数学分析讲义（省身班）

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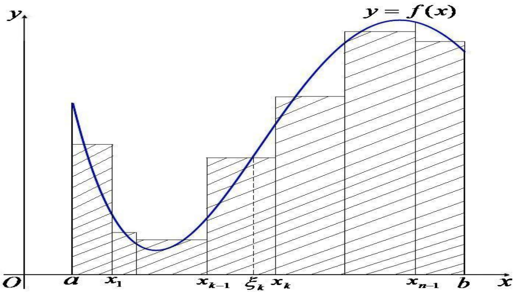
曲边梯形面积

设函数$f(x) \geqslant 0$在区间$[a,b]$有界（或者设为连续函数）.

由曲线$y=f(x)$和直线$x=a, x=b, y=0$ 所围成的图形称为一个曲边梯形.

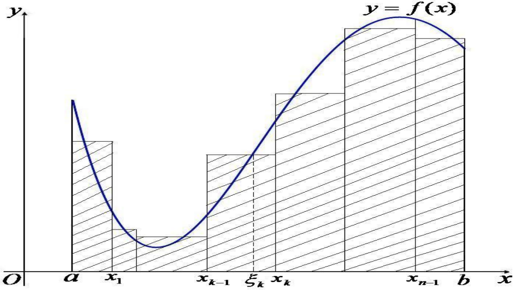


第一步：化整为零

把区间$[a,b]$用分点（一种分割T）

$$a = x_0 < x_1 < \cdots < x_k < \cdots < x_n = b$$

分割成n个小区间$[x_{k-1}, x_k], k = 1, \cdots, n$. 过每个分点作$x$轴的垂线与曲线$y = f(x)$相交，于是将曲边梯形分成了$n$个小曲边梯形.

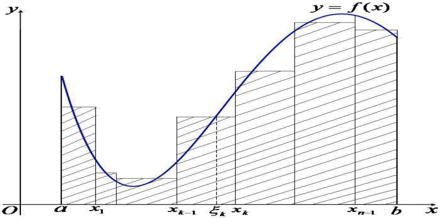


第二步：近似替换

用矩形面积来表示这个小的曲边梯形的面积.

在$[x_{k-1}, x_k]$上任取一点ξ_k,记$\Delta x_k = x_k - x_{k-1}$, 于是第$k$个小矩形的面积为$S_k = f(\xi_k)\Delta x_k$. 求和得近似值（积分和或黎曼和）:

$$S(f;T,\xi) = \sum_{k=1}^{n} S_k = \sum_{k=1}^{n} f(\xi_k)\Delta x_k. \tag{1}$$



第三步：取极限（求精确值）

分割T,分点$\xi=(\xi_1,\xi_2,\cdots,\xi_n)$.

记$\Delta(T)=\max\{\Delta x_k|k=1,\cdots,n\}$为分割$T$的直径

曲边梯形的面积就是

$$S=\lim_{\Delta(T)\to 0}\sum_{k=1}^{n}S_k=\lim_{\Delta(T)\to 0}\sum_{k=1}^{n}f(\xi_k)\Delta x_k.$$

注记: 积分和$\sum\limits_{k=1}^{n}S_k$依赖于分割T与点ξ_k的选择。

Definition

设函数$f(x)$在$[a,b]$上有定义. 如果存在$I\in\mathbb{R}$, 使$\forall\varepsilon>0$, $\exists\delta>0$, 对任意(T,ξ), 只要$\Delta(T)<\delta$, 都有

$$\left|\sum_{k=1}^{n} f(\xi_k)\Delta x_k - I\right| < \varepsilon, \tag{2}$$

则称$f(x)$在$[a,b]$上可积,并称I是$f(x)$在$[a,b]$上的定积分, 记为

$$I=\int_a^b f(x)\mathrm{d}x,$$

其中$f(x)$-被积函数, $f(x)\mathrm{d}x$-被积表达式, x-积分变量,a,b-下、上限.

Theorem

函数$f(x)$在区间$[a,b]$上可积$\Rightarrow$ $f(x)$在$[a,b]$上有界.

证明思路：将区间$[a,b]$均分成n等分。在定义中,取$\varepsilon_0=1$，存在$\delta>0$ 使得$\frac{b-a}{n}<\delta$(n充分大即可)。对任意j，$1\leqslant j\leqslant n$，除ξ_j外，其他ξ_k取小区间的左端点x_k,则

$$\left|\left(\sum_{k\neq j} f(x_k)+f(\xi_j)\right)\frac{b-a}{n}-I\right|<1,$$

$$|f(\xi_j)|<\frac{n(|I|+1)}{b-a}+\sum_{k\neq j}|f(x_k)|. \tag{4}$$

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反证法. 假设f在$[a,b]$上无界,则对于任意分割T,都至少存在某一区间$[x_{k-1},x_k]$,使得f在此区间上无界。不妨设

$$|\sum_{i\neq k} f(\xi_i)\Delta x_i| = F < \infty.$$

任取正数$M>0$,因为f在$[x_{k-1},x_k]$上无界,故存在$\xi_k\in[x_{k-1},x_k]$使得$|f(\xi_k)|\geq \frac{M+F}{\Delta x_k}$. 从而

$$\begin{aligned} |\sum_{i=1}^{n} f(\xi_i)\Delta x_i| &\geq |f(\xi_k)\Delta x_k| - |\sum_{i\neq k} f(\xi_i)\Delta x_i| \\ &\geq \frac{M+F}{\Delta x_k}\Delta x_k - F = M. \end{aligned} \tag{1}$$

这与f在$[a,b]$上可积矛盾.

[例子]：狄利克雷函数

$$D(x)=\begin{cases}1, & x\in\mathbb{Q},\\ 0, & x\notin\mathbb{Q}\end{cases}$$

在区间$[0,1]$上有界，但不可积.

事实上，对任意k,

$$\begin{cases}\sum\limits_{k=1}^{n} D(\xi_k)\Delta x_k=1, & \xi_k\in\mathbb{Q},\\ \sum\limits_{k=1}^{n} D(\xi_k)\Delta x_k=0, & \xi_k\notin\mathbb{Q}.\end{cases}$$

用定义说明

$$f(x)=\begin{cases}\frac{1}{\sqrt{x}}, & \text{当}0<x\le 1,\\ 0, & \text{当}x=0.\end{cases}$$

在区间$[0,1]$上不可积.

Theorem (Newton-Leibniz formula)

设函数$f(x)$在区间$[a,b]$上可积, 如果$F(x)$在$[a,b]$上连续且$F'(x)=f(x)$在(a,b)内成立, 则有

$$\int_a^b f(x)\mathrm{d}x=\int_a^b F'(x)\mathrm{d}x=F(b)-F(a).$$

注记1: 如果$F'(x)=f(x)$在除去有限个第一类间断点外成立,结论同样成立.

注记2：若$f(x)$在$[a,b]$上可积，则$F(x)=\displaystyle\int_a^x f(t)\mathrm{d}t$在$[a,b]$上连续.

注记3：定积分的值与积分变量用什么字母表示无关.

