数学分析讲义（省身班）

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第11.1节

偏导数

Definition

设$z=f(x,y)$在(x_0,y_0)点的某个邻域内有定义. 如果$f(x,y_0)$在x_0点的导数存在, 即

$$\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$$

收敛, 则称二元函数$z=f(x,y)$在(x_0,y_0)点关于x的**偏导数**存在, 极限值称为$z=f(x,y)$在(x_0,y_0)点关于x的偏导数.

记号：$z'_x(x_0,y_0)$, $f'_x(x_0,y_0)$, $f'_1(x_0,y_0)$或者$\dfrac{\partial z}{\partial x}(x_0,y_0)$, $\dfrac{\partial f}{\partial x}(x_0,y_0)$等等.

Example

设$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0.\end{cases}$ 求$f'_x(x,y)$和$f'_y(x,y)$.

Theorem

若$z=f(x,y)$在区域D内所有偏导数处处都存在且有界,
则$f(x,y)$在D内连续.

Theorem

记$g(x,y)=f(u(x,y),v(x,y))$, $(x,y)\in N_2$,满足复合函数条件,且$u_0=u(x_0,y_0),v_0=v(x_0,y_0)$. 如果$f(u,v)$在$(u_0,v_0)$某邻域内偏导数存在且在$(u_0,v_0)$处连续, $u(x,y),v(x,y)$在(x_0,y_0)所有的偏导数都存在. 则$g(x,y)$在(x_0,y_0)的偏导数存在且

$$g_x'(x_0,y_0)=f_u'(u_0,v_0)u_x'(x_0,y_0)+f_v'(u_0,v_0)v_x'(x_0,y_0),$$

$$g_y'(x_0,y_0)=f_u'(u_0,v_0)u_y'(x_0,y_0)+f_v'(u_0,v_0)v_y'(x_0,y_0).$$

$$
\begin{aligned}
& [g(x_0+\Delta x, y_0) - g(x_0, y_0)] \\
= \ & [f(u(x_0+\Delta x, y_0), v(x_0+\Delta x, y_0)) - f(u(x_0, y_0), v(x_0, y_0)) \\
& \quad - f(u(x_0, y_0), v(x_0+\Delta x, y_0)) + f(u(x_0, y_0), v(x_0+\Delta x, y_0))] \\
= \ & f'_u(\xi_u, v(x_0+\Delta x, y_0))[u(x_0+\Delta x, y_0) - u(x_0, y_0)] \\
& \quad f'_v(u(x_0, y_0), \eta_v)[v(x_0+\Delta x, y_0) - v(x_0, y_0)].
\end{aligned}
$$

若$g(X)=f(u_1(X),\cdots,u_m(X))$, 对于任意$j=1,2,\cdots,n$,有

$$\frac{\partial g}{\partial x_j}(X_0)=\sum_{i=1}^{m}\frac{\partial f}{\partial u_i}(U_0)\frac{\partial u_i}{\partial x_j}(X_0).$$

例题2

设$f(x,y)$的偏导数连续,且$u=f(\frac{y-x}{xy},\frac{z-x}{xz})$. 证明:

$$x^2\frac{\partial u}{\partial x}+y^2\frac{\partial u}{\partial y}+z^2\frac{\partial u}{\partial z}=0.$$

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right), \cdots.$$

Example

$$f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0.\end{cases}$$

$$f'_x(x,y)=\begin{cases} y\dfrac{x^2-y^2}{x^2+y^2}+\dfrac{2x^2y}{x^2+y^2}-\dfrac{2x^2y(x^2-y^2)}{(x^2+y^2)^2}=y\dfrac{x^4+4x^2y^2-y^4}{(x^2+y^2)^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0.\end{cases}$$

$$f'_y(x,y)=\begin{cases} x\dfrac{x^2-y^2}{x^2+y^2}-\dfrac{2xy^2}{x^2+y^2}-\dfrac{2xy^2(x^2-y^2)}{(x^2+y^2)^2}=x\dfrac{x^4-4x^2y^2-y^4}{(x^2+y^2)^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0.\end{cases}$$

