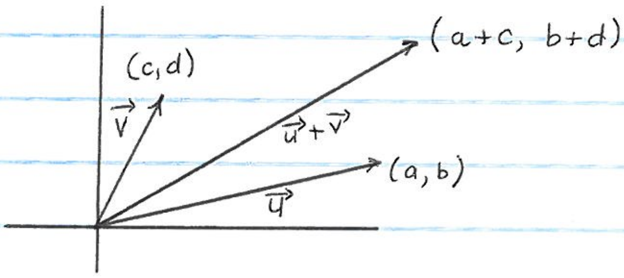
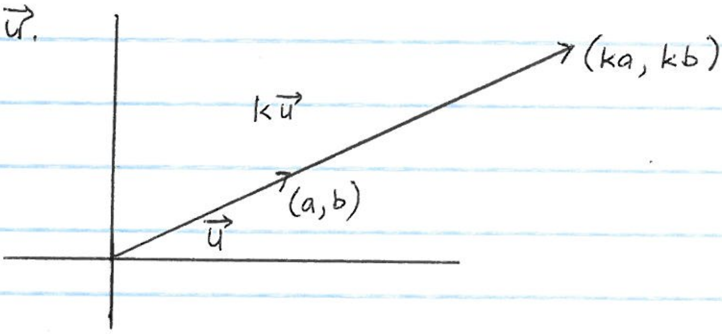
Vectors in $\mathbb{R}^n$

Certain quantities such as temperature and speed possess only magnitude and are called scalars. Other quantities such as force and velocity possess both magnitude and direction and are called vectors. Vectors are usually represented by arrows. If the origin is the initial point of the vector, then every vector is uniquely determined by the coordinates of its terminal point.

(i) Addition: If (a, b) and (c, d) are the terminal points of vectors $\vec{u}$ and $\vec{v}$ respectively, then $(a+c, b+d)$ will be the terminal point of $\vec{u}+\vec{v}$.



(ii) Scalar multiplication: If (a, b) is the terminal point of the vector $\vec{u}$, then (ka, kb) will be the terminal point of the vector $k\vec{u}$.



We can identify a vector with its terminal point: the ordred pair (a, b) can be identified with the vector $\vec{u} = (a, b)$. We genralise this notion so an n-tuple $(a_1, a_2, \dots, a_n)$ can be identified with the vector $\vec{v} = (a_1, a_2, \dots, a_n)$. The coordinates or components of the vector $\vec{v}$ may be real numbers or complex numbers.

Vectors in $\mathbb{R}^n$

Let $\mathbb{R}$ denote the set of all real numbers. The set of all n-tuples of real numbers is called n-dimensional space and is denoted by $\mathbb{R}^n$. Let $u = (u_1, u_2, \dots, u_n)$ be an n-tuple in $\mathbb{R}^n$. u is the point in $\mathbb{R}^n$ with coordinates u_i, $1 \leq i \leq n$. $\vec{u}$ is the vector with components u_i, $1 \leq i \leq n$. The term scalar is used for the elements of $\mathbb{R}$.

Example: Consider the following vectors:

$$(0, 1), \quad (1, -3), \quad (1, 2, \sqrt{3}, 4), \quad (-5, \tfrac{1}{2}, 0, \pi).$$

The first two vectors have two components and can be identified with points in $\mathbb{R}^2$, and the last two vectors have four components and can be identified with points in $\mathbb{R}^4$.

Two vectors $\vec{u} = (u_1, u_2, \dots, u_n)$ and $\vec{v} = (v_1, v_2, \dots, v_n)$ are equal if and only if $u_i = v_i$ for all $1 \leq i \leq n$.

Example: Suppose $(x-y, x+y, z-1) = (4, 2, 3)$. Then by the definition of equality of vectors, $x-y=4$, $x+y=2$, $z-1=3$. Solving this system gives $x=3$, $y=-1$, $z=4$.

