Class groups and units of certain number fields

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Introduction of class groups

A number field K is a finite extension of $\mathbb{Q}$, so it is isomorphic to $\mathbb{Q}[x]/(f(x))$ for some irreducible $\mathbb{Q}$-efficient polynomial. Let $\mathcal{O}_K$ be the ring of integers of K. Example:
$K = \mathbb{Q}(i) = \mathbb{Q}[x]/(x^2+1)$, $\mathcal{O}_K = \mathbb{Z}[i]$;
$K = \mathbb{Q}(\sqrt{223})$, $\mathcal{O}_K = \mathbb{Z}[\sqrt{223}]$.
Algebraic number theory began with the study of the unique factorization and its failure in the ring $\mathcal{O}_K$. Consider the exact sequence

$$1 \to \mathcal{O}_K^\times \to K^\times \to I_K \to \mathrm{Cl}_K \to 1.$$

We know that $I_K = \oplus_{\mathfrak{p}} \mathfrak{p}^{\mathbb{Z}}$. If Cl_K is trivial, then every fractional ideal is principal and $K^\times = \mathcal{O}_K^\times \times \prod_{\mathfrak{p}} \pi_{\mathfrak{p}}^{\mathbb{Z}}$ where $\mathfrak{p} = (\pi_{\mathfrak{p}})\mathcal{O}_K$. Example:
$\mathbb{Q}^\times = \{\pm 1\} \times \oplus_p p^{\mathbb{Z}}$; $-10/3 = -1 \times 2^1 \times 3^{-1} \times 5^1$.

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Fundamental theorems

1. The class group $\mathrm{Cl}(K)$ is finite.
2. Let H be the maximal abelian unramified extension of K. Then $\mathrm{Cl}(K) \cong \mathrm{Gal}(H/K)$.
3. The unit group $\mathcal{O}_K^\times$ is isomorphic to $\mathbb{Z}^{r_1+r_2-1} \times \{$ finite cyclic group$\}$. Here r_1 is the number of real roots of $f(x)$ and r_2 is the number of pairs of complex roots. (Suppose $K = \mathbb{Q}[x]/(f(x))$.)

A problem of Alibaba Mathematics Competition (2018)

Prove that the equation

$$x^3 + 3y^3 + 9z^3 - 9xyz = 1$$

has infinitely many integer solutions (x, y, z). (Hint: decompose the left-hand side as a complex polynomial.)

In fact,

$$LHS = N_{K/\mathbb{Q}}(x + y\sqrt[3]{3} + z(\sqrt[3]{3})^2).$$

Here $K = \mathbb{Q}(\sqrt[3]{2}) = \mathbb{Q}[x]/(x^3 - 2)$. We have $\mathcal{O}_K = \mathbb{Z}[\sqrt[3]{3}]$ and $\mathcal{O}_K^\times \cong \mathbb{Z} \times \{\pm 1\}$ by Dirichlet's unit theorem.

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Examples

1. The equation $x^2 - 223y^2 = -3$ has solution modulo every prime p. In fact it has solution in $\mathbb{Q}_p$ for every prime p and it has $\mathbb{Q}$-solution, for example $x = 14/3, y = 1/3$. However it has no solution in integers. The reason for that such primes exist is that $\mathbb{Q}(\sqrt{223})$ has class number 3.
2. The equation $x^2 + 14y^2 = 71$ has $\mathbb{Q}$-solution (for example: $x = 25/3, y = 1/3$) but has no $\mathbb{Z}$-solution. The reason for that such primes exist is $\mathrm{Cl}(\mathbb{Q}(\sqrt{-14})) \cong \mathbb{Z}/4\mathbb{Z}$.
3. (Kummer's theorem) If a prime p does not divide the class number of $\mathbb{Q}(\mu_p)$, then FLT holds for this p.

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Big open problems

1. Are there infinitely many real quadratic fields with class number 1?
2. Are there infinitely many number fields with class number 1?
3. Let $p \geq 3$ be a prime. Let $\mathbb{Q}_n$ be the subfield with $\mathbb{Q}(\mu_{p^{n+1}})$ such that $[\mathbb{Q}_n : \mathbb{Q}] = p^n$. (For $p = 2$, $\mathbb{Q}_n = \mathbb{Q}(\zeta_{2^{n+1}} + \zeta_{2^{n+1}}^{-1})$.) Coates conjecture that the class number of $\mathbb{Q}_n$ is 1 for any n and any prime p. This is the so-called Weber class number problem.

