ESC384 Assignment 5

Due Monday, 2 December 2019, at 9:10

A hard copy of the assignment is due in class at the specified time. For problems that require coding, please turn in a hard copy of the code and also upload the code to Quercus to facilitate the grading process. Finally, please adhere to the collaboration policy: the final write up must be prepared individually without consulting others. (See the syllabus for details.)

Problem 1. Green's function (25%)

Consider Laplace's equation in the first quadrant $\Omega \equiv \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ with the bottom boundary $\Gamma_B \equiv \mathbb{R}_{>0} \times \{x_2 = 0\}$ and the left boundary $\Gamma_L \equiv \{x_1 = 0\} \times \mathbb{R}_{>0}$:

$$\begin{aligned} -\Delta u &= 0 \quad \text{on } \Omega, \\ u &= g_B \quad \text{on } \Gamma_B, \\ u &= g_L \quad \text{on } \Gamma_L. \end{aligned}$$

Answer the following questions.

(a) (10%) Find the Green's function $G : \Omega \times \Omega \to \mathbb{R}$ such that, for any given $\xi \in \Omega$,

$$\begin{aligned} -\Delta G(x, \xi) &= \delta(x - \xi) \quad \forall x \in \Omega, \\ G(x, \xi) &= 0 \quad \forall x \in \Gamma_B \cup \Gamma_L. \end{aligned}$$

Identify an appropriate number of image points, and express them as $\tilde{\xi}^{(1)} = (\tilde{\xi}_1^{(1)}, \tilde{\xi}_2^{(1)})$, $\tilde{\xi}^{(2)} = (\tilde{\xi}_1^{(2)}, \tilde{\xi}_2^{(2)})$, ... in terms of $\xi = (\xi_1, \xi_2)$. Then express the solution in terms of the fundamental solution Φ of Laplace's equation and the image points.

(b) (10%) Let $g_L = 0$. Find an integral expression for the solution $u(x = (x_1, x_2))$ in terms of g_B.

Hint. Note that the outward directional derivative on Γ_B is $\nu_B \cdot \nabla_\xi G(x, \xi) = -\frac{\partial G(x,\xi)}{\partial \xi_2}$. Also note that $-\frac{\partial \Phi(x-\xi)}{\partial \xi_2} = \frac{\partial}{\partial \xi_2} \left(\frac{1}{2\pi} \log(\sqrt{(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2}) \right)$.

Hint 2. The final solution can be succinctly expressed in terms of Poisson's kernel for the upper half plane.

(c) (5%) Let $g_L = 0$, and

$$g_B(x_1) = \begin{cases} 1, & x_1 \in (0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Find an integral representation of the solution. The solution may contain only an integral over a finite interval.

Problem 2. Wave equation: separation of variables (40%)

Consider the initial-boundary value problem on $\Omega \equiv (0,1)$:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= 0 \quad \text{in } \Omega \times \mathbb{R}_{>0}, \\ \frac{\partial u}{\partial x} &= 0 \quad \text{on } \partial\Omega \times \mathbb{R}_{>0}, \\ u &= g \quad \text{on } \Omega \times \{t = 0\}, \\ \frac{\partial u}{\partial t} &= h \quad \text{on } \Omega \times \{t = 0\}. \end{aligned}$$

Answer the following questions.

(a) (10%) Find a family of separable solutions $u_n(x,t) = \phi_n(x)T_n(t)$, each of which satisfies the PDE and boundary condition, but not necessarily the initial conditions. Each separable solution may contain unknown coefficients.

(b) (5%) Find a series representation of the solution in terms of appropriate (generalized) Fourier coefficients of g and h.

(c) (5%) Let $g(x) = 3\cos(2\pi x)$ and $h(x) = \cos(5\pi x)$. Find the solution $u(x,t)$.

(d) (5%) Let $h = 0$. Find the traveling wave (i.e., d'Alembert's) form of the solution in terms of an appropriate periodic extension of g.

Hint: $\cos(a)\cos(b) = \frac{1}{2}(\cos(a+b) + \cos(a-b))$.

For the following questions, consider the initial condition

$$g(x) = \begin{cases} 0, & x \in [0, 3/8), \\ 8(x - 3/8), & x \in [3/8, 1/2), \\ 8(-x + 5/8), & x \in [1/2, 5/8), \\ 0, & x \in [5/8, 1], \end{cases} \qquad \text{and} \qquad h(x) = 0.$$

(The function g is a "hat" function with a peak height of 1 at $x = 1/2$ and a half-width of $1/8$.)

(e) (7%) Sketch the solution for time $t = 0$, $1/4$, $1/2$, $3/4$, and 1.

