

Class groups in Kummer towers

Jianing Li

University of Science and Technology of China

Joint with Y. Ouyang, Y. Xu and S. Zhang

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The Kummer towers

Let $K_{n,m} = \mathbb{Q}(\sqrt[\ell^n]{p}, \mu_{2\ell^m})$ where ℓ and p are two primes.

$$\begin{array}{ccc}
 K_{0,\infty} := \mathbb{Q}(\mu_{2\ell^\infty}) & \xrightarrow{\mathbb{Z}_\ell} & K_{\infty,\infty} := \bigcup_{n,m} K_{n,m} \\
 \mathbb{Z}_\ell \downarrow & \nearrow \mathbb{Z}_\ell \rtimes \mathbb{Z}_\ell & \downarrow \mathbb{Z}_\ell \\
 K_{0,1} = \mathbb{Q}(\mu_{2\ell}) & \xrightarrow{\text{non-galois}} & K_{\infty,1} := \mathbb{Q}(\sqrt[\ell^\infty]{p}, \mu_{2\ell}) \\
 \downarrow & & \downarrow \\
 \mathbb{Q} & \xrightarrow{\text{non-galois}} & K_{\infty,0} := \mathbb{Q}(\sqrt[\ell^\infty]{p})
 \end{array}$$

Warm up: class numbers in cyclotomic towers

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Theorem (Iwasawa, Ferrono-Washington)

- If $\ell \nmid h_{0,1}$, then $\ell \nmid h_{0,m}$ for any $m \geq 1$.
- If $\ell \mid h_{0,1}$, then there exists $\lambda \in \mathbb{Z}_{>0}$ and $\nu \in \mathbb{Z}_{\geq 0}$ such that $\text{ord}_{\ell}(h_{0,m}) = \lambda m + \nu$ for sufficiently large m .

Example: $\ell = 2$

So $K_{n,m} = \mathbb{Q}(\sqrt[n]{p}, \mu_{2^{m+1}})$.

Let us look at the class numbers of the non-Galois tower

$$K_{n,0} = \mathbb{Q}(\sqrt[n]{p}).$$

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Gauss genus theory

We have $h_{1,0} := h(\mathbb{Q}(\sqrt{p}))$ is odd.

Remark

Let h^+ denote the narrow class numbers, then

$$\begin{cases} 2 \parallel h^+(\mathbb{Q}(\sqrt{p})), & \text{if } p \equiv 3 \pmod{4}, \\ 2 \nmid h^+(\mathbb{Q}(\sqrt{p})), & \text{if } p \equiv 1 \pmod{4}. \end{cases}$$

Results on class groups of $K_{2,0} = \mathbb{Q}(\sqrt[4]{p})$

In 1980s, Parry showed that $A(\mathbb{Q}(\sqrt[4]{p})) := \text{Cl}(\mathbb{Q}(\sqrt[4]{p}))[2^\infty]$ is cyclic.

- (i) (Parry) If $p = 2$ or $p \equiv 3, 5 \pmod{8}$, then $2 \nmid h_{2,0}$.
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- (iii) (Parry) If $p \equiv 7 \pmod{16}$, then $2 \parallel h_{2,0}$.
- (v) (Lemmermeyer 2010) If $p \equiv 9 \pmod{16}$, then $2 \parallel h_{2,0}$.
- (vi) (L. 2019) If $p \equiv 15 \pmod{16}$, then $4 \mid h_{2,0}$.

Remark

We do not have congruence conditions to describe the 2-divisibility when $p \equiv 1, 15 \pmod{16}$.

Our Observation(2018)

Recall $h_{n,0} = h(\mathbb{Q}(\sqrt[n]{p}))$ where p is a prime.

Observation(2018):

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Ingredients of the proof

- Genus theory; Chevalley's ambiguous class number formula and Gras' generalization.

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$$2^{n+1}\sqrt[p]{p} \xrightarrow{\mathbf{N}} -2^n\sqrt[p]{p} \xrightarrow{\mathbf{N}} 2^{n-1}\sqrt[p]{p} \dots$$

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$${}^{2^{n+1}}\sqrt{p} \xrightarrow{\mathbf{N}} - {}^{2^n}\sqrt{p} \xrightarrow{\mathbf{N}} {}^{2^{n-1}}\sqrt{p} \dots$$

- The norm functoriality, i.e. the diagram commutes.

$$\begin{array}{ccc} M^\times & \xrightarrow{\phi_M} & \text{Gal}(M^{ab}/M) \\ \downarrow \mathbf{N}_{M/L} & & \downarrow \text{res} \\ L^\times & \xrightarrow{\phi_L} & \text{Gal}(L^{ab}/k) \end{array}$$

Here M/L are extension of p -adic fields and ϕ_M is the local reciprocity map.

