20级数学伯苓班模拟选拔考试数学分析试卷参考解答

一、（30分）设函数$f(x)$在$[a,b]$两次可导，记$M_0=\sup\limits_{x\in[a,b]}|f(x)|$, $M_1=\sup\limits_{x\in[a,b]}|f'(x)|$,

$M_2=\sup\limits_{x\in[a,b]}|f''(x)|$.

(1) 证明：对任意$x\in[a,b]$, 有$|f'(x)|\leqslant\dfrac{2}{b-a}M_0+\dfrac{(x-a)^2+(x-b)^2}{2(b-a)}M_2$.

(2) 证明：若$(b-a)^2M_2\geqslant 4M_0$, 则$M_1\leqslant 2\sqrt{M_0M_2}$.

证 (1) 由泰勒公式，有

$$\begin{aligned}f(a)&=f(x)+f'(x)(a-x)+\frac{f''(\xi_1)}{2}(a-x)^2,\\ f(b)&=f(x)+f'(x)(b-x)+\frac{f''(\xi_2)}{2}(b-x)^2,\end{aligned}$$

其中ξ_1介于a, x之间，ξ_2介于x, b之间. 后式减前式，得

$$f(b)-f(a)=f'(x)(b-a)+\frac{f''(\xi_2)}{2}(b-x)^2-\frac{f''(\xi_1)}{2}(a-x)^2.$$

于是

$$f'(x)=\frac{f(b)-f(a)}{b-a}+\frac{f''(\xi_1)(x-a)^2-f''(\xi_2)(x-b)^2}{2(b-a)}.$$

因此

$$\begin{aligned}|f'(x)|&\leqslant\frac{|f(a)|+|f(b)|}{b-a}+\frac{|f''(\xi_1)|(x-a)^2+|f''(\xi_2)|(x-b)^2}{2(b-a)}\\ &\leqslant\frac{2}{b-a}M_0+\frac{(x-a)^2+(x-b)^2}{2(b-a)}M_2.\end{aligned}$$

(2) 注意到对任意$x\in[a,b]$, 有

$$(b-a)^2-(x-a)^2-(x-b)^2=-2ab-2x^2+2ax+2bx=2(b-x)(x-a)\geqslant 0,$$

结合(1)就得到

$$|f'(x)|\leqslant\frac{2}{b-a}M_0+\frac{b-a}{2}M_2. \tag{$*$}$$

若$M_2=0$, 则由$(b-a)^2M_2\geqslant 4M_0$知$M_0=0$, 从而$f(x)\equiv 0$, 此时$M_1=0=2\sqrt{M_0M_2}$, 命题自然成立. 下设$M_2>0$. 由$f(x)$在$[a,b]$两次可导知$|f'(x)|$在$[a,b]$连续，故$|f'(x)|$在$[a,b]$取得最值. 设$u\in[a,b]$是$|f'(x)|$在$[a,b]$上的一个最大值点，由$(b-a)^2M_2\geqslant 4M_0$知$b-a\geqslant 2\sqrt{\frac{M_0}{M_2}}$, 于是存在$[c,d]\subseteq[a,b]$, 使得$u\in[c,d]$且$d-c=2\sqrt{\frac{M_0}{M_2}}$. 在上面的$(*)$式中用$u$代替$x$, 用$[c,d]$代替$[a,b]$，就得到

$$|f'(u)|\leqslant\frac{2}{d-c}\sup_{x\in[c,d]}|f(x)|+\frac{d-c}{2}\sup_{x\in[c,d]}|f''(x)|.$$

因此，

$$M_1=|f'(u)|\leqslant\frac{2}{2\sqrt{\frac{M_0}{M_2}}}M_0+\frac{2\sqrt{\frac{M_0}{M_2}}}{2}M_2=2\sqrt{M_0M_2}.$$

□

二、（30分）

(1) 设$\{A_n\}$和$\{\varphi_n\}$都是数列，$\lim\limits_{n\to\infty} A_n=+\infty$. 证明：对任意实数$a<b$, 都存在$x\in(a,b)$, 使得$\varlimsup\limits_{n\to\infty}\cos(A_nx+\varphi_n)=1$.

