第13讲

A lemma about ordinals

Lemma 2.11.

(i) $0 = \emptyset$ *is an ordinal.*

(ii) *If* α *is an ordinal and* $\beta \in \alpha$*, then* β *is an ordinal.*

(iii) *If* $\alpha \neq \beta$ *are ordinals and* $\alpha \subset \beta$*, then* $\alpha \in \beta$*.*

(iv) *If* α*,* β *are ordinals, then either* $\alpha \subset \beta$ *or* $\beta \subset \alpha$*.*

Proof of (ii)

Proof. Let α be an ordinal and let $x \in \alpha$. First we prove that x is transitive. Let u and v be such that $u \in v \in x$; we wish to show that $u \in x$. Since α is transitive and $x \in \alpha$, we have $v \in \alpha$ and therefore, also $u \in \alpha$. Thus u, v, and x are all elements of α and $u \in v \in x$. Since $\in_\alpha$ linearly orders α, we conclude that $u \in x$.

Second, we prove that $\in_x$ is a well-ordering of x. But by transitivity of α we have $x \subseteq \alpha$ and therefore, the relation $\in_x$ is a restriction of the relation $\in_\alpha$. Since $\in_\alpha$ is a well-ordering, so is $\in_x$. $\square$

Some facts about ordinals

(2.1) $<$ is a linear ordering of the class Ord.

(2.2) For each α, $\alpha = \{\beta : \beta < \alpha\}$.

(2.3) If C is a nonempty class of ordinals, then $\bigcap C$ is an ordinal, $\bigcap C \in C$ and $\bigcap C = \inf C$.

(2.4) If X is a nonempty set of ordinals, then $\bigcup X$ is an ordinal, and $\bigcup X = \sup X$.

(2.5) For every α, $\alpha \cup \{\alpha\}$ is an ordinal and $\alpha \cup \{\alpha\} = \inf\{\beta : \beta > \alpha\}$.

We thus define $\alpha + 1 = \alpha \cup \{\alpha\}$ (the *successor* of α). In view of (2.4), the class Ord is a proper class; otherwise, consider $\sup Ord + 1$.

Remark 6.3.3 Historically, in the context of belief in Global Comprehension, this result was seen as a paradox, the so-called ***Burali-Forti paradox***. Hence it may be seen as an alternative to Russell's paradox for demonstrating the inconsistency of Global Comprehension.

Order types of well-ordered sets

Theorem 2.12. *Every well-ordered set is isomorphic to a unique ordinal number.*

Proof. The uniqueness follows from Lemma 2.7. Given a well-ordered set W, we find an isomorphic ordinal as follows: Define $F(x) = \alpha$ if α is isomorphic to the initial segment of W given by x. If such an α exists, then it is unique. By the Replacement Axioms, $F(W)$ is a set. For each $x \in W$, such an α exists (otherwise consider the least x for which such an α does not exist). If γ is the least $\gamma \notin F(W)$, then $F(W) = \gamma$ and we have an isomorphism of W onto γ. $\square$

Successor ordinal and limit ordinal

If $\alpha = \beta + 1$, then α is a *successor ordinal*. If α is not a successor ordinal, then $\alpha = \sup\{\beta : \beta < \alpha\} = \bigcup \alpha$; α is called a *limit ordinal*. We also consider 0 a limit ordinal and define $\sup \emptyset = 0$.

The existence of limit ordinals other than 0 follows from the Axiom of Infinity; see Exercise 2.3.

Natural numbers

Definition 2.13 (Natural Numbers). We denote the least nonzero limit ordinal ω (or $\boldsymbol{N}$). The ordinals less than ω (elements of $\boldsymbol{N}$) are called *finite ordinals*, or *natural numbers*. Specifically,

$$0 = \emptyset, \qquad 1 = 0 + 1, \qquad 2 = 1 + 1, \qquad 3 = 2 + 1, \qquad \text{etc.}$$

A set X is *finite* if there is a one-to-one mapping of X onto some $n \in \boldsymbol{N}$. X is *infinite* if it is not finite.

We use letters n, m, l, k, j, i (most of the time) to denote natural numbers.

Transfinite induction

Theorem 2.14 (Transfinite Induction). *Let C be a class of ordinals and assume that:*

(i) $0 \in C$;
(ii) *if* $\alpha \in C$, *then* $\alpha + 1 \in C$;
(iii) *if* α *is a nonzero limit ordinal and* $\beta \in C$ *for all* $\beta < \alpha$, *then* $\alpha \in C$.

Then C is the class of all ordinals.