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- (ii) there exists a system $\{a_i \in K_i(\mu_\ell)\}_{0 \leq i < n}$ such that $K_{i+1}(\mu_\ell) = K_i(\mu_\ell, \sqrt[\ell]{a_i})$ and $a_i \equiv \mathbf{N}_{K_{i+1}(\mu_\ell)/K_i(\mu_\ell)}(a_{i+1}) \bmod K_i(\mu_\ell)^{\times \ell}$.

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An infinite tower $K_0 \subset K_1 \subset K_2 \subset \cdots$ is called ℓ -norm-compatible if $K_0 \subset K_1 \subset \cdots \subset K_n$ is ℓ -norm-compatible for each $n \geq 2$.

Definition of Norm-Compatible

$$\begin{array}{c} K_3 \\ \left| \mathbb{Z}/\ell \right. \\ K_2 \\ \left| \mathbb{Z}/\ell \right. \\ K_1 \\ \left| \mathbb{Z}/\ell \right. \\ K_0 \end{array}$$

Definition of Norm-Compatible

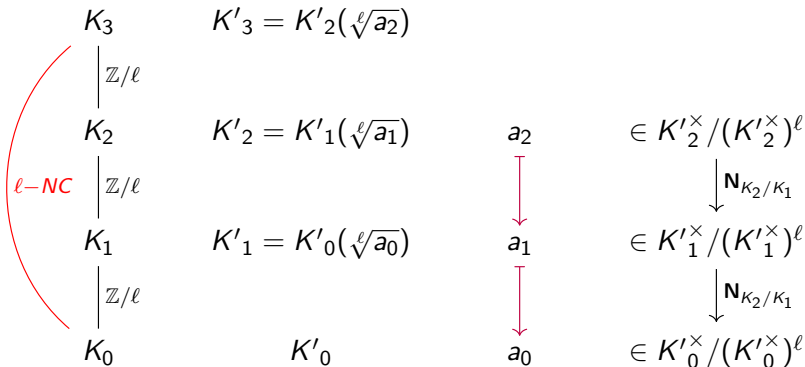
$$\begin{array}{ccc} K_3 & & K'_3 \\ \downarrow \mathbb{Z}/\ell & & \downarrow \mathbb{Z}/\ell \\ K_2 & & K'_2 \\ \downarrow \mathbb{Z}/\ell & & \downarrow \mathbb{Z}/\ell \\ K_1 & & K'_1 \\ \downarrow \mathbb{Z}/\ell & & \downarrow \mathbb{Z}/\ell \\ K_0 & & K'_0 \end{array}$$

Here $K' = K(\mu_\ell)$.

Definition of ℓ -Norm-Compatible Towers

$$\begin{array}{ccc}
 K_3 & K'_3 = K'_2(\sqrt[\ell]{a_2}) & \\
 \downarrow \mathbb{Z}/\ell & & \\
 K_2 & K'_2 = K'_1(\sqrt[\ell]{a_1}) & a_2 \in K'^{\times}_2 / (K'^{\times}_2)^{\ell} \\
 \downarrow \mathbb{Z}/\ell & & \downarrow \mathbf{N}_{K_2/K_1} \\
 K_1 & K'_1 = K'_0(\sqrt[\ell]{a_0}) & a_1 \in K'^{\times}_1 / (K'^{\times}_1)^{\ell} \\
 \downarrow \mathbb{Z}/\ell & & \downarrow \mathbf{N}_{K_2/K_1} \\
 K_0 & K'_0 & a_0 \in K'^{\times}_0 / (K'^{\times}_0)^{\ell}
 \end{array}$$

Definition of ℓ -Norm-Compatible Towers



Here ℓ -NC = ℓ -norm-compatible.

Properties and Examples:

Cyclic ℓ -extension is ℓ -norm compatible

If K_n/K_0 is a $\mathbb{Z}/\ell^n\mathbb{Z}$ -extension, then $K_0 \subset K_1 \subset \cdots \subset K_n$ is ℓ -norm-compatible.

If K_∞/K_0 is a \mathbb{Z}_ℓ -extension, then $K_0 \subset K_1 \subset K_2 \subset \cdots$ is ℓ -norm-compatible.

