d}x=\int\tan^2x\sec^4x\mathrm{d}x=\int\tan^2x(\tan^2x+1)\mathrm{d}(\tan x)$$
$$=\ \frac{1}{5}\tan^5x+\frac{1}{3}\tan^3x+C.$$
□

7.6 无理函数的积分

一、基本方法

通过换元化为有理函数的积分.

例 1 求$\displaystyle\int\frac{\mathrm{d}x}{x(1+2\sqrt{x}+\sqrt[3]{x})}$.

解　$t=\sqrt[6]{x}$换元，原式$=\dfrac{3}{4}\ln\dfrac{x^{\frac{4}{3}}}{(1+\sqrt[6]{x})^2(1-\sqrt[6]{x}+2\sqrt[3]{x})^3}-\dfrac{3\sqrt{7}}{14}\arctan\left(\dfrac{4\sqrt[6]{x}-1}{\sqrt{7}}\right)+C.$

□

二、例题

例 2 求$\displaystyle\int\frac{\mathrm{d}x}{x^2\sqrt{x^2+1}}$.

解　令$x=\tan t$换元，其中$t\in\left(-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right)$, 就有

$$\begin{aligned}&\int\frac{\mathrm{d}x}{x^2\sqrt{x^2+1}}=\int\frac{\sec^2 t\mathrm{d}t}{\tan^2 t\cdot\sec t}=\int\frac{\cos t\mathrm{d}t}{\sin^2 t}=\int\frac{\mathrm{d}\sin t}{\sin^2 t}\\=&-\frac{1}{\sin t}+C=-\frac{\sqrt{x^2+1}}{x}+C.\end{aligned}$$

□

另解

$$\begin{aligned}&\int\frac{\mathrm{d}x}{x^2\sqrt{x^2+1}}=-\operatorname{sgn}x\cdot\int\frac{t\mathrm{d}t}{\sqrt{1+t^2}}\ \left(t=\frac{1}{x}\right)\\=&-\operatorname{sgn}x\cdot\sqrt{1+t^2}+C=-\frac{\sqrt{x^2+1}}{x}+C.\end{aligned}$$

□

例 3 求$\displaystyle\int\frac{\mathrm{d}x}{(1+x^2)\sqrt{x^2-1}}$.

解　令$x=\sec t$换元，其中$t\in\left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right)$, 就有

$$\begin{aligned}&\int\frac{\mathrm{d}x}{(1+x^2)\sqrt{x^2-1}}=\operatorname{sgn}x\cdot\int\frac{\tan t\sec t\mathrm{d}t}{(1+\sec^2 t)\tan t}=\operatorname{sgn}x\cdot\int\frac{\cos t\mathrm{d}t}{\cos^2 t+1}\\=&\operatorname{sgn}x\cdot\int\frac{\mathrm{d}\sin t}{2-\sin^2 t}=\operatorname{sgn}x\cdot\frac{\sqrt{2}}{4}\ln\left|\frac{\sqrt{2}+\sin t}{\sqrt{2}-\sin t}\right|=\frac{\sqrt{2}}{4}\ln\left|\frac{\sqrt{x^2-1}+\sqrt{2}x}{\sqrt{x^2-1}-\sqrt{2}x}\right|+C.\end{aligned}$$

□

例 4 求$\displaystyle\int\frac{x^2-1}{x^2+1}\cdot\frac{\mathrm{d}x}{\sqrt{1+x^2+x^4}}$.

解

$$
\begin{aligned}
&\int \frac{x^2-1}{x^2+1}\cdot\frac{\mathrm{d}x}{\sqrt{1+x^2+x^4}} = \operatorname{sgn} x\cdot\int\frac{\left(1-\frac{1}{x^2}\right)\mathrm{d}x}{\left(x+\frac{1}{x}\right)\sqrt{\left(x+\frac{1}{x}\right)^2-1}}\\
=\ & \operatorname{sgn} x\cdot\int\frac{\mathrm{d}t}{t\sqrt{1-t^2}}\ \left(t=x+\frac{1}{x}\right) = \operatorname{sgn} x\operatorname{sgn} t\cdot\int\frac{-\mathrm{d}\left(\frac{1}{t}\right)}{\sqrt{1-\left(\frac{1}{t}\right)^2}}\\
=\ & -\arcsin\frac{1}{t}+C\ (\text{注意}\operatorname{sgn} x\operatorname{sgn} t=1) = -\arcsin\frac{x}{x^2+1}+C.
\end{aligned}
$$

□

例 5 求$\int\sqrt{\frac{\mathrm{e}^x-1}{\mathrm{e}^x+1}}\mathrm{d}x$.