Proof. Otherwise, let α be the least ordinal $\alpha \notin C$ and apply (i), (ii), or (iii). $\square$

超限归纳法的另一种形式如下：设*C*是一个序数的类，满足“对任意序数α, 只要所有小于α的序数都属于*C*, 就有$\alpha \in C$”, 则$C = \mathit{Ord}$.

Sequence

A function whose domain is the set $\boldsymbol{N}$ is called an *(infinite) sequence* (A *sequence in* X is a function $f : \boldsymbol{N} \to X$.) The standard notation for a sequence is

$$\langle a_n : n < \omega \rangle$$

or variants thereof. A *finite sequence* is a function s such $\mathrm{dom}(s) = \{i : i < n\}$ for some $n \in \boldsymbol{N}$; then s is a *sequence of length* n.

Transfinite sequence

A *transfinite sequence* is a function whose domain is an ordinal:

$$\langle a_\xi : \xi < \alpha \rangle.$$

It is also called an α-*sequence* or a *sequence of length* α. We also say that a sequence $\langle a_\xi : \xi < \alpha \rangle$ is an *enumeration* of its range $\{a_\xi : \xi < \alpha\}$. If s is a sequence of length α, then $s^\frown x$ or simply sx denotes the sequence of length $\alpha + 1$ that extends s and whose αth term is x:

$$s^\frown x = sx = s \cup \{(\alpha, x)\}.$$

Sometimes we shall call a "sequence"

$$\langle a_\alpha : \alpha \in Ord \rangle$$

a function (a proper class) on Ord.

Definition by transfinite recursion

"Definition by transfinite recursion" usually takes the following form: Given a function G (on the class of transfinite sequences), then for every θ there exists a unique θ-sequence

$$\langle a_\alpha : \alpha < \theta \rangle$$

such that

$$a_\alpha = G(\langle a_\xi : \xi < \alpha \rangle)$$

for every $\alpha < \theta$.

We shall give a general version of this theorem, so that we can also construct sequences $\langle a_\alpha : \alpha \in Ord \rangle$.

Transfinite recursion

Theorem 2.15 (Transfinite Recursion). *Let G be a function (on V), then (2.6) below defines a unique function F on Ord such that*

$$F(\alpha) = G(F\restriction\alpha)$$

for each α.

In other words, if we let $a_\alpha = F(\alpha)$, then for each α,

$$a_\alpha = G(\langle a_\xi : \xi < \alpha \rangle).$$

(Note that we tacitly use Replacement: $F\restriction\alpha$ is a set for each α.)

Corollary 2.16. *Let X be a set and θ an ordinal number. For every function G on the set of all transfinite sequences in X of length $< \theta$ such that* $\operatorname{ran}(G) \subset X$ *there exists a unique θ-sequence $\langle a_\alpha : \alpha < \theta \rangle$ in X such that $a_\alpha = G(\langle a_\xi : \xi < \alpha \rangle)$ for every $\alpha < \theta$.* $\square$

The limit of the sequence

Definition 2.17. Let $\alpha > 0$ be a limit ordinal and let $\langle \gamma_\xi : \xi < \alpha \rangle$ be a *nondecreasing* sequence of ordinals (i.e., $\xi < \eta$ implies $\gamma_\xi \leq \gamma_\eta$). We define the *limit* of the sequence by

$$\lim_{\xi \to \alpha} \gamma_\xi = \sup\{\gamma_\xi : \xi < \alpha\}.$$

A sequence of ordinals $\langle \gamma_\alpha : \alpha \in Ord \rangle$ is *normal* if it is increasing and *continuous*, i.e., for every limit α, $\gamma_\alpha = \lim_{\xi \to \alpha} \gamma_\xi$.

Addition, multiplication and exponentiation of ordinals

Definition 2.18 (Addition). For all ordinal numbers α

(i) $\alpha + 0 = \alpha$,
(ii) $\alpha + (\beta + 1) = (\alpha + \beta) + 1$, for all β,
(iii) $\alpha + \beta = \lim_{\xi \to \beta}(\alpha + \xi)$ for all limit $\beta > 0$.

Definition 2.19 (Multiplication). For all ordinal numbers α

(i) $\alpha \cdot 0 = 0$,
(ii) $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha$ for all β,
(iii) $\alpha \cdot \beta = \lim_{\xi \to \beta} \alpha \cdot \xi$ for all limit $\beta > 0$.

Definition 2.20 (Exponentiation). For all ordinal numbers α

(i) $\alpha^0 = 1$,
(ii) $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$ for all β,
(iii) $\alpha^\beta = \lim_{\xi \to \beta} \alpha^\xi$ for all limit $\beta > 0$.