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Radical extension is ℓ -norm compatible

Let ℓ be odd and $a \in \mathbb{Z}$.

$$\begin{array}{ccc} K_n = \mathbb{Q}(\mu_\ell, \sqrt[n]{a}) = K_{n-1}(\sqrt[n]{a_{n-1}}) & a_{n-1} = \sqrt[n-1]{a} \\ \mid & \\ K_{n-1} = \mathbb{Q}(\mu_\ell, \sqrt[n-1]{a}) = K_{n-2}(\sqrt[n-1]{a_{n-2}}) & a_{n-2} = \sqrt[n-2]{a} \\ \vdots & \vdots \end{array}$$

Our main results

Assume $K_0 \subset \cdots \subset K_n$ is an ℓ -norm-compatible tower satisfying the following hypothesis:

RamHyp: *Every place of K_0 ramified in K_n/K_0 is totally ramified and at least one prime is ramified.*

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Let S_0 be a set of prime ideals of K_0 . Let S_i be the set of prime ideals of K_i above S_0 for each $i \geq 0$.

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Let S_0 be a set of prime ideals of K_0 . Let S_i be the set of prime ideals of K_i above S_0 for each $i \geq 0$. Suppose that every prime ideal of S_1 splits completely in K_n/K_1 or ramifies in K_n/K_1 .

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Theorem (L-Ouyang-Xu-Zhang 2019)

If $A_{K_1} = \langle \text{cl}(S_1) \rangle$, then $A_{K_i} = \langle \text{cl}(S_i) \rangle$ for any $i \geq 1$.

Corollary

Assume $K_0 \subset \cdots \subset K_n$ is ℓ -norm-compatible and satisfies

RamHyp.

(1) If A_{K_1} is generated by the ramified primes of K_1 , then A_{K_m} is generated by the ramified primes of K_m for each m .

Corollary

Assume $K_0 \subset \cdots \subset K_n$ is ℓ -norm-compatible and satisfies

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- (1) If A_{K_1} is generated by the ramified primes of K_1 , then A_{K_m} is generated by the ramified primes of K_m for each m .
- (2) If A_{K_1} is trivial, then A_{K_m} is trivial for each m .

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- (2) If $p \equiv 5 \pmod{8}$, then $h_{n,0}$ and $h_{1,m}$ are odd for $n, m \geq 0$ and $2 \parallel h_{n,m}$ for $n \geq 2$ and $m \geq 1$.

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- (3) If $p \equiv 7 \pmod{16}$, then $A_{n,0} \cong \mathbb{Z}/2\mathbb{Z}$, $A_{n,1} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ for $n \geq 2$, and $A_{1,m} \cong \mathbb{Z}/2^{m-1}\mathbb{Z}$ for $m \geq 1$.

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Remark

In the above cases, $A_{n,m}$ is generated by the prime ideals above 2.

Applications: Odd Regular ℓ

Recall $K_{n,m} = \mathbb{Q}(\sqrt[n]{p}, \mu_{2\ell^m})$.

Recall ℓ is called regular if $\ell \nmid h_{0,1} = h(\mathbb{Q}(\mu_\ell))$.

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Theorem (L-Ouyang-Xu-Zhang 2019)

Let ℓ be an odd regular prime. Assume that p is a prime generating the group $(\mathbb{Z}/\ell^2\mathbb{Z})^\times$ or $p = \ell$. Then $\ell \nmid h_{n,m}$ for any $n, m \geq 0$.

Theorem (L-Ouyang-Xu-Zhang 2019)

Let p be a prime number, $K_{n,m} = \mathbb{Q}(\sqrt[n]{p}, \mu_{3^m})$. (1) If $p = 3$ or $p \equiv 2, 5 \pmod{9}$, then $3 \nmid h_{n,m}$ for $n, m \geq 0$.

Theorem (L-Ouyang-Xu-Zhang 2019)

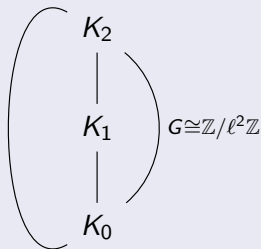
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(2) If $p \equiv 4, 7 \pmod{9}$ and $\left(\frac{3}{p}\right)_3 \neq 1$, then $A_{n,m} \cong \mathbb{Z}/3\mathbb{Z}$ and $A_{n,m}$ is generated by the prime ideals above 3 for $n \geq 1, m \geq 0$.

Another proof for the above results on class groups of $K_{n,m}$

Lemma (Stable lemma)

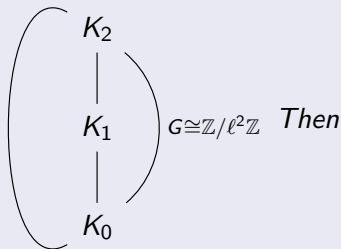
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$|A_{K_0}| = |A_{K_1}|$ implies that

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Remarks on the Stable lemma

Apply to \mathbb{Z}_ℓ -extension

Let K_∞/K be a \mathbb{Z}_ℓ -extension and K_n its n -th layer. It is well known there exists n_0 such that K_∞/K_{n_0} satisfies **RamHyp**. Then we recover Fukuda's result on Stable theorems in \mathbb{Z}_ℓ extensions.