注记4：约定$\displaystyle\int_a^b f(x)\mathrm{d}x=-\int_b^a f(x)\mathrm{d}x,\ \int_a^a f(x)\mathrm{d}x=0.$

应用拉格朗日中值定理，存在$\xi_k \in (x_{k-1}, x_k)$使得

$$F(x_k) - F(x_{k-1}) = f(\xi_k)(x_k - x_{k-1}).$$

因$f(x)$在$[a,b]$上可积, 故

$$\begin{aligned}
\int_a^b f(x)\mathrm{d}x &= \lim_{\Delta(T)\to 0} \sum_{k=1}^{n} f(\xi_k)\Delta x_k \\
&= \lim_{\Delta(T)\to 0} \sum_{k=1}^{n} (F(x_k) - F(x_{k-1})) \\
&= F(b) - F(a).
\end{aligned}$$

习题： 用定义证明$[0,1]$上的黎曼函数可积，

$$R(x)=\begin{cases}\frac{1}{q}, & x=\frac{p}{q},\text{其中},p,q\text{为互素的整数},\text{且}q\neq 0,\\ 0, & x\in\mathbf{Q}.\end{cases}$$

证明思路：任取$\epsilon>0$,记

$$N(\epsilon)=\#\left\{\frac{p}{q}\in[0,1]\mid R(\frac{p}{q})=\frac{1}{q}\geq\frac{\epsilon}{2}\right\}$$

取$\delta=\frac{\epsilon}{4N(\epsilon)}$,则对于任意满足的分割$T$,包含满足$R(\frac{p}{q})\geq\frac{\epsilon}{2}$的点$\frac{p}{q}$的小区间至多$2N(\epsilon)$个,其余区间上,都满足$R(x)<\frac{\epsilon}{2}$,故

$$|\sum_{i=1}^{n}R(\xi_i)\Delta x_i|\leq 2N(\epsilon)\Delta(T)+\frac{\epsilon}{2}\sum\Delta x_k<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

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1. 若函数f在$[a,b]$上存在原函数,那么f在$[a,b]$上是否可积?

反例: $$F(x)=\begin{cases} x^2\sin\frac{1}{x^2}, & x\neq 0,\\ 0, & x=0.\end{cases}$$

$$f(x)=\begin{cases} 2x\sin\frac{1}{x^2}-\frac{2}{x}\cos\frac{1}{x^2}, & x\neq 0,\\ 0, & x=0.\end{cases}$$

注意到,F为f的一个原函数,但f在$[-1,1]$上无界,故不可积.

2. 设$f(x)$在$[a,b]$上可积, $g(x)$与$f(x)$仅在$[a,b]$上有限个点处不相同. 证明$g(x)$在$[a,b]$上可积, 且

$$\int_a^b g(x)\mathrm{d}x=\int_a^b f(x)\mathrm{d}x.$$

3. 计算极限 $\lim\limits_{n\to\infty}(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n})$.

4. 计算极限 $\lim\limits_{n\to\infty}(\frac{1}{n^2+1}+\frac{2}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2})$.

5. 计算极限 $\lim\limits_{n\to\infty}[(1+\frac{1}{n})(1+\frac{2}{n})\cdots(1+\frac{n}{n})]^{1/n}$.

Answer:

$$\int_0^1 \frac{1}{1+x}dx, \quad \int_0^1 \frac{x}{1+x^2}dx, \quad \exp(\int_0^1 \ln(1+x)dx).$$

6. 设f为区间I上的下凸函数,任取区间内部的两点$a<b$,证明

$$\int_a^b f(x)dx \geq (b-a)f(\frac{a+b}{2}).$$

第8.2节

黎曼可积的充要条件

设函数$f(x)$在区间$[a,b]$上有界，分割T把$[a,b]$分成n个小区间$[x_{k-1},x_k], k=1,\cdots,n$. 令$\Delta x_k = x_k - x_{k-1}$,

$$M_k = \sup\{f(x)|x\in[x_{k-1},x_k]\},\ \ m_k = \inf\{f(x)|x\in[x_{k-1},x_k]\},$$
$$\omega_k = M_k - m_k = \sup\{|f(s)-f(t)| \mid s,t\in[x_{k-1},x_k]\}$$

称ω_k为函数$f(x)$在区间$[x_{k-1},x_k]$的振幅.

注：(复习) $f(x)$在x_0连续 $\Leftrightarrow \lim\limits_{\delta\to0^+}\omega_{x_0}(\delta)=0.$

f在区间I上一致连续 $\Leftrightarrow \lim\limits_{\delta\to0^+}\omega_f(\delta)=0.$

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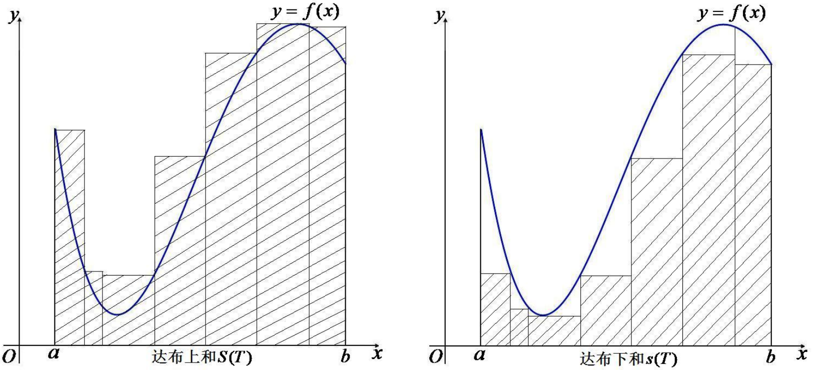
作和式

$$S(T)=\sum_{k=1}^{n} M_k \Delta x_k,\ \ s(T)=\sum_{k=1}^{n} m_k \Delta x_k,$$

分别称之为函数$f(x)$的对应于分割T的达布上和与达布下和.



Definition

若分割$T=\{x_k\}_{0\leqslant k\leqslant n}$与$T'=\{x'_k\}_{0\leqslant k\leqslant n'}$满足$T\subseteq T'$, 则称$T'$是$T$的加细.

Theorem

(i) 若T'是T的加细, 则$s(T)\leqslant s(T')\leqslant S(T')\leqslant S(T)$.

(ii) 设T_1和T_2是$[a,b]$上的任意两个分割, 则$s(T_1)\leqslant S(T_2)$.

记

$$I_*=\sup_T\{s(T)\},\qquad I^*=\inf_T\{S(T)\}.$$

I_*和I^*分别称为$f(x)$在区间$[a,b]$ 上的下积分与上积分.

Theorem

$$\lim_{\Delta(T)\to 0} s(T) = I_*, \quad \lim_{\Delta(T)\to 0} S(T) = I^*.$$

证明思路：由$I_* = \sup\limits_{T}\{s(T)\}$知: $\forall \varepsilon > 0$ 存在分割T_0使得$s(T_0) > I_* - \varepsilon$.

(1) 存在$\delta > 0$(待确定),对满足$\Delta(T) < \delta$的任意分割T,令$T_1 = T \cup T_0$, 则T_1是T_0和T的加细, 则$s(T_1) \geq s(T_0) > I_* - \varepsilon$.

(2) 估计$s(T_1) - s(T)$值：令$\omega = \sup\limits_{s,t\in[a,b]} |f(s) - f(t)| \geq 0$,

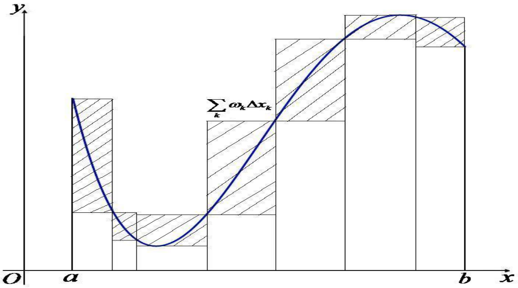
$$0 \leq s(T_1) - s(T) \leq (\#T_0)\omega\Delta(T) < (\#T_0)\omega\delta < \varepsilon.$$

故只要取$\delta = \frac{\varepsilon}{(\#T_0)\omega}$,由(1)和(2)即可得$0 \leq I_* - s(T) < 2\varepsilon$.

