注记：有界函数$f(x)$在区间$[a,b]$上可积 $\Leftrightarrow$ $\forall \eta,\ \sigma>0$，存在分割T，使得 $\sum\limits_{\omega_k>\eta}\Delta x_k<\sigma$.

"$\Rightarrow$" $\eta\sigma=\varepsilon>\sum\limits_{k=1}^{n}\omega_k\Delta x_k \geqslant \sum\limits_{\omega_k>\eta}\omega_k\Delta x_k>\eta\sum\limits_{\omega_k>\eta}\Delta x_k$.

"$\Leftarrow$" $\forall\varepsilon>0$，取$\eta=\dfrac{\varepsilon}{2(b-a)}$，$\sigma=\dfrac{\varepsilon}{2\omega}$，存在分割$T$，使得 $\sum\limits_{\omega_k>\eta}\Delta x_k<\sigma$. 从而可得

$$\begin{aligned}\sum_{k=1}^{n}\omega_k\Delta x_k &= \sum_{\omega_k>\eta}\omega_k\Delta x_k+\sum_{\omega_k\leqslant\eta}\omega_k\Delta x_k \\ &< \omega\sum_{\omega_k>\eta}\Delta x_k+\eta\sum_{\omega_k\leqslant\eta}\Delta x_k<\omega\sigma+\eta(b-a)=\varepsilon.\end{aligned}$$

例题： $[0,1]$上的黎曼函数可积.

$$R(x) = \begin{cases} \frac{1}{q}, & 当x = \frac{p}{q}, p \in \mathbb{Z}, q \in \mathbb{N}^*, (p,q) = 1, \\ 0, & 当x为无理数. \end{cases}$$

$\forall\, \eta > 0, \sigma > 0$, 我们有

$$R\left(\frac{p}{q}\right) = \frac{1}{q} > \eta \Rightarrow q < \frac{1}{\eta}.$$

使上式成立的点$\dfrac{p}{q}$只有有限多个: $x_1, \cdots, x_N$. 取$\delta = \dfrac{\sigma}{2N}$, 于是当$\Delta(T) < \delta$时, 使$\omega_k > \eta$的小区间不超过$2N$个. 所以其长度之和

$$\sum_{\omega_k > \eta} \Delta x_k < 2N\delta = \sigma.$$

故$R(x)$在$[0,1]$上可积.

讨论下列函数在$[0,1]$的可积性:

(1) $$f(x)=\begin{cases}\frac{1}{x}-\left[\frac{1}{x}\right], & x\neq 0,\\ 0, & x=0.\end{cases}$$

(2) $$f(x)=\begin{cases}0, & x\in \mathbf{Q},\\ x, & x\notin \mathbf{Q}.\end{cases}$$

(3) $$f(x)=\begin{cases}\mathrm{sgn}(\sin\frac{\pi}{x}), & x\neq 0,\\ 0, & x=0.\end{cases}$$

(4) 设f在$[a,b]$上连续, g在$[\alpha,\beta]$上可积, $g([\alpha,\beta])\subseteq[a,b]$. 则$f\circ g$在$[\alpha,\beta]$上仍可积.

习题：讨论$f(x)=\begin{cases}0, & x\in\mathbf{Q},\\ x, & x\notin\mathbf{Q}.\end{cases}$在$[0,1]$上的可积性.

分析：取$[0,1]$的n等分分割$T=\{\frac{1}{k}\}_{k=1}^{n-1}$，记

$$X_k=[\frac{k-1}{n},\frac{k}{n}],\quad k=1,2,\cdots,n.$$

取$\varepsilon_0=\frac{1}{2}$，对于任意$\delta>0$，取$n$充分大,则$\Delta(T)=\frac{1}{n}<\delta$，注意到$\omega_k\geq\frac{k}{n}$，我们有

$$\sum_{k=1}^{n}\omega_n\Delta x_k\geq\sum_{k=2}^{n}\omega_k\Delta x_k\geq\sum_{k=2}^{n}\frac{k}{n^2}=\frac{n^2+n-2}{2n^2}>\frac{1}{2}=\varepsilon_0.$$

由可积的第一充要条件知$f(x)$在$[0,1]$上不可积.