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Some old results on certain quadratic fields

The class number of $\mathbb{Q}(\sqrt{\ell})$ is odd for any prime ℓ. Let $K = \mathbb{Q}(\sqrt{-\ell})$ where ℓ is a prime. Let $\mathrm{Cl}_2(K)$ be the 2-Sylow subgroup of $\mathrm{Cl}(K)$. Then

1. We have $2 \nmid h_K$ iff $\ell \equiv 3 \bmod 4$.
2. If $\ell \equiv 1 \bmod 4$, then $\mathrm{Cl}_2(K)$ is a cyclic 2-group and
3. If $\ell \equiv 5 \bmod 8$, then $\mathrm{Cl}_2(K) \cong \mathbb{Z}/2\mathbb{Z}$.
4. For $\ell \equiv 1 \bmod 8$, $4 \mid h_K$. And $8 \mid h_K$ iff ℓ splits completely in $\mathbb{Q}(\zeta_8, \sqrt{1+i})$.
5. Conjecturally, the density of ℓ such that $2^k \mid h_K$ among all the primes is $1/2^k$ for any $k \geq 1$.
6. Very recently, Koyman(2020) proves the density of ℓ such that $16 \mid h_K$ is $1/16$ among all the primes.

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Previous results on certain pure quartic fields

Let $K = \mathbb{Q}(\sqrt[4]{\ell})$ where ℓ is an odd prime. Parry in 1980's proved that $\mathrm{Cl}_2(K)$ is cyclic and

- $2 \nmid h_K$ if and only if $\ell \equiv 3, 5 \bmod 8$;
- $2 \parallel h_K$ if $\ell \equiv 7 \bmod 16$;
- (Lemmermeyer 2010) $2 \parallel h_K$ if $\ell \equiv 9 \bmod 16$;
- $2 \parallel h_K$ if $\ell \equiv 1 \bmod 16$ and $\left(\frac{2}{p}\right)_4 = -1$;
- (L. 2019) $4 \mid h_K$ if $\ell \equiv 15 \bmod 16$.

There is no congruence condition to characterize when $8 \mid h_K$ in the case of $\ell \equiv 15 \bmod 16$.

New results on Kummer towers

Let $K_{n,m} = \mathbb{Q}(\sqrt[2^n]{\ell}, \zeta_{2^{m+1}})$ where ℓ is an odd prime as before. Let $h_{n,m}$ be the class number of $K_{n,m}$. Let $A_{n,m}$ be the Sylow-2 subgroup of $\mathrm{Cl}_{n,m}$. Note that $\mathrm{Gal}(K_{\infty,\infty}/\mathbb{Q}) \cong \mathbb{Z}_2 \rtimes \mathbb{Z}_2^\times$.

Theorem (L.–Ouyang–Xu–Zhang)

We have
(1) *If* $\ell \equiv 3 \bmod 8$, *then* $2 \nmid h_{n,m}$ *for* $n, m \geq 0$.
(2) *If* $\ell \equiv 5 \bmod 8$, *then* $h_{n,0}$ *and* $h_{1,m}$ *are odd for* $n, m \geq 0$ *and* $2 \parallel h_{n,m}$ *for* $n \geq 2$ *and* $m \geq 1$.
(3) *If* $\ell \equiv 7 \bmod 16$, $A_{n,0} \cong \mathbb{Z}/2\mathbb{Z}$ *and* $A_{n,1} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ *for any* $n \geq 2$.

We also proved some results in p-adic Kummer towers for odd p.

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Fundamental units of real quadratic fields

Let $\ell \equiv 3 \bmod 4$. Let $\epsilon = a + b\sqrt{\ell}$ be the fundamental unit of $\mathbb{Q}(\sqrt{\ell})$. Then Yue-Zhang proved the following:

- $2 \mid a$. Moreover,
- $2 \parallel a$ iff $\ell \equiv 3 \bmod 8$;
- $8 \parallel a$ iff $\ell \equiv 7 \bmod 16$;
- $16 \mid a$ if $\ell \equiv 15 \bmod 16$.
- There is no congruence condition to characterize when $16 \parallel a$. But numerical data indicates a density heuristic.