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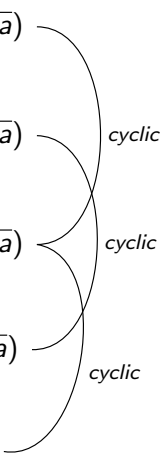
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Apply to radical extension

Suppose $\mu_{\ell^2} \subset K$. Let $K_n = K(\sqrt[\ell^n]{a})$ where $a \in K$. Then $\text{Gal}(K_{m+2}/K_m) \cong \mathbb{Z}/\ell^2\mathbb{Z}$ for any m . We show that there exists some n_0 such that K_∞/K_{n_0} satisfies **RamHyp**. If $|\text{Cl}_{K_m}[\ell^\infty]| = |\text{Cl}_{K_{m+1}}[\ell^\infty]|$ for some $m \geq n_0$, repeatedly applying the Stable lemma, then one can get $|\text{Cl}_{K_{m+i}}[\ell^\infty]| = |\text{Cl}_{K_m}[\ell^\infty]|$ for any $i \geq 0$.

Remarks on the Stable lemma

$$\begin{array}{lcl} K_4 = K_0(\sqrt[4]{a}) & & \\ K_3 = K_0(\sqrt[3]{a}) & & \text{cyclic} \\ K_2 = K_0(\sqrt[2]{a}) & & \text{cyclic} \\ K_1 = K_0(\sqrt[4]{a}) & & \text{cyclic} \\ \mu_{\ell^2} \subset K_0 & & \end{array}$$


Stable results on class groups of $K_{n,m}$

Proposition

Assume $\ell \neq 2$. Assume that all the primes above ℓ in K_{n_0,m_0} are totally ramified in K_{n_0+1,m_0+1} for some integers $n_0 \geq 0$ and $m_0 \geq 1$. Then

- 1 All primes above ℓ in K_{n_0,m_0} are totally ramified in $K_{n,m}/K_{n_0,m_0}$ for all $n \geq n_0$ and $m \geq m_0$;
- 2 If $|A_{n_0,m_0}| = |A_{n_0+1,m_0+1}|$, then $A_{n,m} \cong A_{n_0,m_0}$ for all $n \geq n_0$ and $m \geq m_0$.
- 3 If $\ell \nmid h_{n_0+1,m_0+1}$, then $\ell \nmid h_{n,m}$ for all $n \geq n_0$ and $m \geq m_0$.

The case $K_{n,m} = \mathbb{Q}(\sqrt[n]{p}, \mu_{2\ell^m})$

Stable results in Kummer extension

Suppose we have that the ℓ -class groups of the red fields are same.

$$\begin{array}{ccccc} K_{n_0, m_0+2} & \text{---} & K_{n_0+1, m_0+2} & \text{---} & K_{n_0+2, m_0+2} \\ | & & | & & | \\ K_{n_0, m_0+1} & \text{---} & K_{n_0+1, m_0+1} & \text{---} & K_{n_0+2, m_0+1} \\ | & & | & & | \\ K_{n_0, m_0} & \text{---} & K_{n_0+1, m_0} & \text{---} & K_{n_0+2, m_0} \end{array}$$

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Stable results in Kummer extension

By ramification and class field theory, we have

$$\begin{array}{ccccc} K_{n_0, m_0+2} & \text{---} & K_{n_0+1, m_0+2} & \text{---} & K_{n_0+2, m_0+2} \\ | & & | & & | \\ K_{n_0, m_0+1} & \text{---} & K_{n_0+1, m_0+1} & \text{---} & K_{n_0+2, m_0+1} \\ | & & | & & | \\ K_{n_0, m_0} & \text{---} & K_{n_0+1, m_0} & \text{---} & K_{n_0+2, m_0} \end{array}$$

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Stable results in Kummer extension

Apply our stable lemma to the two left vertical lines,

$$\begin{array}{ccccc} K_{n_0, m_0+2} & \text{---} & K_{n_0+1, m_0+2} & \text{---} & K_{n_0+2, m_0+2} \\ | & & | & & | \\ K_{n_0, m_0+1} & \text{---} & K_{n_0+1, m_0+1} & \text{---} & K_{n_0+2, m_0+1} \\ | & & | & & | \\ K_{n_0, m_0} & \text{---} & K_{n_0+1, m_0} & \text{---} & K_{n_0+2, m_0} \end{array}$$

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C. J. Parry, *A genus theory for quartic fields*. J. Reine Angew. Math. **314**(1980), 40–71.

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Thanks for your attention