习题：讨论$f(x)=\begin{cases}\frac{1}{x}-\left[\frac{1}{x}\right], & x\neq 0,\\ 0, & x=0.\end{cases}$在$[0,1]$上的可积性.

分析：函数在$[0,1]$的间断点为$x_0=0, x_k=\frac{1}{k}, k=1,2,\cdots$.
$\forall\eta>0, 1>\sigma>0$, 取$n=[\frac{2}{\sigma}]$,则$n\leq\frac{2}{\sigma}<n+1$,即$\frac{1}{n+1}<\frac{\sigma}{2}\leq\frac{1}{n}$.
记

$$\mathscr{I}=[0,1]\setminus[0,\frac{\sigma}{2})\bigcup_{k=1}^{n}X_k,\quad X_k\equiv(\frac{1}{k}-\frac{\sigma}{2^{k+2}},\frac{1}{k}+\frac{\sigma}{2^{k+2}}).$$

则$f(x)$ 在$\mathscr{I}$上一致连续, 则$\exists\delta>0$,
当$x_1,x_2\in\mathscr{I}$且$|x_1-x_2|<\delta$时, 有$|f(x_1)-f(x_2)|<\eta$. 则

$$\sum_{\omega_k>\eta}\Delta x_k\leq\frac{\sigma}{2}+\sum_{k=1}^{n}|X_k|=\frac{\sigma}{2}+\sum_{k=1}^{n}\frac{\sigma}{2^{k+1}}=\frac{\sigma}{2}+(\frac{1}{2}-(\frac{1}{2})^{n+1})\sigma<\sigma.$$

Theorem

(i) $[a, b]$上的连续函数;

(ii) $[a, b]$上的单调函数;

(iii) $[a, b]$上只有有限多个间断点的有界函数.

(i) 函数$f(x)$在$[a,b]$上连续. 则$f(x)$在$[a,b]$上一致连续, 故$\forall\varepsilon>0$, 有$\delta>0$, 使当$x_1,x_2\in[a,b]$且$|x_1-x_2|<\delta$时,

$$|f(x_1)-f(x_2)|<\frac{\varepsilon}{b-a}.$$

对于满足$\Delta(T)<\delta$的任意分割$T=\{x_k\}_{0\leqslant k\leqslant n}$, 可得

$$\omega_k=M_k-m_k<\frac{\varepsilon}{b-a},\quad k=1,\cdots,n,$$

于是有

$$\sum_{k=1}^{n}\omega_k\Delta x_k<\frac{\varepsilon}{b-a}\sum_{k=1}^{n}\Delta x_k=\varepsilon.$$

(ii) 不妨设$f(x)$在$[a,b]$上递减. 也不妨设$f(a)>f(b)$.

由于$f(x)$递减, 对于任何分割$T=\{x_k\}_{0\leqslant k\leqslant n}$, 都有$\omega_k=f(x_{k-1})-f(x_k)$.

任给$\varepsilon>0$, 取$\delta=\dfrac{\varepsilon}{f(a)-f(b)}$, 当$\Delta(T)<\delta$时, 就有

$$\begin{aligned}\sum_{k=1}^{n}\omega_k\Delta x_k &< \delta\sum_{k=1}^{n}(f(x_{k-1})-f(x_k))\\ &= \frac{\varepsilon}{f(a)-f(b)}(f(a)-f(b))=\varepsilon.\end{aligned}$$

(iii) 设函数$f(x)$在(a,b)上有k个间断点$\{x_j\}_{j=1}^k$，不妨设$a=x_0<x_1<x_2<\cdots<x_k<x_{k+1}=b$. 记

$$\hat{\delta}=\min\{x_j-x_{j-1}|j=1,2,\cdots,k+1\}.$$

任给$\varepsilon>0$，取$\delta_1=\min\left\{\frac{\varepsilon}{4(k+2)\omega},\frac{\hat{\delta}}{3}\right\}$，其中$\omega$为$f(x)$在$[a,b]$上的振幅. $\mathscr{I}=[a,b]\setminus\bigcup_{j=0}^{k+1}(x_j-\delta_1,x_j+\delta_1)$. $f(x)$在$\mathscr{I}$上一致连续，故$\exists\delta>0$，当$x_1,x_2\in\mathscr{I}$且$|x_1-x_2|<\delta$时，有