(f) (8%) Recall the solution of the wave equation with homogeneous Dirichlet boundary conditions. Sketch the solution for time $t = 0$, $1/4$, $1/2$, $3/4$, and 1. Compare and comment on the way the wave reflects at the boundaries for the homogeneous Neumann and homogeneous Dirichlet boundary conditions.

Note. You do not need to re-derive the solution to the wave equation with homogeneous Dirichlet boundary conditions.

Problem 3. Finite difference method for L-shaped domain (35%)

In this problem, we consider the wave equation on the L-shaped domain

$$\Omega \equiv (0,1)^2 \setminus ([0,1/2] \times [1/2,1]).$$

This is a unit square domain with the top left subdomain missing. (Note in particular the location of the missing subdomain is different from the L-shaped domain in the course notes.) We are interested in the wave equation

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \Delta u &= 0 \quad \text{in } \Omega \times \mathbb{R}_{>0}, \\ u &= 0 \quad \text{on } \partial\Omega \times \mathbb{R}_{>0}, \\ u &= g \quad \text{on } \Omega \times \{t = 0\}, \\ \frac{\partial u}{\partial t} &= 0 \quad \text{on } \Omega \times \{t = 0\}. \end{aligned}$$

The solution to this wave equation can be expressed as a linear combination of standing waves

$$u(x,t) = \sum_{n=1}^{\infty} a_n u_n(x,t),$$

where

$$u_n(x,t) = \phi_n(x) T_n(t).$$

Here T_n is a function that depends only on time, and ϕ_n is the n-th eigenfunction of the eigenproblem on the L-shaped domain:

$$\begin{aligned} -\Delta \phi_n &= \lambda_n^2 \phi_n \quad \text{in } \Omega, \\ \phi_n &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Physically, the n-th eigenfunction and eigenvalue represents the vibration mode and the frequency of the L-shaped "drum" (i.e., membrane). It can be shown all eigenvalues are (strictly) positive. Answer the following question.

(a) (6%) Find the expression for $T_n(t)$ of the separable function $u_n(x,t) = \phi_n(x) T_n(t)$ in terms of λ_n, so that u_n satisfies (i) the PDE, (ii) the boundary condition, and (iii) $\frac{\partial u_n}{\partial t}(x, t = 0) = 0$ (but not necessarily the other initial condition).

Hint. We can use separation of variables; we separate the time and spatial coordinates, but keep the two spatial coordinates unseparated. As usual, substitute $u_n(x,t) = \phi_n(x) T_n(t)$ into the PDE, and appeal to the relationship between the Laplacian and eigenvalues for eigenfunctions.

Unfortunately the eigenproblem on the L-shaped domain is not solvable by analytical means. We hence consider a finite difference approximation. The discretized problem is of the form

$$\hat{A} \tilde{\phi}_n = \tilde{\lambda}_n^2 \tilde{\phi}_n,$$

where $\hat{A}$ is the finite difference matrix associated with the Laplacian, and $\tilde{\phi}_n$ and $\tilde{\lambda}_n$ are the finite difference approximation of the n-th eigenfunction and eigenvalue. Answer the following questions:

(b) (15%) Complete the `ldrum.m` function. See the template for detailed instructions.

Note. Please include a hard copy of the code in the assignment and also upload the code to Quercus to facilitate the grading process.

Note 2. You do *not* have to use the template if you would rather code everything from scratch.

(c) (6%) We wish to verify the code is working correctly. For the square domain case (i.e., `ldomain = false`), the analytical value of the first and fourth eigenvalues are $\lambda_1 = \sqrt{2\pi^2}$ and $\lambda_4 = \sqrt{8\pi^2}$. Compute the finite difference approximation of the eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_4$ for the grid spacing (i.e., the reciprocal of `ns1`) of 1/8, 1/16, and 1/32. Report the error in the eigenvalues in a table, and verify the error converges. What is the observed convergence rate?

Note. The table should have n_s, $|\lambda_1 - \tilde{\lambda}_1|$, and $|\lambda_4 - \tilde{\lambda}_4|$ as the column heading. Please include the table i n the hard copy of the assignment.

(d) (6%) We now solve the eigenproblem on the L-shaped domain (i.e., set `ldomain = true`). Using whatever the grid spacing (i.e., `ns1`) necessary, compute the first four eigenvalues accurately to at least 0.1. Report (i) the grid spacing used and (ii) the four eigenvalues. Also (iii) plot, in four separate figures, the associated eigenfunctions in two-dimensional plot (i.e., set `use_pretty_plot = false`).

Hint. Try different values of grid spacing until the eigenvalues are "sufficiently converged".

(e) (2%) Plot the first three eigenfunction of the L-shaped domain using `use_pretty_plot = true`. Where have you seen one of the figures?