ESC384 Assignment 1

Due Monday, 23 September 2019, at 9:10

A hard copy of the assignment is due in class at the specified time. For problems that require coding, please turn in a hard copy of the code and also upload the code to Quercus to facilitate the grading process. Finally, please adhere to the collaboration policy: the final write up must be prepared individually without consulting others. (See the syllabus for details.)

Update: please skip Problem 3(d) and (e). This assignment will be graded out of 90.

Problem 1. Classification of PDEs (22%)

Consider the following PDEs:

(i) $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \sin(4x) \quad \text{in } \mathbb{R} \times \mathbb{R}_{>0}$

(ii) $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = -\frac{u}{|1+x|} \quad \text{in } \mathbb{R} \times \mathbb{R}_{>0}$

(iii) $\frac{\partial u^2}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \sin(u) = 0 \quad \text{in } \mathbb{R}^2 \times \mathbb{R}_{>0}$

(iv) $-2\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} - 2\frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in } \mathbb{R}^2$

For each PDE, answer the following questions and briefly justify your answers:

(a) (3%) State the order.

(b) (4%) State if the PDE is linear homogeneous, linear nonhomogeneous, or nonlinear.

(c) (5%) If the equation is second-order, classify the equation as elliptic, parabolic, or hyperbolic.

Hint: the standard analysis of the "A matrix" applies even if the equation is nonlinear.

(d) (5%) State whether the following statement (S1) holds: if v and w satisfies the PDE, then $v+w$ also satisfies the PDE.

(e) (5%) State whether the following statement (S2) holds: if u satisfies the PDE, then w obtained by shifting u by π in the x direction (i.e., $w(x,t) = u(x-\pi,t)$ in $\mathbb{R} \times \mathbb{R}_{>0}$ and $w(x,y,t) = u(x-\pi,y,t)$ in $\mathbb{R}^2 \times \mathbb{R}_{>0}$) also satisfies the PDE.

Problem 2. Odd, even, and periodic functions (18%)

State if each of the following statements is true or false, and then prove the statement or provide a counterexample:

(a) (5%) If $f : \mathbb{R} \to \mathbb{R}$ is even and continuously differentiable, then $g(x) \equiv f'(x)$ is odd.

(b) (5%) If $f : \mathbb{R} \to \mathbb{R}$ is even and integrable, then $g(x) \equiv \int_{\xi=0}^{x} f(\xi) d\xi$ is odd.

(c) (8%) If $f : \mathbb{R} \to \mathbb{R}$ is z-periodic and integrable, then $\int_{x=y}^{y+z} f(x) dx = \int_{x=0}^{z} f(x) dx$ for any $y \in \mathbb{R}$.

Problem 3. Fourier series solution of ODE ~~(25%)~~ (15%)

We wish to solve an ODE boundary value problem

$$-u'' + u = f \quad \text{in } (0, L),$$
$$u(x = 0) = u(x = L) = 0,$$

where f is some square-integrable function, and $L > 0$ is the length of the domain. Our goal is to express the solution u as a series; i.e., $u(x) \sim \sum_{n=1}^{\infty} \hat{u}_n \phi_n(x)$ where $(\phi_n)_{n=1}^{\infty}$ is an appropriate sequence of functions. Answer the following questions:

(a) (3%) Choose either (i) $\phi_n(x) = \sin(\lambda_n x)$ or (ii) $\phi_n(x) = \cos(\lambda_n x)$ and an appropriate λ_n so that the boundary conditions are satisfied for any set of coefficients $\hat{u}_n$, $n = 1, 2, \dots$. (From hereon, $\phi_n(x)$ should be replaced by $\sin(\lambda_n x)$ or $\cos(\lambda_n x)$ with an appropriate λ_n.)

(b) (8%) Substitute the expression found in (a) to the ODE, perform the necessary differentiation (assuming the solution is sufficiently smooth), and invoke the orthogonality relationship to find the expression for the coefficient $\hat{u}_n$ in terms of the coefficient $\hat{f}_n \equiv \frac{1}{2L} \int_0^L f(x)\phi_n(x)dx$.

(c) (4%) Let $f = x$. Find coefficients $\hat{u}_n$, $n = 1, 2, \dots$, associated with the solution u.

(d) ~~(5%) Prove or provide a counterexample to the following statement: if~~ $\cancel{f}$ ~~is square-integrable, then the series~~ $\cancel{\sum_{n=1}^{\infty} \hat{u}_n \phi_n}$ ~~converges uniformly to~~ $\cancel{u}$.

~~*Hint:* to prove the statement, we need to prove it for all square-integrable~~ $\cancel{f}$. ~~To disprove the statement, we need to find just one counterexample.~~

(e) ~~(5%) Prove or provide a counterexample to the following statement: if~~ $\cancel{f}$ ~~is square-integrable, then the series~~ $\cancel{\sum_{n=1}^{\infty} \hat{u}_n \phi_n'}$ ~~converges uniformly to~~ $\cancel{u'}$.

Problem 4. Convergence of Fourier series (35%)

Throughout this problem, let $f(x) = x^2$, $x \in [0, 1]$. This problem requires the use MATLAB or another comparable programming language.

(a) (5%) Find, analytically, the Fourier cosine coefficients and the Fourier sine coefficients of f. We will hereafter refer to the cosine (resp. sine) series as f^{c} (resp. f^{s}) and the associated N-term truncated series as f^{c}_N (resp. f^{s}_N).

(b) (7%) Using MATLAB, plot, in a single figure, (i) $f^{\mathrm{c}}_{N=3}$, (ii) $f^{\mathrm{c}}_{N=30}$, (iii) $f^{\mathrm{c}}_{N=300}$, and (iv) the appropriate periodic extension of f that the series approximates, for $x \in [-1, 2]$.

(c) (7%) Repeat (b) for the Fourier sine series.

(d) (6%) Report, in a table, the pointwise error in the truncated Fourier cosine and sine series $|f(x) - f^{\mathrm{c/s}}_N(x)|$ at $x = 0.5$ for $N = 3$, 30, and 300. (There should be six entries in the table.) Repeat the exercise for $x = 0.9$. Discuss if the observed behavior is consistent with the expected pointwise convergence of the series.

(e) (6%) Report, in a table, the maximum error in the truncated Fourier cosine series $\max_{x \in [0,1]} |f(x) - f_N^{\mathrm{c}}(x)|$ for $N = 3$, 30, and 300. (Approximate the "max" on a sufficiently fine set of evaluation points.) Discuss if the observed behavior is consistent with the expected uniform convergence behavior of the series.

(f) (4%) Would the maximum error in the truncated Fourier sine series $\max_{x \in [0,1]} |f(x) - f_N^{\mathrm{s}}(x)|$ behave differently from that for the Fourier cosine series as $N \to \infty$? Describe the expected behavior.

Hint: we can find the answer analytically without carry out the computation.