Vector addition and scalar multiplication

Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$,

$$\vec{u} = (u_1, u_2, \dots, u_n), \quad \vec{v} = (v_1, v_2, \dots, v_n).$$

The sum of $\vec{u}$ and $\vec{v}$, written $\vec{u} + \vec{v}$, is the vector obtained by adding corresponding components:

$$\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$$

The product of the real number k by the vector $\vec{u}$, written $k\vec{u}$, is the vector obtained by multiplying each component by k:

$$k\vec{u} = (ku_1, ku_2, \dots, ku_n)$$

Note that $k\vec{u}$ and $\vec{u}+\vec{v}$ are also vectors in $\mathbb{R}^n$.

We also define

$$-\vec{u} = -1\vec{u} = (-u_1, -u_2, \ldots, -u_n)$$

and $\vec{u}-\vec{v} = \vec{u} + (-\vec{v}) = (u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n)$.

The sum of vectors having different numbers of components is not defined.

<u>Example</u>:

Let $\vec{u} = (1, -3, 2, 4)$, $\vec{v} = (3, 5, -1, -2)$. Then

$$\vec{u}+\vec{v} = (1+3, -3+5, 2-1, 4-2) = (4, 2, 1, 2)$$

$$-2\vec{u} = -2(1, -3, 2, 4) = (-2, 6, -4, -8)$$

$$\vec{u} - 3\vec{v} = (1, -3, 2, 4) - (9, 15, -3, -6) = (-8, -18, 5, 10).$$

<u>Example</u>:

The vector $(0, 0, \ldots, 0)$ in $\mathbb{R}^n$, denoted by $\vec{0}$, is called the zero vector. Note that for $\vec{u} = (u_1, u_2, \ldots, u_n)$

$$\vec{u} + \vec{0} = (u_1 + 0, u_2 + 0, \ldots, u_n + 0) = (u_1, u_2, \ldots, u_n) = \vec{u}$$

$$\vec{0} + \vec{u} = (0 + u_1, 0 + u_2, \ldots, 0 + u_n) = (u_1, u_2, \ldots, u_n) = \vec{u}.$$

Basic properties of vectors in $\mathbb{R}^n$ under the operations addition and scalar multiplication are as follows:

<u>Theorem</u>: For any vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and any scalars $k, k' \in \mathbb{R}$:

(i) $(\vec{u}+\vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

(ii) $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$

(iii) $\vec{u} + (-\vec{u}) = \vec{0}$

(iv) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(v) $k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v}$

(vi) $(k+k')\vec{u} = k\vec{u} + k'\vec{u}$

(vii) $(kk')\vec{u} = k(k'\vec{u})$

(viii) $1\vec{u} = \vec{u}$

<u>Dot Product</u>

Let $\vec{u}, \vec{v}$ be vectors in $\mathbb{R}^n$,

$$\vec{u} = (u_1, u_2, \ldots, u_n), \quad \vec{v} = (v_1, v_2, \ldots, v_n).$$

The dot product or inner product of $\vec{u}$ and $\vec{v}$, denoted by $\vec{u}.\vec{v}$, is the scalar obtained by multiplying the components and then adding the products:

$$\vec{u}.\vec{v} = (u_1, u_2, \ldots, u_n).(v_1, v_2, \ldots, v_n)$$
$$= u_1v_1 + u_2v_2 + \ldots + u_nv_n$$

$\vec{u}$ and $\vec{v}$ are orthogonal (perpendicular) iff $\vec{u}.\vec{v} = 0$.

Example: Let $\vec{u} = (1, -2, 3, -4)$, $\vec{v} = (6, 7, 1, -2)$, $\vec{w} = (5, -4, 5, 7)$

Then $\vec{u}.\vec{v} = (1)(6) + (-2)(7) + (3)(1) + (-4)(-2) = 3$

$\vec{u}.\vec{w} = (1)(5) + (-2)(-4) + (3)(5) + (-4)(7) = 0$

So $\vec{u}$ and $\vec{w}$ are orthogonal.