Theorem

有界函数$f(x)$在区间$[a,b]$上可积 $\Leftrightarrow$

$$\lim_{\Delta(T)\to 0}(S(T)-s(T)) = \lim_{\Delta(T)\to 0}\sum_{k=1}^{n}\omega_k\Delta x_k = 0 \Leftrightarrow I^* = I_*.$$



"$\Leftarrow$" 对任何分割$T=\{x_k\}_{0\leqslant k\leqslant n}$，成立

$$I_* \leftarrow s(T) \leqslant \sum_{k=1}^{n} f(\xi_k)\Delta x_k \leqslant S(T) \to I^* \qquad (\text{当}\Delta(T)\to 0).$$

"$\Rightarrow$" 由M_k的定义, $\exists \xi_k \in [x_{k-1}, x_k]$使得$f(\xi_k) > M_k - \frac{\varepsilon}{b-a}$.

从而

$$\sum_{k=1}^{n} f(\xi_k)\Delta x_k > \sum_{k=1}^{n} M_k\Delta x_k - \varepsilon = S(T) - \varepsilon.$$

因此

$$S(T) - I < \sum_{k=1}^{n} f(\xi_k)\Delta x_k - I + \varepsilon < \varepsilon + \varepsilon = 2\varepsilon.$$

同理$I - s(T) < 2\varepsilon$. 于是

$$0 \leqslant S(T) - s(T) < 4\varepsilon.$$

Theorem

有界函数$f(x)$在区间$[a,b]$上可积 $\Leftrightarrow$ 对任意$\varepsilon > 0$, 存在分割T, 使得

$$\sum_{k=1}^{n} \omega_k \Delta x_k < \varepsilon.$$

"$\Leftarrow$" 注意到$s(T) \le I_* \le I^* \le S(T)$,故

$$0 \leqslant I^* - I_* \leqslant S(T) - s(T) = \sum_{k=1}^{n} \omega_k \Delta x_k < \varepsilon.$$

问题: 在定积分的定义中,分割采用等分但是保证小区间中ξ_k的任意选取,是否可行?

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有界函数$f(x)$在区间$[a,b]$上可积 $\Leftrightarrow$
$\forall\eta,\ \sigma>0$, $\exists\delta>0$, 存在分割T,使得当$\Delta(T)<\delta$时, $\sum\limits_{\omega_k>\eta}\Delta x_k<\sigma$.

$$\text{“}\Rightarrow\text{”}\quad \eta\sigma=\varepsilon>\sum_{k=1}^{n}\omega_k\Delta x_k\geqslant\sum_{\omega_k>\eta}\omega_k\Delta x_k>\eta\sum_{\omega_k>\eta}\Delta x_k.$$

“$\Leftarrow$” $\forall\varepsilon>0$, 取$\eta=\dfrac{\varepsilon}{2(b-a)}$, $\sigma=\dfrac{\varepsilon}{2\omega}$($\omega$为振幅), $\exists\delta>0$,存在分割T, 使得当$\Delta(T)<\delta$时, $\sum\limits_{\omega_k>\eta}\Delta x_k<\sigma$. 从而可得

$$\begin{aligned}\sum_{k=1}^{n}\omega_k\Delta x_k &= \sum_{\omega_k>\eta}\omega_k\Delta x_k+\sum_{\omega_k\leqslant\eta}\omega_k\Delta x_k\\ &< \omega\sum_{\omega_k>\eta}\Delta x_k+\eta\sum_{\omega_k\leqslant\eta}\Delta x_k<\omega\sigma+\eta(b-a)=\varepsilon.\end{aligned}$$

注记：有界函数$f(x)$在区间$[a,b]$上可积 $\Leftrightarrow$ $\forall \eta,\ \sigma>0$，存在分割T，使得 $\sum\limits_{\omega_k>\eta}\Delta x_k<\sigma$.

"$\Rightarrow$" $\eta\sigma=\varepsilon>\sum\limits_{k=1}^{n}\omega_k\Delta x_k\geqslant\sum\limits_{\omega_k>\eta}\omega_k\Delta x_k>\eta\sum\limits_{\omega_k>\eta}\Delta x_k$.

"$\Leftarrow$" $\forall\varepsilon>0$，取$\eta=\dfrac{\varepsilon}{2(b-a)}$，$\sigma=\dfrac{\varepsilon}{2\omega}$，存在分割$T$，使得 $\sum\limits_{\omega_k>\eta}\Delta x_k<\sigma$. 从而可得

$$
\begin{aligned}
\sum_{k=1}^{n}\omega_k\Delta x_k &= \sum_{\omega_k>\eta}\omega_k\Delta x_k+\sum_{\omega_k\leqslant\eta}\omega_k\Delta x_k\\
&< \omega\sum_{\omega_k>\eta}\Delta x_k+\eta\sum_{\omega_k\leqslant\eta}\Delta x_k<\omega\sigma+\eta(b-a)=\varepsilon.
\end{aligned}
$$

例题： $[0,1]$上的黎曼函数可积.

$$R(x)=\begin{cases}\frac{1}{q}, & \text{当}x=\frac{p}{q}, p\in\mathbb{Z}, q\in\mathbb{N}^*, (p,q)=1,\\ 0, & \text{当}x\text{为无理数}.\end{cases}$$

$\forall\,\eta>0,\sigma>0$，我们有

$$R\left(\frac{p}{q}\right)=\frac{1}{q}>\eta\Rightarrow q<\frac{1}{\eta}.$$

使上式成立的点$\dfrac{p}{q}$只有有限多个: $x_1,\cdots,x_N$. 取$\delta=\dfrac{\sigma}{2N}$，于是当$\Delta(T)<\delta$时，使$\omega_k>\eta$的小区间不超过$2N$个. 所以其长度之和

$$\sum_{\omega_k>\eta}\Delta x_k<2N\delta=\sigma.$$

故$R(x)$在$[0,1]$上可积.

讨论下列函数在$[0,1]$的可积性:

(1) $f(x)=\begin{cases}\frac{1}{x}-\left[\frac{1}{x}\right], & x\neq 0,\\ 0, & x=0.\end{cases}$

(2) $f(x)=\begin{cases}0, & x\in\mathbf{Q},\\ x, & x\notin\mathbf{Q}.\end{cases}$

(3) $f(x)=\begin{cases}\mathrm{sgn}(\sin\frac{\pi}{x}), & x\neq 0,\\ 0, & x=0.\end{cases}$

(4) 设f在$[a,b]$上连续, g在$[\alpha,\beta]$上可积, $g([\alpha,\beta])\subseteq[a,b]$. 则$f\circ g$在$[\alpha,\beta]$上仍可积.

习题：讨论$f(x)=\begin{cases}0, & x\in \mathbf{Q},\\ x, & x\notin \mathbf{Q}.\end{cases}$在$[0,1]$上的可积性.

分析：取$[0,1]$的n等分分割$T=\{\frac{1}{k}\}_{k=1}^{n-1}$，记

$$X_k=[\frac{k-1}{n},\frac{k}{n}],\quad k=1,2,\cdots,n.$$

取$\varepsilon_0=\frac{1}{2}$，对于任意$\delta>0$，取$n$充分大,则$\Delta(T)=\frac{1}{n}<\delta$，注意到$\omega_k\geq\frac{k}{n}$，我们有

$$\sum_{k=1}^{n}\omega_n\Delta x_k\geq\sum_{k=2}^{n}\omega_k\Delta x_k\geq\sum_{k=2}^{n}\frac{k}{n^2}=\frac{n^2+n-2}{2n^2}>\frac{1}{2}=\varepsilon_0.$$

由可积的第一充要条件知$f(x)$在$[0,1]$上不可积.