(2) 设$\{a_n\}$和$\{b_n\}$都是数列，(α,β)是一个开区间. 证明：若对任意$x\in(\alpha,\beta)$, 都有$\lim\limits_{n\to\infty}(a_n\cos nx+b_n\sin nx)=0$, 则$\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}b_n=0$.

证 (1) 因为$\lim\limits_{n\to\infty}A_n=+\infty$, 所以存在正整数$n_1$, 使得$\dfrac{2\pi}{A_{n_1}}<b-a$, 于是存在$[c_1,d_1]\subseteq(a,b)$, 使得$d_1-c_1=\dfrac{2\pi}{A_{n_1}}$. 因为$\cos(A_{n_1}x+\varphi_{n_1})$的周期为$\dfrac{2\pi}{A_{n_1}}$, 所以存在$[\alpha_1,\beta_1]\subseteq[c_1,d_1]$, 使得$\beta_1-\alpha_1\leqslant\dfrac{1}{2}(d_1-c_1)\leqslant\dfrac{1}{2}(b-a)$, 且对任意$x\in[\alpha_1,\beta_1]$, 有$\cos(A_{n_1}x+\varphi_{n_1})\geqslant\dfrac{1}{2}$. 类似地，存在正整数$n_2>n_1$, 使得$\dfrac{2\pi}{A_{n_2}}<\beta_1-\alpha_1$, 于是存在$[c_2,d_2]\subseteq[\alpha_1,\beta_1]$, 使得$d_2-c_2=\dfrac{2\pi}{A_{n_2}}$. 因为$\cos(A_{n_2}x+\varphi_{n_2})$的周期为$\dfrac{2\pi}{A_{n_2}}$, 所以存在$[\alpha_2,\beta_2]\subseteq[c_2,d_2]$, 使得$\beta_2-\alpha_2\leqslant\dfrac{1}{2}(d_2-c_2)\leqslant\dfrac{1}{2}(\beta_1-\alpha_1)\leqslant\dfrac{1}{4}(b-a)$, 且对任意$x\in[\alpha_2,\beta_2]$, 有$\cos(A_{n_2}x+\varphi_{n_2})\geqslant\dfrac{2}{3}$. 一直这样做下去，得到$n_1<n_2<\cdots$和区间套$\{[\alpha_k,\beta_k]\}$, 满足$\beta_k-\alpha_k\leqslant\dfrac{1}{2^k}(b-a)$, 且对$x\in[\alpha_k,\beta_k]$, 有$\cos(A_{n_k}x+\varphi_{n_k})\geqslant\dfrac{k}{k+1}$, $k=1,2,\cdots$. 由区间套定理，存在唯一的$x_0\in[\alpha_k,\beta_k]$, $k=1,2\cdots$. 于是$x_0\in[c_1,d_1]\subseteq(a,b)$, 由

$$\frac{k}{k+1}\leqslant\cos(A_{n_k}x_0+\varphi_{n_k})\leqslant 1,\ k=1,2,\cdots,$$

根据两边夹定理得$\lim\limits_{k\to\infty}\cos(A_{n_k}x_0+\varphi_{n_k})=1$. 结合$\cos(A_nx+\varphi_n)\leqslant 1$知$\varlimsup\limits_{n\to\infty}\cos(A_nx+\varphi_n)=1$.