可得

$$f''_{xy}(0,0)=\lim_{y\to 0}\frac{f'_x(0,y)-f'_x(0,0)}{y}=-1,$$

$$f''_{yx}(0,0)=\lim_{x\to 0}\frac{f'_y(x,0)-f'_y(0,0)}{x}=1.$$

Theorem

设$f(x,y)$在(x_0,y_0)某邻域偏导数处处存在，若f''_{xy}和f''_{yx}其中之一在某邻域内处处存在且在(x_0,y_0)连续，则另一个必在(x_0,y_0)存在，且$f''_{xy}(x_0,y_0)=f''_{yx}(x_0,y_0)$.

Proof. 设f''_{yx}存在且在(x_0,y_0)连续,考虑

$$
\begin{aligned}
\varphi(\Delta x,\Delta y) &= \frac{1}{\Delta x\,\Delta y}[f(x_0+\Delta x,y_0+\Delta y)-f(x_0+\Delta x,y_0) \\
&\qquad -f(x_0,y_0+\Delta y)+f(x_0,y_0)] \\
&\equiv \frac{1}{\Delta x\,\Delta y}(H(y_0+\Delta y)-H(y_0)).
\end{aligned}
$$

Example

设$f(r)$二阶导数连续，记$u=f(r), r=(x_1^2+\cdots+x_n^2)^{\frac{1}{2}}$，求证:

$$\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}=f''(r)+\frac{n-1}{r}f'(r).$$

特别地

$$u(X)=\begin{cases}-\ln r, & n=2,\\ \dfrac{1}{n-2}r^{-(n-2)}, & n>2.\end{cases}$$

$$\Delta u\equiv\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}=0.$$

例题5

Example (热传导(抛物)方程)

验证$u(x,t)=\frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}$在$D=\{(x,t)\in\mathbb{R}^2 \mid t>0\}$上满足$\frac{\partial u}{\partial t}-a^2\frac{\partial^2 u}{\partial x^2}=0$.

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{1}{2a\sqrt{\pi}}\left(-\frac{1}{2t^{\frac{3}{2}}}+\frac{x^2}{4a^2t^{\frac{5}{2}}}\right)e^{-\frac{x^2}{4a^2t}},\\
\frac{\partial u}{\partial x} &= \frac{1}{2a\sqrt{\pi t}}\left(-\frac{x}{2a^2t}\right)e^{-\frac{x^2}{4a^2t}},\\
\frac{\partial^2 u}{\partial x^2} &= \frac{1}{2a\sqrt{\pi t}}\left(\frac{x^2}{4a^4t^2}-\frac{1}{2a^2t}\right)e^{-\frac{x^2}{4a^2t}}.
\end{aligned}$$

Example (波动(双曲)方程)

验证$u(x,t)=\varphi(x+at)+\psi(x-at)$满足方程$\frac{\partial^2 u}{\partial t^2}-a^2\frac{\partial^2 u}{\partial x^2}=0$,其中$\varphi,\psi$具有二阶连续导数.

Example

设一阶偏导连续函数$f:\mathbb{R}^n \to \mathbb{R}$满足:对于任意$X \neq 0, \lambda \in \mathbb{R}$和正实数$t$ 有$f(tX) = t^\lambda f(X)$. 则$\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(X) = \lambda f(X)$.