习题：讨论$f(x)=\begin{cases}\frac{1}{x}-\left[\frac{1}{x}\right], & x\neq 0,\\ 0, & x=0.\end{cases}$在$[0,1]$上的可积性.

分析：函数在$[0,1]$的间断点为$x_0=0, x_k=\frac{1}{k}, k=1,2,\cdots$.
$\forall\eta>0, 1>\sigma>0$, 取$n=[\frac{2}{\sigma}]$,则$n\leq\frac{2}{\sigma}<n+1$,即$\frac{1}{n+1}<\frac{\sigma}{2}\leq\frac{1}{n}$.
记

$$\mathscr{I}=[0,1]\setminus[0,\frac{\sigma}{2})\bigcup_{k=1}^{n}X_k,\quad X_k\equiv(\frac{1}{k}-\frac{\sigma}{2^{k+2}},\frac{1}{k}+\frac{\sigma}{2^{k+2}}).$$

则$f(x)$ 在$\mathscr{I}$上一致连续，则$\exists\delta>0$,
当$x_1,x_2\in\mathscr{I}$且$|x_1-x_2|<\delta$时，有$|f(x_1)-f(x_2)|<\eta$. 则

$$\sum_{\omega_k>\eta}\Delta x_k\leq\frac{\sigma}{2}+\sum_{k=1}^{n}|X_k|=\frac{\sigma}{2}+\sum_{k=1}^{n}\frac{\sigma}{2^{k+1}}=\frac{\sigma}{2}+(\frac{1}{2}-(\frac{1}{2})^{n+1})\sigma<\sigma.$$

Theorem

(i) $[a,b]$上的连续函数;

(ii) $[a,b]$上的单调函数;

(iii) $[a,b]$上只有有限多个间断点的有界函数.

(i) 函数$f(x)$在$[a,b]$上连续. 则$f(x)$在$[a,b]$上一致连续, 故$\forall\varepsilon>0$, 有$\delta>0$, 使当$x_1,x_2\in[a,b]$且$|x_1-x_2|<\delta$时,

$$|f(x_1)-f(x_2)|<\frac{\varepsilon}{b-a}.$$

对于满足$\Delta(T)<\delta$的任意分割$T=\{x_k\}_{0\leqslant k\leqslant n}$, 可得

$$\omega_k=M_k-m_k<\frac{\varepsilon}{b-a},\quad k=1,\cdots,n,$$

于是有

$$\sum_{k=1}^{n}\omega_k\Delta x_k<\frac{\varepsilon}{b-a}\sum_{k=1}^{n}\Delta x_k=\varepsilon.$$

(ii) 不妨设$f(x)$在$[a,b]$上递减. 也不妨设$f(a)>f(b)$.

由于$f(x)$递减, 对于任何分割$T=\{x_k\}_{0\leqslant k\leqslant n}$, 都有$\omega_k=f(x_{k-1})-f(x_k)$.

任给$\varepsilon>0$, 取$\delta=\dfrac{\varepsilon}{f(a)-f(b)}$, 当$\Delta(T)<\delta$时, 就有

$$\begin{aligned}\sum_{k=1}^{n}\omega_k\Delta x_k &< \delta\sum_{k=1}^{n}(f(x_{k-1})-f(x_k))\\ &= \frac{\varepsilon}{f(a)-f(b)}(f(a)-f(b))=\varepsilon.\end{aligned}$$

(iii) 设函数$f(x)$在(a,b)上有k个间断点$\{x_j\}_{j=1}^k$, 不妨设$a=x_0<x_1<x_2<\cdots<x_k<x_{k+1}=b$. 记

$$\hat{\delta}=\min\{x_j-x_{j-1}|j=1,2,\cdots,k+1\}.$$

任给$\varepsilon>0$, 取$\delta_1=\min\left\{\frac{\varepsilon}{4(k+2)\omega},\frac{\hat{\delta}}{3}\right\}$, 其中$\omega$为$f(x)$在$[a,b]$上的振幅. $\mathscr{I}=[a,b]\setminus\bigcup_{j=0}^{k+1}(x_j-\delta_1,x_j+\delta_1)$. $f(x)$在$\mathscr{I}$上一致连续, 故$\exists\delta>0$, 当$x_1,x_2\in\mathscr{I}$且$|x_1-x_2|<\delta$时, 有

$$|f(x_1)-f(x_2)|<\frac{\varepsilon}{2(b-a)}.$$

取$\mathscr{I}$的一个分割T'使$\Delta(T')<\delta$, 可以看成是$[a,b]$的一个分割, 记之为T. 对应于这个分割T, 有

$$\sum_{k=1}^n\omega_k\Delta x_k=\sum_{\mathscr{I}}\omega_i\Delta x_i+\sum_{[a,b]\setminus\mathscr{I}}\omega_j\Delta x_j<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

(iii) 记$\hat{\delta}=\min\{x_j-x_{j-1}|j=1,2,\cdots,k+1\}$. 对任意$\sigma>0$, 取$\delta_1=\min\{\hat{\delta},\sigma\}$.令

$$\mathscr{I}=[a,b]\setminus\bigcup_{j=0}^{k+1}X_j,\quad X_j\equiv(x_j-\frac{\delta_1}{2^{j+2}},x_j+\frac{\delta_1}{2^{j+2}}).$$

则$f(x)$ 在$\mathscr{I}$上一致连续. $\forall\ \eta>0$, $\exists\delta>0$,
当$x_1,x_2\in\mathscr{I}$且$|x_1-x_2|<\delta$时, 有$|f(x_1)-f(x_2)|<\eta$.

取$\mathscr{I}$的一个分割T'使$\Delta(T')<\delta$, 然后任取每个小区间X_j的分割,使得最大区间长度小于δ. 这样构成$[a,b]$的一个满足$\Delta(T)<\delta$的分割T.

$$\sum_{\omega_k>\eta}\Delta x_k\le\sum_{j=0}^{k+1}|X_j|=\sum_{j=0}^{k+2}\frac{\delta_1}{2^{j+1}}=(1-(\frac{1}{2})^{k+2})\delta_1<\sigma.$$

第8.3节

定积分的性质

Theorem

设函数$f(x)$和$g(x)$都在$[a,b]$上可积, λ为常数, 则$\lambda f(x)$, $f(x)\pm g(x)$也都在$[a,b]$上可积, 且有

(i) $\int_a^b \lambda f(x)\mathrm{d}x=\lambda\int_a^b f(x)\mathrm{d}x$;

(ii) $\int_a^b [f(x)\pm g(x)]\mathrm{d}x=\int_a^b f(x)\mathrm{d}x\pm\int_a^b g(x)\mathrm{d}x$;

(iii) 若$f(x)\geq 0,\forall x\in[a,b]$, 则$\int_a^b f(x)\mathrm{d}x\geqslant 0$;

(iv) 若$f(x)\geq g(x),\forall x\in[a,b]$,则$\int_a^b f(x)\mathrm{d}x\geq\int_a^b g(x)\mathrm{d}x$.

Proof. 由定积分的定义直接得到.

