§二次型习题解答

1、(1) 用非退化线性替换化下列二次型为标准形，并利用矩阵验算所得结果

(2) $x_1^2+2x_1x_2+2x_2^2+4x_2x_3+4x_3^2$

$= (x_1+x_2)^2+(x_2+2x_3)^2$

令 $z_1 = x_1+x_2 \quad z_2=x_2+2x_3 \quad z_3=x_3$

则 $f(x_1,x_2,x_3)= z_1^2+z_2^2$

$x_2= z_2-2z_3$

$x_1= z_1-z_2+2z_3$

$x_3= z_3$

$$\begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix} 1&-1&2\\ 0&1&-2\\ 0&0&1\end{pmatrix}\begin{pmatrix} z_1\\ z_2\\ z_3\end{pmatrix} \quad 令B=\begin{pmatrix} 1&0&0\\ 0&1&0\\ 0&0&0\end{pmatrix} \quad C=\begin{pmatrix} 1&-1&2\\ 0&1&-2\\ 0&0&1\end{pmatrix} \quad A=\begin{pmatrix} 1&1&0\\ 1&2&2\\ 0&2&4\end{pmatrix}$$

$B= C^TAC$

(3) $x_1^2-3x_2^2-2x_1x_2+2x_1x_3-6x_2x_3$

$= (x_1-x_2+x_3)^2-4x_2^2-x_3^2-4x_2x_3$

$=(x_1-x_2+x_3)^2-(2x_2+x_3)^2$

令 $y_1=x_1-x_2+x_3 \quad y_2=2x_2+x_3 \quad y_3=x_3$

$\Rightarrow \quad x_2=\dfrac{y_2-y_3}{2}$

$x_1= y_1+\dfrac{y_2-y_3}{2}-y_3 = y_1+\dfrac{y_2}{2}-\dfrac{3y_3}{2}$

$x_3=y_3$

$$令C=\begin{pmatrix} 1&\frac{1}{2}&-\frac{3}{2}\\ 0&\frac{1}{2}&-\frac{1}{2}\\ 0&0&1\end{pmatrix} \quad A=\begin{pmatrix} 1&-1&1\\ -1&-3&-3\\ 1&-3&0\end{pmatrix} \quad B=\begin{pmatrix} 1&0&0\\ 0&-1&0\\ 0&0&0\end{pmatrix}$$

$B=C^TAC$

(4) $8x_1x_4+2x_3x_4+2x_2x_3+8x_2x_4$

令 $x_1=y_1+y_2$ $\quad x_4=y_1-y_2$ $\quad x_3=y_3$ $\quad x_2=y_4$

$$f(x_1,x_2,x_3,x_4)=8y_1^2-8y_2^2+2y_3(y_1-y_2)+2y_4y_3+8y_4(y_1-y_2)$$

$$=8y_1^2-8y_2^2+2y_1y_3-2y_2y_3+2y_3y_4+8y_1y_4-8y_2y_4$$

$$=\left(2\sqrt{2}y_1+\frac{\sqrt{2}}{4}y_3+\sqrt{2}y_4\right)^2-\left(2\sqrt{2}y_2+\frac{\sqrt{2}}{4}y_3+\sqrt{2}y_4\right)^2+2y_3y_4$$

令 $z_1=2\sqrt{2}y_1+\frac{\sqrt{2}}{4}y_3+\sqrt{2}y_4$ $\quad z_2=2\sqrt{2}y_2+\frac{\sqrt{2}}{4}y_3+\sqrt{2}y_4$ $\quad z_3=y_3$ $\quad z_4=y_4$

则原式 $=z_1^2+z_2^2+2z_3z_4$ $\quad y_1=\dfrac{z_1-\frac{\sqrt{2}}{4}z_3-\sqrt{2}z_4}{2\sqrt{2}}$ $\quad y_2=\dfrac{z_2-\frac{\sqrt{2}}{4}z_3-\sqrt{2}z_4}{2\sqrt{2}}$ $\quad y_3=z_3$ $\quad y_4=z_4$

令 $\sqrt{2}z_3=w_3-w_4$ $\quad \sqrt{2}z_4=w_3+w_4$ $\quad z_1=w_1$ $\quad z_2=w_2$

则原式 $=w_1^2+w_2^2+w_3^2-w_4^2$

$w_1=z_1$ $\quad w_2=z_2$ $\quad w_3=\dfrac{z_3+z_4}{\sqrt{2}}$ $\quad w_4=\dfrac{z_4-z_3}{\sqrt{2}}$

$$\begin{pmatrix} \quad \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$$