ESC384 Assignment 4

Due Monday, 18 November 2019, at 9:10

A hard copy of the assignment is due in class at the specified time. For problems that require coding, please turn in a hard copy of the code and also upload the code to Quercus to facilitate the grading process. Finally, please adhere to the collaboration policy: the final write up must be prepared individually without consulting others. (See the syllabus for details.)

Problem 1. Heat equation: method of reflection (20%)

Consider the heat equation on the half line:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \mathbb{R}_{>0} \times \mathbb{R}_{>0},$$
$$u(x, t = 0) = g(x) \equiv \begin{cases} 1 - x, & x \le 1, \\ 0, & x > 1. \end{cases}$$

We consider two different boundary conditions and the associated solutions:

D. The solution u_D associated with the homogeneous Dirichlet boundary condition at $x = 0$.

N. The solution u_N associated with the homogeneous Neumann boundary condition at $x = 0$.

Answer the following questions:

(a) (5%) We wish to identify a problem on the whole line (i.e., $\mathbb{R}$) whose solution restricted to $x \ge 0$ is identical to u_D or u_N for $t > 0$. Sketch, in two separate figures, the initial conditions on the whole line required to yield u_D and u_N.

(b) (15%) For each of the following statements, state whether the statement holds and justify your answer:

(i) $u_N(x,t) \ge u_D(x,t)$ for all $x \in \mathbb{R}_{>0}$ and $t \in \mathbb{R}_{>0}$.

(ii) $u_N(x,t) > u_D(x,t)$ for all $x \in \mathbb{R}_{>0}$ and $t \in \mathbb{R}_{>0}$.

(iii) $\int_0^\infty u_N(x,t)dx = \int_0^\infty g(x)dx$ for all $t \in \mathbb{R}_{>0}$.

(iv) $\int_0^\infty u_D(x,t)dx = \int_0^\infty g(x)dx$ for all $t \in \mathbb{R}_{>0}$.

Problem 2. Heat equation: finite difference (50%)

Consider the heat equation

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= f \quad \text{in } (0,1) \times (0,T], \\ u(x=0,t) &= 0 \quad \text{on } \{x=0\} \times (0,T], \\ \frac{\partial u}{\partial x}(x=1,t) + u(x=1,t) &= 0 \quad \text{on } \{x=1\} \times (0,T], \\ u &= g \quad \text{on } (0,1) \times \{t=0\} \end{aligned}$$

for some initial condition function $g : (0,1) \to \mathbb{R}$ and source function $f : (0,1) \times (0,T] \to \mathbb{R}$. Note that a Robin boundary condition is imposed on the right boundary.

We will implement a finite difference heat equation solver in MATLAB. We will use $n+1$ grid points $0 \equiv x_0 < x_1 < \cdots < x_n \equiv 1$ to discretize the spatial domain, and $J+1$ time points $0 \equiv t^0 < t^1 \cdots < t^J = T$ to discretize the time interval. We assume both spatial and temporal points are equispaced. Our goal is to approximate the solution $u(x_i, t^j)$ for $i = 1, \dots, n$ and $j = 1, \dots, J$.

We first consider the semi-discrete form of the equation associated with the second-order accurate finite difference approximation in space.

(a) (10%) Find the semi-discrete equations for (i) an interior point $i \in [2, n-1]$, (ii) the first point $i = 1$, and (iii) the last point $i = n$. Also identify the equation for (iv) the initial condition. Express the answer in terms of $\tilde{x}_i$, $\tilde{u}_i(t)$, $\hat{f}_i(t) \equiv f(x_i, t)$, and $\hat{g}_i \equiv g(x_i)$.

(b) (6%) The semi-discrete equations found in (a) can be expressed as

$$\frac{d\tilde{u}}{dt}(t) + \hat{A}\tilde{u}(t) = \hat{f}(t) \quad \text{in } \mathbb{R}^n,$$
$$\tilde{u}(t=0) = \hat{g} \quad \text{in } \mathbb{R}^n,$$

where $\tilde{u}(t) \in \mathbb{R}^n$, $\hat{A} \in \mathbb{R}^{n\times n}$, $\hat{f}(t) \in \mathbb{R}^n$, and $\hat{g} \in \mathbb{R}^n$. Find the expressions for the matrix $\hat{A}$ and vectors $\hat{f}(t)$ and $\hat{g}$.