An analogous result on units in $\mathbb{Q}(\sqrt[4]{\ell})$

Theorem

(L-Ouyang-Xu-Zhang) Let $\ell \equiv 7 \bmod 16$ *be a prime. Let* ϵ *be the fundamental unit of* $\mathbb{Q}(\sqrt{\ell})$.

(1) *There exists a totally positive unit* η *of* $\mathbb{Q}(\sqrt[4]{\ell})$ *such that* $N(\eta) = \epsilon$ *and the group of units* $\mathcal{O}^{\times}_{\mathbb{Q}(\sqrt[4]{\ell})} = \langle \eta, \epsilon, -1 \rangle$.

(2) *We have* $v_{\mathfrak{q}}(\mathrm{Tr}_{\mathbb{Q}(\sqrt[4]{\ell})/\mathbb{Q}(\sqrt{\ell})}(\eta)) = 3$ *and* $\eta \equiv -1 \bmod \sqrt[4]{\ell}$, *where* $\mathfrak{q}$ *is the unique prime of* $\mathbb{Q}(\sqrt{\ell})$ *above* 2.

A question of Coates-Li

In a recent paper of Coates-Li where they study the arithmetic of elliptic curves, they conjecture the following statement and I prove it last month.

Theorem (L.)

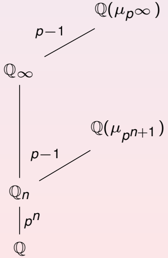
Let $\ell \equiv 15 \bmod 16$ be a prime. Let η be the fundamental unit of $K = \mathbb{Q}(\sqrt[4]{-\ell})$. Let $\mathfrak{P}$ be the unique ramified prime ideal of K above 2. Then $\nu_{\mathfrak{P}}(\mathrm{Tr}(\eta)) \geq 6$. Here $\mathrm{Tr} : K \to \mathbb{Q}(\sqrt{-\ell})$ is the trace map.

The $\mathcal{T}_p$-groups

Let $\hat{K}$ be the maximal abelian p-extension of K which is unramified outside p. Then

$$\mathrm{Gal}(\hat{K}/K) \cong \mathbb{Z}_p^{r_2+1+\delta} \times \mathcal{T}_p(K)$$

with $\mathcal{T}_p(K)$ finite. Leopoldt's conjecture states that $\delta = 0$.
Example: $K = \mathbb{Q}$, $\mathcal{T}_p(\mathbb{Q}) = 0$ for any p. For $p > 2$, we have the diagram



Formulas for $\mathcal{T}_p$

1. Let $K \neq \mathbb{Q}$ be a totally real number field. Assume that Leopoldt Conjecture holds. Then Gras proved

$$|\mathcal{T}_p(K)| = (p\text{-adic unit}) \cdot \frac{p \cdot [K \cap \mathbb{Q}^\infty : \mathbb{Q}] \cdot h(K) \cdot R_p(K)}{\sqrt{D_K} \cdot \prod_{\mathfrak{p}|p} N\mathfrak{p}}.$$

   Here $R_p(K)$ is the p-adic regulator.

2. If K is real quadratic, then $R_p(K) = \log_p(\epsilon)$.
3. Let $K = \mathbb{Q}(\sqrt{\ell})$ be real quadratic with $\ell \equiv \pm 1 \bmod 8$. Let $\epsilon = a + b\sqrt{\ell}$ be the fundamental unit. Then $\mathcal{T}_2(K)$ is cyclic and $\nu_2(|\mathcal{T}_2(K)|) = \nu_2(a) - 1$.

Cohen-Lenstra Heuristics

Definition

Let $\mathcal{M}_{\mathbb{Z}_p}$ be the set of isomorphic classes of finite abelian p-groups. For $u \in \mathbb{Z}_{\geq 0}$, define the weight function $\omega_u : \mathcal{M}_{\mathbb{Z}_p} \longrightarrow \mathbb{R}_{\geq 0}$ by

$$\omega_u(G) := \frac{1}{|G|^u \cdot |\mathrm{Aut}(G)|}.$$

The following is a paragraph of the article of Cohen-Lenstra.