(2) 记$\rho_n=\sqrt{a_n^2+b_n^2}$, $n=1,2,\cdots$. 用反证法. 若不然，则$\lim\limits_{n\to\infty}\rho_n\neq 0$. 于是存在$\varepsilon_0>0$, 存在数列$\{\rho_n\}$的子列$\{\rho_{n_k}\}$, 使得$\rho_{n_k}\geqslant\varepsilon_0$, $k=1,2,\cdots$. 令φ_n满足$\cos\varphi_n=\dfrac{a_n}{\rho_n}$, $\sin\varphi_n=-\dfrac{b_n}{\rho_n}$, 则$a_n\cos nx+b_n\sin nx=\rho_n\cos(nx+\varphi_n)$. 因为对任意$x\in(\alpha,\beta)$, 都有$\lim\limits_{n\to\infty}(a_n\cos nx+b_n\sin nx)=0$, 所以由$\rho_{n_k}\geqslant\varepsilon_0$知对任意$x\in(\alpha,\beta)$, 都有$\lim\limits_{k\to\infty}\cos(n_kx+\varphi_{n_k})=0$. 但另一方面，由(1)知存在$x_0\in(\alpha,\beta)$, 使得$\varlimsup\limits_{k\to\infty}\cos(n_kx_0+\varphi_{n_k})=1$, 与$\lim\limits_{k\to\infty}\cos(n_kx_0+\varphi_{n_k})=0$矛盾. $\square$

三、（30分）设函数$f(x)$在$[0,a]$连续可微，$f(0)=0$.

(1) 证明：$\displaystyle\int_0^a |f(x)f'(x)|\mathrm{d}x \leqslant \frac{a}{2}\int_0^a [f'(x)]^2\mathrm{d}x$.

(2) 证明：$\displaystyle\int_0^a f^2(x)[f'(x)]^2\mathrm{d}x \leqslant \frac{a^2}{2}\int_0^a [f'(x)]^4\mathrm{d}x$.

证 (1) 令$\varphi(x) = \displaystyle\int_0^x |f'(t)|\mathrm{d}t$, $x \in [0,a]$, 则由微积分基本定理知$\varphi'(x) = |f'(x)|$. 注意到$|f(x)| = \left|\displaystyle\int_0^x f'(t)\mathrm{d}t\right| \leqslant \varphi(x)$, 就有

$$\begin{aligned}
&\int_0^a |f(x)f'(x)|\mathrm{d}x \leqslant \int_0^a \varphi(x)\varphi'(x)\mathrm{d}x = \frac{1}{2}\varphi^2(x)\Big|_0^a = \frac{1}{2}\varphi^2(a)\\
= \ & \frac{1}{2}\left(\int_0^a |f'(x)|\mathrm{d}x\right)^2 \leqslant \frac{1}{2}\left(\int_0^a 1^2\mathrm{d}x\right)\left(\int_0^a [f'(x)]^2\mathrm{d}x\right)\\
= \ & \frac{a}{2}\int_0^a [f'(x)]^2\mathrm{d}x.
\end{aligned}$$

(2) 证法一 令$\psi(x) = \displaystyle\int_0^x [f'(t)]^2\mathrm{d}t$, $x \in [0,a]$, 则$\psi'(x) = [f'(x)]^2$. 由牛顿-莱布尼茨公式和施瓦兹不等式得

$$f^2(x) = \left(\int_0^x f'(t)\mathrm{d}t\right)^2 \leqslant \left(\int_0^x 1^2\mathrm{d}t\right)\left(\int_0^x [f'(t)]^2\mathrm{d}t\right) = x\psi(x).$$

于是

$$\begin{aligned}
&\int_0^a f^2(x)[f'(x)]^2\mathrm{d}x \leqslant \int_0^a x\psi(x)\psi'(x)\mathrm{d}x\\
\leqslant \ & a\int_0^a \psi(x)\psi'(x)\mathrm{d}x \ (\text{这是因为}\psi(x)\text{和}\psi'(x))\text{都非负})\\
= \ & \frac{a}{2}\psi^2(x)\Big|_0^a = \frac{a}{2}\psi^2(a) = \frac{a}{2}\left(\int_0^a [f'(t)]^2\mathrm{d}t\right)^2\\
\leqslant \ & \frac{a}{2}\left(\int_0^a 1^2\mathrm{d}t\right)\left(\int_0^a [f'(t)]^4\mathrm{d}t\right)\\
= \ & \frac{a^2}{2}\int_0^a [f'(x)]^4\mathrm{d}x.
\end{aligned}$$