Theorem: For any vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and any scalar $k \in \mathbb{R}$:

(i) $(\vec{u} + \vec{v}).\vec{w} = \vec{u}.\vec{w} + \vec{v}.\vec{w}$

(ii) $(k\vec{u}).\vec{v} = k(\vec{u}.\vec{v})$

(iii) $\vec{u}.\vec{v} = \vec{v}.\vec{u}$

(iv) $\vec{u}.\vec{u} \geq 0$ and $\vec{u}.\vec{u} = 0$ if and only if $\vec{u} = 0$.

Norm and distance in $\mathbb{R}^n$

Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$,

$$\vec{u} = (u_1, u_2, \ldots, u_n), \quad \vec{v} = (v_1, v_2, \ldots, v_n).$$

The distance between the points u and v, written $d(u, v)$, is given by

$$d(u,v) = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2 + \ldots + (v_n - u_n)^2}$$

The norm (or length) of the vector $\vec{u}$, written $\vec{u}$, is defined to be the nonnegative square root of $\vec{u}.\vec{u}$:

$$\|\vec{u}\| = \sqrt{\vec{u}.\vec{u}} = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$$

Observe that

$$\|\vec{v} - \vec{u}\| = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2 + \ldots + (v_n - u_n)^2} = d(u, v).$$

Example: Let $\vec{u} = (1, -2, 4, 1)$ and $\vec{v} = (3, 1, -5, 0)$.

$$d(u,v) = \sqrt{(3-1)^2 + (1+2)^2 + (-5-4)^2 + (0-1)^2} = \sqrt{95}$$
$$\|\vec{u}\| = \sqrt{1^2 + (-2)^2 + 4^2 + 1^2} = \sqrt{22}$$
$$\|\vec{v}\| = \sqrt{3^2 + 1^2 + (-5)^2 + 0^2} = \sqrt{35}$$

A vector is called a unit vector if its norm is 1. For example, the vectors $\vec{e}_1 = (1, 0, \dots, 0)$, $\vec{e}_2 = (0, 1, 0, \dots, 0)$ $\dots, \vec{e}_n = (0, \dots, 0, 1)$ are all unit vectors. For any non-zero vector $u \in \mathbb{R}^n$, the vector $\frac{1}{\|\vec{u}\|}(\vec{u})$ is a unit vector since $\|\frac{1}{\|\vec{u}\|}\vec{u}\| = \frac{1}{\|\vec{u}\|}(\|\vec{u}\|) = 1$. It has the same direction as the vector $\vec{u}$.

<u>Theorem</u>: (Cauchy - Schwarz Inequality)
For any vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$, $|\vec{u}.\vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$.

<u>Proof</u>: Let $\vec{u} = (u_1, u_2, \dots, u_n)$, $\vec{v} = (v_1, v_2, \dots, v_n)$.
We shall prove the stronger statement $|\vec{u}.\vec{v}| \leq \sum_{i=1}^{n} |u_i v_i| \leq \|\vec{u}\| \|\vec{v}\|$
If $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$, then the inequality reduces to $0 \leq 0 \leq 0$ which is true. So we only need to consider the case when $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0}$ ie when $\|\vec{u}\| \neq 0$ and $\|\vec{v}\| \neq 0$. Now

$|\vec{u}.\vec{v}| = |u_1 v_1 + u_2 v_2 + \dots + u_n v_n| \leq |u_1 v_1| + |u_2 v_2| + \dots + |u_n v_n| = \sum_{i=1}^{n} |u_i v_i|$

So we only need to prove the second inequality.
For any real numbers x, y,

$$0 \leq (x - y)^2 = x^2 - 2xy + y^2$$

$$\text{ie } 2xy \leq x^2 + y^2 \quad \dots\dots\dots\dots \quad (1)$$

Set $x = \frac{|u_i|}{\|\vec{u}\|}$ and $y = \frac{|v_i|}{\|\vec{v}\|}$ in (1) to obtain

$$2 \frac{|u_i|}{\|\vec{u}\|} \frac{|v_i|}{\|\vec{v}\|} \leq \frac{|u_i|^2}{\|\vec{u}\|^2} + \frac{|v_i|^2}{\|\vec{v}\|^2} \quad \dots \quad (2)$$