解

$$
\begin{aligned}
&\int\sqrt{\frac{\mathrm{e}^x-1}{\mathrm{e}^x+1}}\mathrm{d}x = \int t\cdot\left(\frac{2t}{1+t^2}+\frac{2t}{1-t^2}\right)\mathrm{d}t\left(t=\sqrt{\frac{\mathrm{e}^x-1}{\mathrm{e}^x+1}}\right)\\
=\ & 2\left(\int\frac{\mathrm{d}t}{1-t^2}-\int\frac{\mathrm{d}t}{1+t^2}\right) = \ln\left|\frac{1+t}{1-t}\right|-2\arctan t+C\\
=\ & \ln\left|\frac{1+\sqrt{\frac{\mathrm{e}^x-1}{\mathrm{e}^x+1}}}{1-\sqrt{\frac{\mathrm{e}^x-1}{\mathrm{e}^x+1}}}\right|-2\arctan\sqrt{\frac{\mathrm{e}^x-1}{\mathrm{e}^x+1}}+C = \ln\left(\mathrm{e}^x+\sqrt{\mathrm{e}^{2x}-1}\right)-2\arctan\sqrt{\frac{\mathrm{e}^x-1}{\mathrm{e}^x+1}}+C.
\end{aligned}
$$

□

例 6 求$\int\frac{\mathrm{d}x}{\sqrt{\tan\ x}}$.

解

$$
\begin{aligned}
&\int\frac{\mathrm{d}x}{\sqrt{\tan\ x}} = \int\frac{\frac{2t}{1+t^4}\mathrm{d}t}{t}\left(t=\sqrt{\tan x}\right) = 2\int\frac{\mathrm{d}t}{t^4+1}\\
=\ & \frac{\sqrt{2}}{4}\ln\frac{t^2+\sqrt{2}t+1}{t^2-\sqrt{2}t+1}+\frac{\sqrt{2}}{2}\left[\arctan(\sqrt{2}t-1)+\arctan(\sqrt{2}t+1)\right]+C\\
=\ & \frac{\sqrt{2}}{4}\ln\frac{\tan x+\sqrt{2\tan x}+1}{\tan x-\sqrt{2\tan x}+1}+\frac{\sqrt{2}}{2}\left[\arctan\left(\sqrt{2\tan x}-1\right)+\arctan\left(\sqrt{2\tan x}+1\right)\right]+C.
\end{aligned}
$$

□

例 7 求$\int\frac{\mathrm{d}x}{\sqrt{1+\mathrm{e}^x}+\sqrt{1-\mathrm{e}^x}}$.

解　原式 $=\displaystyle\int\frac{\sqrt{1+\mathrm{e}^x}-\sqrt{1-\mathrm{e}^x}}{(1+\mathrm{e}^x)-(1-\mathrm{e}^x)}\mathrm{d}x=\int\frac{\sqrt{1+\mathrm{e}^x}}{2\mathrm{e}^x}\mathrm{d}x-\int\frac{\sqrt{1-\mathrm{e}^x}}{2\mathrm{e}^x}\mathrm{d}x$. 令$t=\sqrt{1+\mathrm{e}^x}$换元，得