□

证法二 令$h(x)=\dfrac{x^2}{2}\displaystyle\int_0^x[f'(t)]^4\mathrm{d}t-\int_0^x f^2(t)[f'(t)]^2\mathrm{d}t$, $x\in[0,a]$, 则

$$h'(x)=x\int_0^x[f'(t)]^4\mathrm{d}t+\frac{x^2}{2}[f'(x)]^4-f^2(x)[f'(x)]^2.$$

由Hölder不等式得

$$\int_0^x|f'(t)|\mathrm{d}t\leqslant\left(\int_0^x 1^{\frac{4}{3}}\mathrm{d}t\right)^{\frac{3}{4}}\left(\int_0^x[f'(t)]^4\mathrm{d}t\right)^{\frac{1}{4}}=x^{\frac{3}{4}}\left(\int_0^x[f'(t)]^4\mathrm{d}t\right)^{\frac{1}{4}},$$

故

$$\int_0^x[f'(t)]^4\mathrm{d}t\geqslant\frac{\left(\int_0^x|f'(t)|\mathrm{d}t\right)^4}{x^3}\geqslant\frac{f^4(x)}{x^3}.$$

于是

$$\begin{aligned}
h'(x)&\geqslant x\cdot\frac{f^4(x)}{x^3}+\frac{x^2}{2}[f'(x)]^4-f^2(x)[f'(x)]^2\\
&=\frac{f^4(x)}{x^2}+\frac{x^2}{2}[f'(x)]^4-f^2(x)[f'(x)]^2\\
&=\left(\frac{f^2(x)}{x}-\frac{x}{2}[f'(x)]^2\right)^2+\frac{x^2}{4}[f'(x)]^4\\
&\geqslant 0.
\end{aligned}$$

因此，$h(x)$在$[0,a]$单调递增. 于是$h(a)\geqslant h(0)=0$, 故

$$\int_0^a f^2(x)[f'(x)]^2\mathrm{d}x\leqslant\frac{a^2}{2}\int_0^a[f'(x)]^4\mathrm{d}x.$$

□

注 可以证明更一般的结果：设函数$f(x)$在$[0,a]$连续可微，$f(0)=0$, $p\geqslant 0$, $q\geqslant 1$, 则$\displaystyle\int_0^a|f(x)|^p|f'(x)|^q\mathrm{d}x\leqslant\frac{qa^p}{p+q}\int_0^a|f'(x)|^{p+q}\mathrm{d}x$.

四、（10分）设函数$f(x)$在$[1,+\infty)$上两次连续可微，对任意$x\geqslant 1$, 有$f''(x)+xf(x)=0$. 证明：$f(x)$在$[1,+\infty)$上有界.

证法一 令$\varphi(x)=\dfrac{[f'(x)]^2}{x}+f^2(x)$, 则

$$\varphi'(x)=\frac{2f'(x)f''(x)x-[f'(x)^2]}{x^2}+2f(x)f'(x)=\frac{2xf'(x)[f''(x)+xf(x)]-[f'(x)]^2}{x^2}=-\frac{[f'(x)]^2}{x^2}.$$

于是$\varphi'(x)\leqslant 0$, $\forall x\geqslant 1$, 故$\varphi(x)$在$[1,+\infty)$单调递减. 因此，记$M=\varphi(1)=f^2(1)+[f'(1)]^2$, 对任意$x\geqslant 1$, 就有

$$f^2(x)\leqslant\varphi(x)\leqslant\varphi(1)=M.$$

所以对任意$x\geqslant 1$, 有$|f(x)|\leqslant\sqrt{M}$, 即$f(x)$在$[1,+\infty)$上有界. □

证法二 由$f''(x)+xf(x)=0$得$\dfrac{f''(x)f'(x)}{x}+f(x)f'(x)=0$, 在$[1,x]$上积分，得

$$\int_1^x\frac{f'(t)f''(t)}{t}\mathrm{d}t+\int_1^x f(t)f'(t)\mathrm{d}t=0.$$

记$I=\displaystyle\int_1^x\frac{f'(t)f''(t)}{t}\mathrm{d}t$, 由分部积分法，有

$$\begin{aligned}I&=\int_1^x\frac{f'(t)f''(t)}{t}\mathrm{d}t=\int_1^x\frac{f'(t)}{t}\mathrm{d}(f'(t))\\&=\left.\frac{[f'(t)^2]}{t}\right|_1^x-\int_1^x f'(t)\cdot\frac{f''(t)t-f'(t)}{t^2}\mathrm{d}t\\&=\frac{[f'(x)]^2}{x}-[f'(1)]^2-I+\int_1^x\frac{[f'(t)]^2}{t^2}\mathrm{d}t,\end{aligned}$$