Theorem

(i) 若$f(x)$在$[a,b]$上可积，则$f(x)$在$[a,c]$和$[c,b]$上都可积，且有

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x; \tag{1}$$

(ii) 若$f(x)$在$[a,c]$和$[c,b]$上都可积，则$f(x)$在$[a,b]$上可积.

Theorem

设非负函数$f(x)$在$[a,b]$上连续，且$\int_a^b f(x)\mathrm{d}x = 0$. 则在$[a,b]$上$f(x) \equiv 0$.

Proof. 利用反证法可得.

Theorem

若$f(x)$在$[a,b]$上可积, 则$|f(x)|$也在$[a,b]$上可积. 且有

$$\left|\int_a^b f(x)\mathrm{d}x\right| \leqslant \int_a^b |f(x)|\mathrm{d}x.$$

Proof. 注意到振幅:

$$\begin{aligned}
\omega_k(|f|) &= \sup_{s,t\in[x_{k-1},x_k]} \big|\,|f(s)|-|f(t)|\,\big| \\
&\le \sup_{s,t\in[x_{k-1},x_k]} |f(s)-f(t)| \\
&= \omega_k(f).
\end{aligned}$$

Theorem

若$f(x)$和$g(x)$都在$[a,b]$上可积,则$f(x)g(x)$也在$[a,b]$上可积.

Proof. 注意到

$$\begin{aligned}
\omega_k(fg) &= \sup_{s,t} |\, f(s)g(s) - f(t)g(t)\,| \\
&\leq \sup_{s,t} |f(s)g(s) - f(t)g(s)| + \sup_{s,t} |f(t)g(s) - f(t)g(t)| \\
&= \sup_{s,t} (|f(s) - f(t)||g(s)|) + \sup_{s,t} (|f(t)||g(s) - g(t)|) \\
&\leq M(\omega_k(f) + \omega_k(g)).
\end{aligned}$$

习题：设$f(x),g(x)$为$[a,b]$上的可积函数,证明下列Cauchy-Schwartz不等式

$$\left[\int_a^b f(x)g(x)dx\right]^2 \le \int_a^b f^2(x)dx \int_a^b g^2(x)dx. \tag{2}$$

推广：证明下列Hölder不等式

$$\int_a^b |f(x)g(x)|dx \le \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{q}}, \tag{3}$$

其中p,q为满足$\frac{1}{p}+\frac{1}{q}=1$的正数.

Theorem

设$f(x)$在$[a,b]$上连续，$g(x)$在$[a,b]$可积且不变号，则$\exists\xi\in[a,b]$, s.t.

$$\int_a^b f(x)g(x)\mathrm{d}x = f(\xi)\int_a^b g(x)\mathrm{d}x.$$

Proof. 不妨设$g\le 0$. 且当$g\equiv 0$时，显然成立.

下面仅考虑$\int_a^b g(x)\mathrm{d}x<0$. 设$m\le f(x)\le M, \forall x\in[a,b]$.则

$$m\le \frac{\int_a^b f(x)g(x)\mathrm{d}x}{\int_a^b g(x)\mathrm{d}x}\le M.$$

习题: 设$f(x)$为$[a,b]$上的可积函数,且$\int_a^b f(x)dx > 0$, 证明，存在子区间$[\alpha,\beta] \subset [a,b]$和$A > 0$,使得$f(x) > A$, $\forall\ x \in [\alpha,\beta]$.

Remark (积分第一中值定理)

设$f(x)$在$[a,b]$上连续, $g(x)$在(a,b)可积且不变号, 则$\exists \xi \in (a,b)$, *s.t.*

$$\int_a^b f(x)g(x)\mathrm{d}x = f(\xi)\int_a^b g(x)\mathrm{d}x.$$

特别地,若$f(x)$在$[a,b]$上连续，则存在$\xi \in (a,b)$,使得

$$\int_a^b f(x)\mathrm{d}x = f(\xi)(b-a).$$

设$f_n(x)$在$[a,b]$上一列可积函数,且 $\lim\limits_{n\to\infty} f_n(x)=f(x)$ (即逐点收敛), $\forall\ x\in[a,b]$. 则经常考虑下列等式是否成立:

$$\lim_{n\to\infty}\int_a^b f_n(x)\mathrm{d}x=\int_a^b \lim_{n\to\infty} f_n(x)\mathrm{d}x\ ? \tag{4}$$

反例: $f_n(x)=\begin{cases} n, & 0<x\le \frac{1}{n}, \\ 0, & x=0 \text{ 或 } \frac{1}{n}<x\le 1. \end{cases}$

显然,

$$\lim_{n\to\infty}\int_0^1 f_n(x)\mathrm{d}x=1,\quad \int_0^1 \lim_{n\to\infty} f_n(x)\mathrm{d}x=0. \tag{5}$$

Example

求证 $\lim_{n\to\infty}\int_0^{\frac{\pi}{2}}\sin^n x\mathrm{d}x = 0.$

思路： $\forall\, \varepsilon \in (0, \frac{\pi}{2})$，我们有

$$\begin{aligned} 0 \quad &\leqslant \int_0^{\frac{\pi}{2}}\sin^n x\mathrm{d}x = \int_0^{\frac{\pi-\varepsilon}{2}}\sin^n x\mathrm{d}x + \int_{\frac{\pi-\varepsilon}{2}}^{\frac{\pi}{2}}\sin^n x\mathrm{d}x \\ &\leqslant \frac{\pi}{2}\sin^n\frac{\pi-\varepsilon}{2} + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\ n\text{充分大时}. \end{aligned}$$

例2. 设$f(x)$在$[a,b]$可积, 证明存在$[a,b]$上连续函数列$\{\varphi_n(x)\}$, 使得

$$\lim_{n\to\infty}\int_a^b |f(x)-\varphi_n(x)|\mathrm{d}x = 0.$$

思路: 把$[a,b]$ n等份,设$P_k=(x_k,f(x_k))$, $k=0,1,\cdots,n$, 依次连接$P_0,P_1,\cdots,P_n$得折线$\varphi_n(x)$, 显然$\varphi_n(x)$是$[a,b]$上的连续函数. 且当$x\in[x_{k-1},x_k]$时, 成立$m_k\leqslant\varphi_n(x)\leqslant M_k$. 于是

$$|f(x)-\varphi_n(x)|\leqslant M_k-m_k=\omega_k.$$

$$\begin{aligned}0 &\leqslant \lim_{n\to\infty}\int_a^b |f(x)-\varphi_n(x)|\mathrm{d}x\\ &= \lim_{n\to\infty}\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k}|f(x)-\varphi_n(x)|\mathrm{d}x \le \lim_{n\to\infty}\sum_{k=1}^{n}\omega_k\Delta x_k=0.\end{aligned}$$

例2. 设$f(x)$在$[a,b]$可积, 证明存在$[a,b]$上连续函数列$\{\varphi_n(x)\}$, 使得

$$\lim_{n\to\infty}\int_a^b |f(x)-\varphi_n(x)|\mathrm{d}x = 0.$$

思路: 把$[a,b]n$等份,设$P_k=(x_k,f(x_k))$, $k=0,1,\cdots,n$, 依次连接$P_0,P_1,\cdots,P_n$得折线$\varphi_n(x)$, 显然$\varphi_n(x)$是$[a,b]$上的连续函数. 且当$x\in[x_{k-1},x_k]$时, 成立$m_k\leqslant\varphi_n(x)\leqslant M_k$. 于是

$$|f(x)-\varphi_n(x)|\leqslant M_k-m_k=\omega_k.$$

$$\begin{aligned}
0 &\leqslant \lim_{n\to\infty}\int_a^b |f(x)-\varphi_n(x)|\mathrm{d}x\\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\int_{x_{k-1}}^{x_k} |f(x)-\varphi_n(x)|\mathrm{d}x \leq \lim_{n\to\infty}\sum_{k=1}^{n}\omega_k\Delta x_k = 0.
\end{aligned}$$

注记： 进一步可得,

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty}\int_a^b \varphi_n(x)\mathrm{d}x.$$

即对于连续函数成立的定积分命题,可以推广到更一般的可积函数.