$$|f(x_1)-f(x_2)|<\frac{\varepsilon}{2(b-a)}.$$

取$\mathscr{I}$的一个分割T'使$\Delta(T')<\delta$，可以看成是$[a,b]$的一个分割，记之为T. 对应于这个分割T，有

$$\sum_{k=1}^{n}\omega_k\Delta x_k=\sum_{\mathscr{I}}\omega_i\Delta x_i+\sum_{[a,b]\setminus\mathscr{I}}\omega_j\Delta x_j<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

(iii) 证法二

(iii) 记$\hat{\delta} = \min\{x_j - x_{j-1} | j = 1, 2, \cdots, k+1\}$. 对任意$\sigma > 0$, 取$\delta_1 = \min\{\hat{\delta}, \sigma\}$.令

$$\mathscr{I} = [a, b] \setminus \bigcup_{j=0}^{k+1} X_j, \quad X_j \equiv (x_j - \frac{\delta_1}{2^{j+2}}, x_j + \frac{\delta_1}{2^{j+2}}).$$

则$f(x)$ 在$\mathscr{I}$上一致连续. $\forall\ \eta > 0$, $\exists \delta > 0$, 当$x_1, x_2 \in \mathscr{I}$且$|x_1 - x_2| < \delta$时, 有$|f(x_1) - f(x_2)| < \eta$.

取$\mathscr{I}$的一个分割T'使$\Delta(T') < \delta$, 然后任取每个小区间X_j的分割,使得最大区间长度小于δ. 这样构成$[a, b]$的一个满足$\Delta(T) < \delta$的分割T.

$$\sum_{\omega_k > \eta} \Delta x_k \leq \sum_{j=0}^{k+1} |X_j| = \sum_{j=0}^{k+2} \frac{\delta_1}{2^{j+1}} = (1 - (\frac{1}{2})^{k+2})\delta_1 < \sigma.$$

第8.3节

定积分的性质

Theorem

设函数$f(x)$和$g(x)$都在$[a,b]$上可积, λ为常数, 则$\lambda f(x)$, $f(x)\pm g(x)$也都在$[a,b]$上可积, 且有

(i) $\displaystyle\int_a^b \lambda f(x)\mathrm{d}x=\lambda\int_a^b f(x)\mathrm{d}x$;

(ii) $\displaystyle\int_a^b [f(x)\pm g(x)]\mathrm{d}x=\int_a^b f(x)\mathrm{d}x\pm\int_a^b g(x)\mathrm{d}x$;

(iii) 若$f(x)\geq 0, \forall x\in[a,b]$, 则$\displaystyle\int_a^b f(x)\mathrm{d}x\geqslant 0$;

(iv) 若$f(x)\geq g(x), \forall x\in[a,b]$,则$\displaystyle\int_a^b f(x)\mathrm{d}x\geq\int_a^b g(x)\mathrm{d}x$.

Proof. 由定积分的定义直接得到.

Theorem

(i) 若$f(x)$在$[a,b]$上可积，则$f(x)$在$[a,c]$和$[c,b]$上都可积，且有

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x; \tag{1}$$

(ii) 若$f(x)$在$[a,c]$和$[c,b]$上都可积，则$f(x)$在$[a,b]$上可积.

Theorem

设非负函数$f(x)$在$[a,b]$上连续，且$\int_a^b f(x)\mathrm{d}x = 0$. 则在$[a,b]$上$f(x) \equiv 0$.

Proof. 利用反证法可得.

Theorem

若$f(x)$在$[a,b]$上可积, 则$|f(x)|$也在$[a,b]$上可积. 且有

$$\left|\int_a^b f(x)\mathrm{d}x\right| \leqslant \int_a^b |f(x)|\mathrm{d}x.$$

Proof. 注意到振幅:

$$\begin{aligned}\omega_k(|f|) &= \sup_{s,t\in[x_{k-1},x_k]} \big||f(s)|-|f(t)|\big| \\ &\leq \sup_{s,t\in[x_{k-1},x_k]} |f(s)-f(t)| \\ &= \omega_k(f).\end{aligned}$$

Theorem

若$f(x)$和$g(x)$都在$[a,b]$上可积,则$f(x)g(x)$也在$[a,b]$上可积.