Summing (2) with respect to i, we get

$$2 \frac{\sum_{i=1}^{n} |u_i| |v_i|}{\|\vec{u}\| \|\vec{v}\|} \leq \frac{\sum_{i=1}^{n} |u_i|^2}{\|\vec{u}\|^2} + \frac{\sum_{i=1}^{n} |v_i|^2}{\|\vec{v}\|^2}$$

$$\text{ie. } 2 \frac{\sum_{i=1}^{n} |u_i v_i|}{\|\vec{u}\| \|\vec{v}\|} \leq \frac{\sum_{i=1}^{n} |u_i|^2}{\|\vec{u}\|^2} + \frac{\sum_{i=1}^{n} |v_i|^2}{\|\vec{v}\|^2}$$

Now since $\|\vec{u}\|^2 = \sum_{i=1}^{n} u_i^2 = \sum_{i=1}^{n} |u_i|^2$ and $\|\vec{v}\|^2 = \sum_{i=1}^{n} v_i^2 = \sum_{i=1}^{n} |v_i|^2$, we get

$$\frac{2\sum_{i=1}^{n} |u_i v_i|}{\|\vec{u}\|\,\|\vec{v}\|} \leq \frac{\|\vec{u}\|^2}{\|\vec{u}\|^2} + \frac{\|\vec{v}\|^2}{\|\vec{v}\|^2} = 2$$

ie $$\frac{\sum_{i=1}^{n} |u_i v_i|}{\|\vec{u}\|\,\|\vec{v}\|} \leq 1.$$

Multiplying both sides by $\|\vec{u}\|\,\|\vec{v}\|$, we obtain the required inequality.

<u>Theorem</u>: (Minskowski's inequality)
For any vectors $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$, $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$.

<u>Proof</u>: If $\|\vec{u}+\vec{v}\| = 0$, then the inequality holds. So we need only consider the case when $\|\vec{u}+\vec{v}\| \neq 0$.
Now $|u_i + v_i| \leq |u_i| + |v_i|$ for any real numbers u_i, v_i.
Hence
$$\begin{aligned} \|\vec{u}+\vec{v}\|^2 &= \sum_{i=1}^{n} (u_i+v_i)^2 \\ &= \sum_{i=1}^{n} |u_i+v_i|^2 \\ &\leq \sum_{i=1}^{n} |u_i+v_i|\left(|u_i| + |v_i|\right) \end{aligned}$$

So $\|\vec{u}+\vec{v}\| \leq \sum_{i=1}^{n} |u_i+v_i|\,|u_i| + \sum_{i=1}^{n} |u_i+v_i|\,|v_i|$
By the Cauchy-Schwarz inequality

$$\sum_{i=1}^{n} |u_i+v_i|\,|u_i| = \sum_{i=1}^{n} |(u_i+v_i)(u_i)| \leq \|\vec{u}+\vec{v}\|\,\|\vec{u}\| \quad \text{and}$$
$$\sum_{i=1}^{n} |u_i+v_i|\,|v_i| = \sum_{i=1}^{n} |(u_i+v_i)(v_i)| \leq \|\vec{u}+\vec{v}\|\,\|\vec{v}\|.$$

Hence $\|\vec{u}+\vec{v}\|^2 \leq \|\vec{u}+\vec{v}\|\,\|\vec{u}\| + \|\vec{u}+\vec{v}\|\,\|\vec{v}\| = \|\vec{u}+\vec{v}\|\left(\|\vec{u}\| + \|\vec{v}\|\right)$
Dividing both sides by $\|\vec{u}+\vec{v}\|$, we obtain the required inequality.