$$\begin{aligned}&\int\frac{\sqrt{1+\mathrm{e}^x}}{2\mathrm{e}^x}\mathrm{d}x=\int\frac{t}{2(t^2-1)}\cdot\frac{2t}{t^2-1}\mathrm{d}t\\=&\int\frac{t^2}{(t+1)^2(t-1)^2}\mathrm{d}t=\frac{1}{4}\int\left(\frac{1}{t-1}-\frac{1}{t+1}+\frac{1}{(t-1)^2}+\frac{1}{(t+1)^2}\right)\mathrm{d}t\\=&\frac{1}{4}\left(\ln(t-1)-\ln(t+1)-\frac{1}{t-1}-\frac{1}{t+1}\right)+C=\frac{1}{4}\ln\frac{\sqrt{1+\mathrm{e}^x}-1}{\sqrt{1+\mathrm{e}^x}+1}-\frac{\sqrt{1+\mathrm{e}^x}}{2\mathrm{e}^x}+C.\end{aligned}$$

令$t=\sqrt{1-\mathrm{e}^x}$换元，得

$$\begin{aligned}&\int\frac{\sqrt{1-\mathrm{e}^x}}{2\mathrm{e}^x}\mathrm{d}x=\int\frac{t}{2(1-t^2)}\cdot\frac{-2t}{1-t^2}\mathrm{d}t\\=&-\int\frac{t^2}{(t+1)^2(t-1)^2}\mathrm{d}t=-\frac{1}{4}\int\left(\frac{1}{t-1}-\frac{1}{t+1}+\frac{1}{(t-1)^2}+\frac{1}{(t+1)^2}\right)\mathrm{d}t\\=&-\frac{1}{4}\left(\ln(1-t)-\ln(1+t)-\frac{1}{t-1}-\frac{1}{t+1}\right)+C=-\frac{1}{4}\ln\frac{1-\sqrt{1-\mathrm{e}^x}}{1+\sqrt{1-\mathrm{e}^x}}-\frac{\sqrt{1-\mathrm{e}^x}}{2\mathrm{e}^x}+C.\end{aligned}$$

因此有

$$\text{原式}=\frac{1}{4}\ln\frac{\sqrt{1+\mathrm{e}^x}-1}{\sqrt{1+\mathrm{e}^x}+1}-\frac{\sqrt{1+\mathrm{e}^x}}{2\mathrm{e}^x}+\frac{1}{4}\ln\frac{1-\sqrt{1-\mathrm{e}^x}}{1+\sqrt{1-\mathrm{e}^x}}+\frac{\sqrt{1-\mathrm{e}^x}}{2\mathrm{e}^x}+C.$$

□

补充题7

1. 求$\displaystyle\int\frac{\mathrm{e}^{-\frac{1}{x}}}{x^3}\mathrm{d}x$.
2. 求$\displaystyle\int\tan^3x\mathrm{d}x$.
3. 求$\displaystyle\int\frac{\mathrm{d}x}{(x^2+a^2)^{\frac{3}{2}}}$, 其中$a>0$.
4. 求$\displaystyle\int\mathrm{e}^x\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)\mathrm{d}x$.
5. 求$\displaystyle\int\sin x\cdot\sinh x\ \mathrm{d}x$.
6. 求$\displaystyle\int\cos(\sqrt{x}+1)\mathrm{d}x$.
7. 求$\displaystyle\int\frac{x^2\sin x}{(1+\cos x)^2}\mathrm{d}x$.

8. 求$\int x\arctan x\ln(1+x^2)\mathrm{d}x$.

9. 设$f'(\ln x)=\begin{cases}\frac{1}{2\sqrt{x}}-x\ln^2 x, & 0<x\leqslant 1,\\ \frac{1}{\sqrt{1+3x^2},} & x>1\end{cases}$ 且$f(0)=0$，求$f(x)$.

10. 求$\int\frac{x-5}{(x+1)(x-2)^2}\mathrm{d}x$.

11. 求$\int\frac{x^5-x}{x^8+1}dx$.

12. 求$\int\frac{\sin x}{1+\sin x}\mathrm{d}x$.

13. 求$\int\frac{x^5}{\sqrt[4]{x^3+1}}\mathrm{d}x$.

14. 求$\int\sqrt{1-\frac{1}{x^2}}\mathrm{d}x$.