Proof. 注意到

$$
\begin{aligned}
\omega_k(fg) &= \sup_{s,t} |\, f(s)g(s) - f(t)g(t) \,| \\
&\leq \sup_{s,t} |f(s)g(s) - f(t)g(s)| + \sup_{s,t} |f(t)g(s) - f(t)g(t)| \\
&= \sup_{s,t} (|f(s) - f(t)||g(s)|) + \sup_{s,t} (|f(t)||g(s) - g(t)|) \\
&\leq M(\omega_k(f) + \omega_k(g)).
\end{aligned}
$$

习题：设$f(x), g(x)$为$[a, b]$上的可积函数,证明下列Cauchy-Schwartz不等式

$$\left[\int_a^b f(x)g(x)dx\right]^2 \leq \int_a^b f^2(x)dx \int_a^b g^2(x)dx. \tag{2}$$

推广：证明下列Hölder不等式

$$\int_a^b |f(x)g(x)|dx \leq \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{q}}, \tag{3}$$

其中p, q为满足$\frac{1}{p} + \frac{1}{q} = 1$的正数.

Theorem

设$f(x)$在$[a,b]$上连续，$g(x)$在$[a,b]$可积且不变号，则$\exists \xi \in [a,b]$，s.t.

$$\int_a^b f(x)g(x)\mathrm{d}x = f(\xi)\int_a^b g(x)\mathrm{d}x.$$

Proof. 不妨设$g \le 0$. 且当$g \equiv 0$时，显然成立.
下面仅考虑$\int_a^b g(x)\mathrm{d}x < 0$. 设$m \le f(x) \le M, \forall x \in [a,b]$.则

$$m \le \frac{\int_a^b f(x)g(x)\mathrm{d}x}{\int_a^b g(x)\mathrm{d}x} \le M.$$

习题: 设$f(x)$为$[a,b]$上的可积函数,且$\int_a^b f(x)dx > 0$, 证明，存在子区间$[\alpha,\beta] \subset [a,b]$和$A > 0$,使得$f(x) > A$, $\forall\ x \in [\alpha,\beta]$.

Remark (积分第一中值定理)

设$f(x)$在$[a,b]$上连续, $g(x)$在(a,b)可积且不变号, 则$\exists \xi \in (a,b)$, *s.t.*

$$\int_a^b f(x)g(x)\mathrm{d}x = f(\xi)\int_a^b g(x)\mathrm{d}x.$$

特别地,若$f(x)$在$[a,b]$上连续，则存在$\xi \in (a,b)$,使得

$$\int_a^b f(x)\mathrm{d}x = f(\xi)(b-a).$$

设$f_n(x)$在$[a,b]$上一列可积函数,且 $\lim\limits_{n\to\infty} f_n(x) = f(x)$ (即逐点收敛), $\forall\ x \in [a,b]$. 则经常考虑下列等式是否成立:

$$\lim_{n\to\infty}\int_a^b f_n(x)\mathrm{d}x = \int_a^b \lim_{n\to\infty} f_n(x)\mathrm{d}x\ ? \tag{4}$$

反例: $f_n(x) = \begin{cases} n, & 0 < x \le \frac{1}{n}, \\ 0, & x = 0 \text{ 或 } \frac{1}{n} < x \le 1. \end{cases}$

显然,

$$\lim_{n\to\infty}\int_0^1 f_n(x)\mathrm{d}x = 1, \quad \int_0^1 \lim_{n\to\infty} f_n(x)\mathrm{d}x = 0. \tag{5}$$