例3. 若$f(x)$在$[A,B]$可积，$A<a<b<B$，求证

$$\lim_{h\to 0}\int_a^b |f(x+h)-f(x)|\mathrm{d}x=0.$$

思路： 证明连续函数成立，然后过渡到可积函数.

Riemann-Lebesgue引理

例题：设$f(x)$是$[a,b]$上的可积函数,证明

$$\lim_{m\to\infty}\int_a^b f(x)\sin mx\mathrm{d}x=0,\quad \lim_{m\to\infty}\int_a^b f(x)\cos mx\mathrm{d}x=0.$$

思路：$\forall\ \varepsilon>0$, 存在分割$T=\{x_k\}_{k=1}^n$, 使得

$$\begin{aligned}\left|\int_a^b f(x)\sin mx\mathrm{d}x\right| &\le \sum_{k=1}^n\int_{x_{k-1}}^{x_k}|f(x)-f(x_k)||\sin mx|\mathrm{d}x\\ &+\sum_{k=1}^n|f(x_k)|\left|\int_{x_{k-1}}^{x_k}\sin mx\mathrm{d}x\right|\\ &\le \sum_{k=1}^n\omega_k\Delta x_k+\frac{2Mn}{m}<\frac{\varepsilon}{2}+\frac{2Mn}{m}.\end{aligned}$$

例题：设$f(x)$是$[a,b]$上的可积函数,证明

$$\lim_{m\to\infty}\int_a^b f(x)\sin mx\mathrm{d}x=0,\quad \lim_{m\to\infty}\int_a^b f(x)\cos mx\mathrm{d}x=0.$$

思路：$\forall\ \varepsilon>0$, 存在分割$T=\{x_k\}_{k=1}^n$, 使得

$$\begin{aligned}
\left|\int_a^b f(x)\sin mx\mathrm{d}x\right| &\le \sum_{k=1}^n\int_{x_{k-1}}^{x_k}|f(x)-f(x_k)||\sin mx|\mathrm{d}x\\
&+\sum_{k=1}^n|f(x_k)|\left|\int_{x_{k-1}}^{x_k}\sin mx\mathrm{d}x\right|\\
&\le \sum_{k=1}^n\omega_k\Delta x_k+\frac{2Mn}{m}<\frac{\varepsilon}{2}+\frac{2Mn}{m}.
\end{aligned}$$

Theorem (积分第二中值定理)

设$f(x)$在$[a,b]$上可积,$g(x)$为$[a,b]$上单调，则存在$\xi\in[a,b]$,使得

$$\int_a^b f(x)g(x)\mathrm{d}x = g(a)\int_a^{\xi} f(x)\mathrm{d}x + g(b)\int_{\xi}^b f(x)\mathrm{d}x. \tag{6}$$

注记: (1) 设$f(x)$可积,$g(x)$单调减且非负,则存在$\xi\in[a,b]$,使得

$$\int_a^b f(x)g(x)\mathrm{d}x = g(a)\int_a^{\xi} f(x)\mathrm{d}x. \tag{7}$$

(2) 设$f(x)$可积,$g(x)$单调增且非负,则存在$\xi\in[a,b]$,使得

$$\int_a^b f(x)g(x)\mathrm{d}x = g(b)\int_{\xi}^b f(x)\mathrm{d}x. \tag{8}$$

Theorem (积分第二中值定理)

设$f(x)$在$[a,b]$上可积,$g(x)$为$[a,b]$上单调，则存在$\xi \in [a,b]$,使得

$$\int_a^b f(x)g(x)\mathrm{d}x = g(a)\int_a^\xi f(x)\mathrm{d}x + g(b)\int_\xi^b f(x)\mathrm{d}x. \tag{6}$$

注记: (1) 设$f(x)$可积,$g(x)$单调减且非负,则存在$\xi \in [a,b]$,使得

$$\int_a^b f(x)g(x)\mathrm{d}x = g(a)\int_a^\xi f(x)\mathrm{d}x. \tag{7}$$

(2) 设$f(x)$可积,$g(x)$单调增且非负,则存在$\xi \in [a,b]$,使得

$$\int_a^b f(x)g(x)\mathrm{d}x = g(b)\int_\xi^b f(x)\mathrm{d}x. \tag{8}$$

1. 对于定义在$[a,b]$上的非负连续函数$f(x)$,成立

$$\lim_{n\to\infty}\left(\int_a^b f^n(x)\ \mathrm{d}x\right)^{\frac{1}{n}}=\max_{x\in[a,b]}\{f(x)\}.$$

2. 设非负函数$f(x)$在$[a,b]$上可积，且存在$[\alpha,\beta]\subseteq[a,b]$ 使得$f|_{[\alpha,\beta]}>0$,证明$\displaystyle\int_a^b f(x)\ \mathrm{d}x>0$.

3. 设$f(x)$为$[0,\pi]$上的连续函数，且满足

$$\int_0^\pi f(x)\sin x\ \mathrm{d}x=\int_0^\pi f(x)\cos x\ \mathrm{d}x=0,$$

则$f(x)$在$(0,\pi)$至少有两个零点。

第8.4节

微积分基本定理

Theorem

(i) 设$f(x)$在$[a,b]$上可积, 则$F(x)=\int_a^x f(t)\mathrm{d}t$在$[a,b]$上连续;
(ii) *(**微积分基本定理**)* 若$f(x)$在点$x_0\in[a,b]$连续,
则$F(x)=\int_a^x f(t)\mathrm{d}t$在点$x_0$可导且$F'(x_0)=f(x_0)$.
若$f(x)$在$[a,b]$连续, 则$F(x)$是$f(x)$在$[a,b]$的一个原函数

$$\int f(x)\mathrm{d}x=\int_a^x f(t)\mathrm{d}t+C,\quad x\in[a,b].$$

注： 任一区间上的连续函数都有原函数.

(i)的证明:

$$\begin{aligned}|F(x+\Delta x)-F(x)| &= \left|\int_a^{x+\Delta x} f(t)\mathrm{d}t-\int_a^x f(t)\mathrm{d}t\right| \\ &= \left|\int_x^{x+\Delta x} f(t)\mathrm{d}t\right| \leqslant \left|\int_x^{x+\Delta x} |f(t)|\mathrm{d}t\right| \\ &\leqslant M|\Delta x|.\end{aligned}$$

(ii)的证明:

$$\begin{aligned}\left|\tfrac{F(x_0+\Delta x)-F(x_0)}{\Delta x}-f(x_0)\right| &\leqslant \tfrac{1}{|\Delta x|}\left|\int_{x_0}^{x_0+\Delta x} |f(t)-f(x_0)|\mathrm{d}t\right| \\ &< \tfrac{\varepsilon}{|\Delta x|}\left|\int_{x_0}^{x_0+\Delta x} \mathrm{d}t\right| = \varepsilon.\end{aligned}$$

Corollary (牛顿-莱布尼茨公式)

设$f(x)$在$[a,b]$上连续, $F(x)$是$f(x)$在$[a,b]$上的任一原函数, 则有

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a).$$

Corollary

设$f(x)$在$[a,b]$上连续, $u(x), v(x)$是值域属于$[a,b]$的可微函数, 则

$$\left(\int_{v(x)}^{u(x)} f(t)\mathrm{d}t\right)' = f(u(x))u'(x) - f(v(x))v'(x).$$

Example

设函数$f(x)$连续，求极限$\lim\limits_{x\to 0}\dfrac{\displaystyle\int_{x^3}^{x^2} f(t)\mathrm{d}t}{x^2}$.