We now consider the fully discrete form of the equation associated with the Crank-Nicolson approximation in time.

(c) (6%) The fully discrete equation associated with the semi-discrete equation found in (b) can be expressed as

$$\hat{C}\tilde{\tilde{u}}^j = \hat{D}\tilde{\tilde{u}}^{j-1} + \hat{F}(t^j, t^{j-1}), \quad j = 1, \dots, J.$$

where $\hat{C} \in \mathbb{R}^{n\times n}$, $\hat{D} \in \mathbb{R}^{n\times n}$, and $\hat{F}(t^j, t^{j-1}) \in \mathbb{R}^n$. Find the expressions for the matrices and vectors.

We now implement the finite difference solver in MATLAB.

(d) (15%) Starting with the template `heat_fd_temp.m`, implement the finite difference solver.

Note. Please include a hard copy of the code in the assignment and also upload the code to Quercus to facilitate the grading process.

Note 2. You do *not* have to use the template if you would rather code everything from scratch.

(e) (5%) Let

$$f(x,t) = \exp(-3x)\exp(t),$$
$$g(x) = \frac{1}{2}x(2-x) + \frac{1}{6\pi}\sin(3\pi x).$$

Invoke the solver for $T = 1$, $n = 16$, $J = 16$. Plot, in a single figure, the solution at the final time $t = 0$, $t = 1/16$, $t = 1/8$, $t = 1/4$, $1/2$, and $t = 1$.

(f) (8%) We wish to verify the convergence of the solver. To this end, for the functions f and g given in (e), compute the solution for $(n, J) = (8, 8), (16, 16), (32, 32)$, and $(64, 64)$, and then evaluate (the approximation of) the output $s \equiv u(x = 1, t = 1)$ for each of the four discretizations. Also evaluate the reference output s_{ref} associated with $(n, J) = (512, 512)$. Report, in a table, the error associated with the four different levels of discretizations. Report also the value of s_{ref} to at least six significant digits. Does the observed error behavior match your expectation?

Note. The table should have three columns with headings n, J, and $|s_{\text{ref}} - s|$. Please provide both the table and the value of s_{ref} in the hard copy of the assignment to facilitate the grading process.

Problem 3. Laplace's equation in an annular domain (30%)

Let Ω be an two-dimensional annular domain of the inner radius r_1 and the outer radius of r_2: i.e.,

$$\Omega \equiv \{x \in \mathbb{R}^2 \mid r_1 < |x| < r_2\}.$$

We consider Laplace's equation

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega, \\ u(r = r_1, \theta) &= 0, \quad \theta \in [0, 2\pi), \\ u(r = r_2, \theta) &= g(\theta), \quad \theta \in [0, 2\pi). \end{aligned}$$

In words, the homogeneous Dirichlet boundary condition is imposed on the inner boundary, and a nonhomogeneous Dirichlet boundary condition is imposed on the outer boundary. Answer the following questions:

(a) (14%) Find a family of functions of the form $u_n(r, \theta) = r_n(r)\Theta_n(\theta)$ that satisfies (i) the PDE and (ii) the inner boundary condition (but not necessarily the outer boundary condition). Express the answer in terms of r, θ, r_1, r_2, and some unknown coefficients.

(b) (8%) Let $g(\theta) \equiv \cos(m\theta)$, where m is a positive integer. Find the associated solution u as a function of m.

Note. The integer m is positive; i.e., it does not take on 0.

(c) (8%) Let $r_1 = 1$ and $r_2 = 2$. Using MATLAB (or any other software), plot, in a single figure, $u(r, \theta = 0)$ over $r \in [r_1, r_2]$ for $m = 1$ and 10. Comment (i) if the relative decay in the two solutions away from the outer boundary is consistent with your expectation and (ii) if the maximum principle is satisfied.