Example

求证 $\lim\limits_{n\to\infty}\int_0^{\frac{\pi}{2}}\sin^n x\mathrm{d}x = 0.$

思路： $\forall\, \varepsilon \in (0, \frac{\pi}{2})$, 我们有

$$\begin{aligned} 0 \;&\leqslant\; \int_0^{\frac{\pi}{2}}\sin^n x\mathrm{d}x = \int_0^{\frac{\pi-\varepsilon}{2}}\sin^n x\mathrm{d}x + \int_{\frac{\pi-\varepsilon}{2}}^{\frac{\pi}{2}}\sin^n x\mathrm{d}x \\ &\leqslant\; \frac{\pi}{2}\sin^n\frac{\pi-\varepsilon}{2} + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\ \ n\text{充分大时}. \end{aligned}$$

例题2

例2. 设$f(x)$在$[a,b]$可积, 证明存在$[a,b]$上连续函数列$\{\varphi_n(x)\}$, 使得

$$\lim_{n\to\infty}\int_a^b |f(x)-\varphi_n(x)|\mathrm{d}x=0.$$

思路: 把$[a,b]$$n$等份,设$P_k=(x_k,f(x_k))$, $k=0,1,\cdots,n$, 依次连接$P_0,P_1,\cdots,P_n$得折线$\varphi_n(x)$, 显然$\varphi_n(x)$是$[a,b]$上的连续函数. 且当$x\in[x_{k-1},x_k]$时, 成立$m_k\leqslant\varphi_n(x)\leqslant M_k$. 于是

$$|f(x)-\varphi_n(x)|\leqslant M_k-m_k=\omega_k.$$

$$\begin{aligned}0 &\leqslant \lim_{n\to\infty}\int_a^b|f(x)-\varphi_n(x)|\mathrm{d}x\\ &= \lim_{n\to\infty}\sum_{k=1}^n\int_{x_{k-1}}^{x_k}|f(x)-\varphi_n(x)|\mathrm{d}x\le\lim_{n\to\infty}\sum_{k=1}^n\omega_k\Delta x_k=0.\end{aligned}$$

例2. 设$f(x)$在$[a,b]$可积, 证明存在$[a,b]$上连续函数列$\{\varphi_n(x)\}$, 使得

$$\lim_{n\to\infty}\int_a^b |f(x)-\varphi_n(x)|\mathrm{d}x = 0.$$

思路: 把$[a,b]n$等份,设$P_k=(x_k,f(x_k))$, $k=0,1,\cdots,n$, 依次连接$P_0,P_1,\cdots,P_n$得折线$\varphi_n(x)$, 显然$\varphi_n(x)$是$[a,b]$上的连续函数. 且当$x\in[x_{k-1},x_k]$时, 成立$m_k\leqslant\varphi_n(x)\leqslant M_k$. 于是

$$|f(x)-\varphi_n(x)|\leqslant M_k-m_k=\omega_k.$$

$$\begin{aligned}0 &\leqslant \lim_{n\to\infty}\int_a^b |f(x)-\varphi_n(x)|\mathrm{d}x\\ &= \lim_{n\to\infty}\sum_{k=1}^n\int_{x_{k-1}}^{x_k}|f(x)-\varphi_n(x)|\mathrm{d}x \leq \lim_{n\to\infty}\sum_{k=1}^n\omega_k\Delta x_k = 0.\end{aligned}$$

注记： 进一步可得,

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty}\int_a^b \varphi_n(x)\mathrm{d}x.$$

即对于连续函数成立的定积分命题,可以推广到更一般的可积函数.

例3. 若$f(x)$在$[A,B]$可积，$A<a<b<B$，求证

$$\lim_{h\to 0}\int_a^b |f(x+h)-f(x)|\mathrm{d}x=0.$$

思路： 证明连续函数成立，然后过渡到可积函数.

Riemann-Lebesgue引理

例题： 设$f(x)$是$[a,b]$上的可积函数,证明

$$\lim_{m\to\infty}\int_a^b f(x)\sin mx\mathrm{d}x=0,\quad \lim_{m\to\infty}\int_a^b f(x)\cos mx\mathrm{d}x=0.$$