Example

设函数$f(x)$连续，求极限$\lim\limits_{x\to 0}\dfrac{\displaystyle\int_{0}^{x} \sin(x-t)^2\mathrm{d}t}{x^3}$.

$$\int_a^b f'(x)g(x)\mathrm{d}x = [f(x)g(x)]\Big|_a^b - \int_a^b f(x)g'(x)\mathrm{d}x.$$

例题:

$$\int_0^1 x\ln^2 x\mathrm{d}x, \quad \int_e^{2e} \frac{\ln x}{(1-x)^2}\mathrm{d}x.$$

Example

求$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \mathrm{d}x$.

解　$I_0 = \frac{\pi}{2}$, $I_1 = 1$. 当$n \geq 2$时,由分部积分得

$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{2}} \sin^n x \mathrm{d}x = \int_0^{\frac{\pi}{2}} \sin^{n-1} x \cdot \sin x \mathrm{d}x \\ &= -\sin^{n-1} x \cos x \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \cdot (n-1) \sin^{n-2} x \cdot \cos x \mathrm{d}x \\ &= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \mathrm{d}x \\ &= (n-1) I_{n-2} - (n-1) I_n. \end{aligned}$$

由此得递推公式

$$I_n = \frac{n-1}{n} I_{n-2}.$$

$$I_n = \begin{cases} \frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{3}{4}\frac{1}{2}\frac{\pi}{2} = \frac{(n-1)!!}{n!!}\frac{\pi}{2}, & n\text{为偶数}, \\ \frac{n-1}{n}\frac{n-3}{n-2}\cdots\frac{4}{5}\frac{2}{3} = \frac{(n-1)!!}{n!!}, & n\text{为奇数}, \end{cases}$$

注记: Wallis公式

$$\lim_{n\to\infty}\frac{1}{2n+1}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 = \frac{\pi}{2}.$$

求下列各极限:

(1) $\displaystyle\lim_{n\to\infty}\frac{1^p+2^p+\cdots+n^p}{n^{p+1}}\ (p>0)$.

(2) $\displaystyle\lim_{n\to\infty}\frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\cdots+\sin\frac{(n-1)\pi}{n}\right)$.

(3) $\displaystyle\lim_{n\to\infty}n\left(\frac{1}{n^2+x^2}+\frac{1}{n^2+2^2x^2}+\cdots+\frac{1}{n^2+n^2x^2}\right)$.

Example

计算$\displaystyle\int_{-1}^{1}\left(2x\sin\frac{1}{x}-\cos\frac{1}{x}\right)\mathrm{d}x.$

Example

计算$\displaystyle\int_{0}^{\pi}\frac{\mathrm{d}x}{1+\cos^2 x}.$ 原函数$\displaystyle\frac{1}{\sqrt{2}}\arctan\left(\frac{\tan x}{\sqrt{2}}\right).$

Example

设函数$f(x)$在$[a,b]$连续可微,$f(a)=0$. 则

$$\int_a^b f^2(x)\mathrm{d}x\le\frac{(b-a)^2}{2}\int_a^b[f'(x)]^2\mathrm{d}x.$$

1. 设$f(x)$为$[0,2\pi]$上的连续函数,证明:

$$\lim_{n\to\infty}\int_0^{2\pi} f(x)|\sin nx|\mathrm{d}x=\frac{2}{\pi}\int_0^{2\pi} f(x)\mathrm{d}x.$$

2. 设函数$f(x)$为$[0,1]$上二次连续可微, $f(0)=f(1)=0$, $f(x)\neq 0$, $\forall\ x\in(0,1)$. 则

$$\int_0^1\left|\frac{f''(x)}{f(x)}\right|\mathrm{d}x\geq 4.$$

3. 设$f(x)$在$[a,b]$上连续,$g(x)$为$[a,b]$上可导,$g'(x)$可积且不变号，则存在$\xi\in[a,b]$,使得

$$\int_a^b f(x)g(x)\mathrm{d}x=g(a)\int_a^{\xi} f(x)\mathrm{d}x+g(b)\int_{\xi}^b f(x)\mathrm{d}x.$$

Theorem

若$f(x)$在x_0的某邻域有$(n+1)$阶连续导数, 则对此邻域内的任意x, 泰勒公式成立:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x),$$

其中

$$R_n(x) = \frac{1}{n!}\int_{x_0}^{x}(x - t)^n f^{(n+1)}(t)\mathrm{d}t$$

称为积分型余项.

注记1：由积分型余项⟹ 拉格朗日余项

$$R_n(x) = \frac{1}{n!}\int_{x_0}^{x}(x-t)^n f^{(n+1)}(t)\mathrm{d}t$$

因$f^{(n+1)}(t)$连续且$(x-t)^n$不变号，故由积分第一中值定理知，在x_0与x之间存在ξ，使得

$$\begin{aligned}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n\mathrm{d}t &= f^{(n+1)}(\xi)\int_{x_0}^{x}(x-t)^n\mathrm{d}t \\ &= -\frac{1}{n+1}f^{(n+1)}(\xi)(x-t)^{n+1}\Big|_{x_0}^{x} \\ &= \frac{1}{n+1}f^{(n+1)}(\xi)(x-x_0)^{n+1}.\end{aligned}$$

得到拉格朗日余项

$$R_n(x) = \frac{1}{(n+1)!}f^{(n+1)}(\xi)(x-x_0)^{n+1}.$$

$$R_n(x) = \frac{1}{n!}\int_{x_0}^{x} (x-t)^n f^{(n+1)}(t) \cdot 1 \, \mathrm{d}t$$

由积分第一中值定理知,在x_0与x之间存在ξ, 使得

$$\begin{aligned} R_n(x) &= \frac{1}{n!} f^{(n+1)}(\xi)(x-\xi)^n \int_{x_0}^{x} \mathrm{d}t \\ &= \frac{1}{n!} f^{(n+1)}(\xi)(x-\xi)^n (x-x_0). \end{aligned}$$

记$\xi = x_0 + \theta(x-x_0), \theta \in [0,1]$, 则

$$R_n(x) = \frac{1}{n!} f^{(n+1)}\left(x_0+\theta(x-x_0)\right)(1-\theta)^n (x-x_0)^{n+1}.$$

则$R_n(x)$为柯西余项.