第11.2节

全微分

一元函数$y=f(x)$在x_0处可微:

$$\Delta y=f(x_0+\Delta x)-f(x_0)=A(x_0)\Delta x+o(|\Delta x|).$$

类似于一元函数我们问何时有

$$\Delta z=A\Delta x+B\Delta y+o(\sqrt{|\Delta x|^2+|\Delta y|^2}).$$

Definition

设$f(x,y)$为区域D上的函数, 如果存在实数A和B使得

$$\lim_{\rho\to 0}\frac{1}{\rho}\left[f(x_0+\Delta x, y_0+\Delta y)-f(x_0,y_0)-A\Delta x-B\Delta y\right]=0,$$

其中$\rho=\sqrt{\Delta x^2+\Delta y^2}$, 即

$$f(x_0+\Delta x, y_0+\Delta y)-f(x_0,y_0)=A\Delta x+B\Delta y+o(\rho)\ (\rho\to 0).$$

则称f在(x_0,y_0)**可微**.

定义中有序数对(A,B)被称为$f(x,y)$在点(x_0,y_0)处的梯度，记为$\mathrm{grad}\, f(x_0,y_0)=\nabla f(x_0,y_0)$. 我们称线性函数$A\Delta x+B\Delta y$为$f(x,y)$在点$(x_0,y_0)$处的**全微分**，记

$$\mathrm{d}f(x_0,y_0)=A\Delta x+B\Delta y=\langle\nabla f(x_0,y_0),\Delta X\rangle.$$

若记$\Delta x=\mathrm{d}x$，$\Delta y=\mathrm{d}y$，即$\Delta X=\mathrm{d}X=(\mathrm{d}x,\mathrm{d}y)$，从而

$$\mathrm{d}f(x_0,y_0)=A\mathrm{d}x+B\mathrm{d}y.$$

设$X_0=(x_{01},\cdots,x_{0n})\in D$. 如果存在$L=(A_1,\cdots,A_n)\in\mathbb{R}^n$, 使得

$$f(X_0+\Delta X)-f(X_0)=\langle L,\Delta X\rangle+o(|\Delta X|)\ \ (|\Delta X|\to 0),$$

其中$\Delta X=(\Delta x_1,\cdots,\Delta x_n)\in\mathbb{R}^n$, $X_0+\Delta X\in D$, 则称f在X_0可微, L称为f在X_0的梯度, 记为$\nabla f(X_0)$或$\mathrm{grad}\,f(X_0)$. 通常称$\sum_{i=1}^n A_i\Delta x_i$ 为$f(X)$在X_0的全微分. 与二元函数相同写$\Delta x_i=\mathrm{d}x_i$, 则

$$\mathrm{d}f(X_0)=\sum_{i=1}^n A_i\,\mathrm{d}x_i=\langle\nabla f(X_0),\Delta X\rangle=\langle\nabla f(X_0),\mathrm{d}X\rangle.$$

$$f(X_0+\Delta X)-f(X_0)=\langle L,\Delta X\rangle+o(|\Delta X|)\ \ (|\Delta X|\to 0).$$

$$f(X_0+\Delta X)-f(X_0)=\langle L,\Delta X\rangle+o(|\Delta X|)\quad(|\Delta X|\to 0).$$

$\Rightarrow L=\nabla f=(\frac{\partial f}{\partial x_1},\cdots,\frac{\partial f}{\partial x_n}).$

Example

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0.\end{cases}$$

$$f'_x(x,y)=\begin{cases}\dfrac{y^3}{(\sqrt{x^2+y^2})^3}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0,\end{cases}$$

$$f'_y(x,y)=\begin{cases}\dfrac{x^3}{(\sqrt{x^2+y^2})^3}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0,\end{cases}$$

所有偏导数处处存在且有界.

但$f(x,y)$在$(0,0)$点不可微.

事实上, 若可微,由于$f'_x(0,0)=f'_y(0,0)=0$, 从而$\dfrac{xy}{\sqrt{x^2+y^2}}=o\left(\sqrt{x^2+y^2}\right)$, 即

$$\lim_{x^2+y^2\to 0}\frac{xy}{x^2+y^2}=0,$$

但$\lim_{x^2+y^2\to 0}\dfrac{xy}{x^2+y^2}$不存在, 矛盾!