思路：$\forall\,\varepsilon>0$, 存在分割$T=\{x_k\}_{k=1}^n$, 使得

$$\begin{aligned}
\left|\int_a^b f(x)\sin mx\mathrm{d}x\right| &\le \sum_{k=1}^n\int_{x_{k-1}}^{x_k}|f(x)-f(x_k)||\sin mx|\mathrm{d}x\\
&+\sum_{k=1}^n|f(x_k)|\left|\int_{x_{k-1}}^{x_k}\sin mx\mathrm{d}x\right|\\
&\le \sum_{k=1}^n\omega_k\Delta x_k+\frac{2Mn}{m}<\frac{\varepsilon}{2}+\frac{2Mn}{m}.
\end{aligned}$$

例题： 设$f(x)$是$[a,b]$上的可积函数,证明

$$\lim_{m\to\infty}\int_a^b f(x)\sin mx\mathrm{d}x=0,\quad \lim_{m\to\infty}\int_a^b f(x)\cos mx\mathrm{d}x=0.$$

思路： $\forall\ \varepsilon>0$, 存在分割$T=\{x_k\}_{k=1}^n$, 使得

$$\begin{aligned}|\int_a^b f(x)\sin mx\mathrm{d}x| &\le \sum_{k=1}^n\int_{x_{k-1}}^{x_k}|f(x)-f(x_k)||\sin mx|\mathrm{d}x\\ &+\sum_{k=1}^n|f(x_k)||\int_{x_{k-1}}^{x_k}\sin mx\mathrm{d}x|\\ &\le \sum_{k=1}^n\omega_k\Delta x_k+\frac{2Mn}{m}<\frac{\varepsilon}{2}+\frac{2Mn}{m}.\end{aligned}$$

Theorem (积分第二中值定理)

设$f(x)$在$[a,b]$上可积,$g(x)$为$[a,b]$上单调，则存在$\xi \in [a,b]$,使得

$$\int_a^b f(x)g(x)\mathrm{d}x = g(a)\int_a^\xi f(x)\mathrm{d}x + g(b)\int_\xi^b f(x)\mathrm{d}x. \tag{6}$$

注记: (1) 设$f(x)$可积,$g(x)$单调减且非负,则存在$\xi \in [a,b]$,使得

$$\int_a^b f(x)g(x)\mathrm{d}x = g(a)\int_a^\xi f(x)\mathrm{d}x. \tag{7}$$

(2) 设$f(x)$可积,$g(x)$单调增且非负,则存在$\xi \in [a,b]$,使得

$$\int_a^b f(x)g(x)\mathrm{d}x = g(b)\int_\xi^b f(x)\mathrm{d}x. \tag{8}$$

Theorem (积分第二中值定理)

设$f(x)$在$[a,b]$上可积,$g(x)$为$[a,b]$上单调，则存在$\xi\in[a,b]$,使得

$$\int_a^b f(x)g(x)\mathrm{d}x=g(a)\int_a^{\xi} f(x)\mathrm{d}x+g(b)\int_{\xi}^b f(x)\mathrm{d}x. \tag{6}$$

注记: (1) 设$f(x)$可积,$g(x)$单调减且非负,则存在$\xi\in[a,b]$,使得

$$\int_a^b f(x)g(x)\mathrm{d}x=g(a)\int_a^{\xi} f(x)\mathrm{d}x. \tag{7}$$

(2) 设$f(x)$可积,$g(x)$单调增且非负,则存在$\xi\in[a,b]$,使得

$$\int_a^b f(x)g(x)\mathrm{d}x=g(b)\int_{\xi}^b f(x)\mathrm{d}x. \tag{8}$$

1. 对于定义在$[a,b]$上的非负连续函数$f(x)$,成立

$$\lim_{n\to\infty}\left(\int_a^b f^n(x)\ \mathrm{d}x\right)^{\frac{1}{n}}=\max_{x\in[a,b]}\{f(x)\}.$$

2. 设非负函数$f(x)$可积在$[a,b]$上可积，且存在$[\alpha,\beta]\subseteq[a,b]$ 使得$f|_{[\alpha,\beta]}>0$,证明$\int_a^b f(x)\ \mathrm{d}x>0.$

3. 设$f(x)$为$[0,\pi]$上的连续函数，且满足

$$\int_0^\pi f(x)\sin x\ \mathrm{d}x=\int_0^\pi f(x)\cos x\ \mathrm{d}x=0,$$

则$f(x)$在$(0,\pi)$至少有两个零点。