证明 反复使用分部积分公式,可得

$$
\begin{aligned}
f(x)-f(x_0) &= \int_{x_0}^{x} f'(t)\mathrm{d}t = -\int_{x_0}^{x} f'(t)\mathrm{d}(x-t) \\
&= -(x-t)f'(t)\Big|_{x_0}^{x} + \int_{x_0}^{x}(x-t)f''(t)\mathrm{d}t \\
&= f'(x_0)(x-x_0) + \int_{x_0}^{x}(x-t)f''(t)\mathrm{d}t \\
&= f'(x_0)(x-x_0) - \int_{x_0}^{x} f''(t)\mathrm{d}\frac{(x-t)^2}{2} \\
&= f'(x_0)(x-x_0) - \frac{(x-t)^2}{2}f''(t)\Big|_{x_0}^{x} + \frac{1}{2!}\int_{x_0}^{x}(x-t)^2 f'''(t)\mathrm{d}t. \\
&= f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{1}{2!}\int_{x_0}^{x}(x-t)^2 f'''(t)\mathrm{d}t.
\end{aligned}
$$

Example

设$f(x)$在$[a,b]$上连续可微,证明

$$\max_{a\leqslant x\leqslant b}|f(x)|\leqslant\frac{1}{b-a}\left|\int_a^b f(x)\mathrm{d}x\right|+\int_a^b|f'(x)|\mathrm{d}x.$$

Example

设$f(x)$在$[a,b]$上可积且$\int_a^b f(x)\mathrm{d}x\neq 0$. 证明存在$\xi\in(a,b)$, 使得$\int_a^\xi f(x)\mathrm{d}x=\int_\xi^b f(x)\mathrm{d}x$.

Example

设$f(x)$在$[a,b]$上的连续函数，且对任意满足$\int_a^b f(x)g(x)\mathrm{d}x=0$的连续函数$g$总有$\int_a^b g(x)\mathrm{d}x=0$, 证明$f(x)$为常值函数.

第8.5节

换元积分法

Theorem

设函数$f(x)$在$[a,b]$上连续,$\varphi(t)$在$[\alpha,\beta]$上连续可导且$a\leqslant\varphi(t)\leqslant b$,
$\varphi(\alpha)=a$, $\varphi(\beta)=b$，则

$$\int_a^b f(x)\mathrm{d}x=\int_\alpha^\beta f(\varphi(t))\varphi'(t)\mathrm{d}t.$$

Theorem

设函数$f(x)$在$[a,b]$上可积,$\varphi(t)$在$[\alpha,\beta]$上单调可导,且导数φ'在$[\alpha,\beta]$可积,$a\leqslant\varphi(t)\leqslant b$, $\varphi(\alpha)=a$, $\varphi(\beta)=b$，则

$$\int_a^b f(x)\mathrm{d}x=\int_\alpha^\beta f(\varphi(t))\varphi'(t)\mathrm{d}t.$$

Example

求 $\int_0^a \sqrt{a^2 - x^2}\mathrm{d}x \ (a > 0)$.

Example

若$f(x)$在区间$[-a, a]$可积，则

(i) 当$f(x)$为偶函数时，有$\int_{-a}^{a} f(x)\mathrm{d}x = 2\int_0^a f(x)\mathrm{d}x$.

(ii) 当$f(x)$为奇函数时，有$\int_{-a}^{a} f(x)\mathrm{d}x = 0$.

Example

若$f(x)$是以T为周期的周期函数，且在任何有限区间上可积，则对任意实数a，有

$$\int_a^{a+T} f(x)\mathrm{d}x = \int_0^T f(x)\mathrm{d}x.$$

Example

求$\displaystyle\int_0^1 \frac{\ln(1+x)}{1+x^2}\mathrm{d}x$.

解法一 令$x=\tan t$, 于是$\mathrm{d}x=\sec^2 t\mathrm{d}t$. 则

$$\begin{aligned}\int_0^1\frac{\ln(1+x)}{1+x^2}\mathrm{d}x&=\int_0^{\frac{\pi}{4}}\ln(1+\tan t)\mathrm{d}t\\ &=\int_0^{\frac{\pi}{4}}\ln\frac{\cos t+\sin t}{\cos t}\mathrm{d}t=\int_0^{\frac{\pi}{4}}\ln\frac{\sqrt{2}\sin(\frac{\pi}{4}+t)}{\cos t}\mathrm{d}t\\ &=\frac{\pi}{8}\ln 2+\int_0^{\frac{\pi}{4}}\ln\sin\left(\frac{\pi}{4}+t\right)\mathrm{d}t-\int_0^{\frac{\pi}{4}}\ln\cos t\mathrm{d}t\\ &=\frac{\pi}{8}\ln 2.\end{aligned}$$

令$\frac{\pi}{4}+t=\dfrac{\pi}{2}-u$,即$t=-u+\frac{\pi}{4}$,则

$$\int_0^{\frac{\pi}{4}}\ln\sin\left(\frac{\pi}{4}+t\right)\mathrm{d}t=-\int_{\frac{\pi}{4}}^0\ln\cos u\ \mathrm{d}u=\int_0^{\frac{\pi}{4}}\ln\cos u\ \mathrm{d}u.$$

解法二 令$x=\dfrac{1-t}{1+t}$, 于是$1+x=\dfrac{2}{1+t}$, $\mathrm{d}x=-\dfrac{2\mathrm{d}t}{(1+t)^2}$, $1+x^2=1+\left(\dfrac{1-t}{1+t}\right)^2=\dfrac{2(1+t^2)}{(1+t)^2}$,则

$$\begin{aligned}\int_0^1\frac{\ln(1+x)}{1+x^2}\mathrm{d}x &= \int_0^1\ln\frac{2}{1+t}\cdot\frac{(1+t)^2}{2(1+t^2)}\cdot\frac{2\mathrm{d}t}{(1+t)^2}\\ &= \ln 2\int_0^1\frac{\mathrm{d}t}{1+t^2}-\int_0^1\frac{\ln(1+t)}{1+t^2}\mathrm{d}t.\end{aligned}$$

移项得

$$\int_0^1\frac{\ln(1+x)}{1+x^2}\mathrm{d}x=\frac{1}{2}\ln 2\int_0^1\frac{\mathrm{d}t}{1+t^2}=\frac{\pi}{8}\ln 2.$$

Example

求 $$\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\cos x + \sin x} \mathrm{d}x.$$

Example

求 $$\int_0^{\pi} \frac{\cos x}{\sqrt{a \sin^2 x + b \cos^2 x}} \mathrm{d}x \ (a, b > 0).$$

Example

求 $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin^2 x}{1 + e^{-x}} \mathrm{d}x.$$

Example

设$f(x)$在$[0,a]$上可积,令$g(x)=f(x)+f(a-x)$,则

$$\int_0^a f(x)\mathrm{d}x=\int_0^{\frac{a}{2}} g(x)\mathrm{d}x.$$

Example

计算

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\mathrm{d}x.$$

Example

设$f(x)$在$[a,b]$上二次连续可微,且$f(\frac{a+b}{2})=0$,则

$$\left|\int_a^b f(x)\mathrm{d}x\right| \leq \frac{1}{24}M(b-a)^3.$$

Example

设$f(x)$在$[a,b]$上连续可微,且$f(a)=f(b)=0$,证明

$$\max_{a\leq x\leq b}|f'(x)| \geq \frac{4}{(b-a)^2}\int_a^b |f(x)|\mathrm{d}x.$$

1. 求$\displaystyle\int_0^{\frac{\sqrt{2}}{2}} \arcsin x \arccos x \mathrm{d}x$.
2. 设$f(x)$为$[a,b]$上的可积函数，令$F(x)=\int_a^x f(t)\mathrm{d}t$. 证明
$$\int_a^b F(x)\mathrm{d}x=\int_a^b (b-x)f(x)\mathrm{d}x.$$
3. 设$f(x)$为$[0,1]$上的连续函数，证明
$$\lim_{n\to\infty}\int_0^1 \frac{nf(x)}{1+n^2x^2}=\frac{\pi}{2}f(0).$$
4. 设$f(x)=\begin{cases}\int_0^x \sin\frac{1}{t}\mathrm{d}t, & x\neq 0,\\ 0, & x=0.\end{cases}$ 证明$f(x)$在$x=0$处可导.