故$I=\dfrac{[f'(x)]^2}{2x}-\dfrac{1}{2}[f'(1)]^2+\dfrac{1}{2}\displaystyle\int_1^x\frac{[f'(t)]^2}{t^2}\mathrm{d}t$. 又$\displaystyle\int_1^x f(t)f'(t)\mathrm{d}t=\left.\frac{1}{2}f^2(t)\right|_1^x=\frac{1}{2}f^2(x)-\frac{1}{2}f^2(1)$, 记$M=f^2(1)+[f'(1)]^2$, 由上面的讨论即知对任意$x\geqslant 1$, 有

$$\frac{1}{2}f^2(x)-\frac{1}{2}M+\frac{[f'(x)]^2}{2x}+\frac{1}{2}\int_1^x\frac{[f'(t)]^2}{t^2}\mathrm{d}t=0$$

从而对任意$x\geqslant 1$, 有$f^2(x)\leqslant M$, 即$|f(x)|\leqslant\sqrt{M}$, 故$f(x)$在$[1,+\infty)$上有界. □

证法三 由$f''(x)+xf(x)=0$得$f''(x)f'(x)+xf(x)f'(x)=0$, 在$[1,x]$上积分，得

$$\int_1^x f'(t)f''(t)\mathrm{d}t+\int_1^x tf(t)f'(t)\mathrm{d}t=0.$$

由分部积分法，有

$$\int_1^x tf(t)f'(t)\mathrm{d}t=\frac{1}{2}tf^2(t)\Big|_1^x-\frac{1}{2}\int_1^x f^2(t)\mathrm{d}t=\frac{1}{2}xf^2(x)-\frac{1}{2}f^2(1)-\frac{1}{2}\int_1^x f^2(t)\mathrm{d}t.$$

又

$$\int_1^x f'(t)f''(t)\mathrm{d}t=\frac{1}{2}[f'(t)]^2\Big|_1^x=\frac{1}{2}[f'(x)]^2-\frac{1}{2}[f'(1)]^2\geqslant-\frac{1}{2}[f'(1)]^2,$$

记$M=f^2(1)+[f'(1)]^2$, 由上面的讨论就得到

$$0=\int_1^x f'(t)f''(t)\mathrm{d}t+\int_1^x tf(t)f'(t)\mathrm{d}t\geqslant\frac{1}{2}xf^2(x)-\frac{1}{2}M-\frac{1}{2}\int_1^x f^2(t)\mathrm{d}t.$$

记$h(x)=f^2(x)$, 则上面的不等式可以写成

$$xh(x)-\int_1^x h(t)\mathrm{d}t\leqslant M.$$

令$\varphi(x)=\dfrac{1}{x}\displaystyle\int_1^x h(t)\mathrm{d}t$, 则

$$\varphi'(x)=\frac{xh(x)-\int_1^x h(t)\mathrm{d}t}{x^2}\leqslant\frac{M}{x^2}.$$

于是

$$\varphi(x)=\int_1^x\varphi'(t)\mathrm{d}t\leqslant\int_1^x\frac{M}{t^2}\mathrm{d}t=M\left(1-\frac{1}{x}\right).$$

由$xh(x)-\displaystyle\int_1^x h(t)\mathrm{d}t\leqslant M$得$h(x)\leqslant\varphi(x)+\dfrac{M}{x}$, 因此，

$$h(x)\leqslant M\left(1-\frac{1}{x}\right)+\frac{M}{x}=M,$$

从而对任意$x\geqslant1$, 有$|f(x)|\leqslant\sqrt{M}$, 故$f(x)$在$[1,+\infty)$上有界. □