The main idea, due to the second author, is that the scarcity of noncyclic groups can be attributed to the fact that they have many automorphisms. This naturally leads to the heuristic assumption that isomorphism classes G of abelian groups should be weighted with a weight proportional to $1/\#$ Aut G . This is a very natural and common weighting factor, and it is the purpose of this paper to show that the assumption above, plus another one to take into account the units, is sufficient to

The Setting of Local Cohen-Lenstra Heuristic

Definition

Fix a weight ω and subset V of $\mathcal{M}_{\mathbb{Z}_p}$. A function $f : V \to \mathbb{C}$ is called L^1 if $\lim_{N\to\infty} \sum_{\substack{G\in V,\\ |G|\le N}} |f(G)|\omega(G) < \infty$. For $f \in L^1$, the average is defined as

$$M(f, V, \omega) := \lim_{N\to\infty} \frac{\sum_{G\in V, |G|\le N} f(G)\omega(G)}{\sum_{G\in V, |G|\le N} \omega(G)}.$$

Example

Let $H \in V$. Let

$$1_H(G) = \begin{cases} 1 & \text{if } G \cong H, \\ 0 & \text{if } G \ncong H. \end{cases} \quad \text{Then } M(1_H, V, \omega) = \frac{\text{weight of } H}{\text{weight of the space } V}$$

The Setting of Local Cohen-Lenstra Heuristic

Definition

For a sequence $\{M_i\}_{i\geq 1}$ in $\mathcal{M}_{\mathbb{Z}_p}$, let $V = \bigcup\{M_i\} \subset \mathcal{M}_A$. The sequence is equidistributed in (V, ω) if

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} f(M_i)}{n} = M(f, V, \omega)$$

for every $f \in L^1(V, \omega)$.

Remark

Take $f = 1_H$. Then the LHS is the natural density of the subsequence $\{M_j\}$ with each $M_j \cong H$.

The Local Cohen-Lenstra Heuristic for Class groups of quadratic fields

Let $\mathcal{F}_{im}$ (resp. $\mathcal{F}_{re}$) be the family of all imaginary (resp. real) quadratic fields ordered by the absolute value of discriminants.
Let $\mathrm{Cl}_p(K)$ (resp. $\mathrm{Cl}_p^+(K)$) denote the (resp. narrow) class group of K.

Conjecture (Cohen-Lenstra Heuristics)

Let p be a prime.

a) **Imaginary quadratic case**: *The codomain of the sequence of $\mathbb{Z}_p$-modules $\{2\mathrm{Cl}_p(K)\}_{K\in\mathcal{F}_{im}}$ is $\mathcal{M}_{\mathbb{Z}_p}$ and this sequence is equidistributed in $(\mathcal{M}_{\mathbb{Z}_p}, \omega_0)$.*

b) **Real quadratic case**: *The codomain of the sequence of $\mathbb{Z}_p$-modules $\{2\mathrm{Cl}_p^+(K)\}_{K\in\mathcal{F}_{re}}$ is $\mathcal{M}_{\mathbb{Z}_p}$ and this sequence is equidistributed in $(\mathcal{M}_{\mathbb{Z}_p}, \omega_1)$.*

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Recent Breakthrough of Alex Smith

Theorem (Smith 2017)

For each finite abelian 2-group G, one has

$$\lim_{X\to\infty} \frac{\#\{K \in \mathcal{F}_{im}, |D_K| \le X,\ 2\mathrm{Cl}_2(K) \cong G\}}{\#\{K \in \mathcal{F}_{im}, |D_K| \le X\}} = M(1_G, \mathcal{M}_{\mathbb{Z}_2}, \omega_0).$$

Cohen-Lenstra on $\mathcal{T}_p$ in the full family

We propose the following conjecture for $\mathcal{T}_p$-groups of quadratic fields.

Conjecture (Cohen-Lenstra Heuristic on $\mathcal{T}_p$)

Let $p \geq 5$ be a prime.

a) **Imaginary quadratic case**: *The codomain of $\{\mathcal{T}_p(K)\}_{K\in\mathcal{F}_{im}}$ is $\mathcal{M}_{\mathbb{Z}_p}$ and this sequence is equidistributed in $(\mathcal{M}_{\mathbb{Z}_p}, \omega_1)$.*

b) **Real quadratic case** : *The codomain of $\{\mathcal{T}_p(K)\}_{K\in\mathcal{F}_{re}}$ is $\mathcal{M}_{\mathbb{Z}_p}$ and this sequence is equidistributed in $(\mathcal{M}_{\mathbb{Z}_p}, \omega_0)$.*

Remarks

1. The weight functions are reversed comparing to the class group case.
2. For $p = 2$ and 3, it seems that one should consider the group $p\mathcal{T}_p(K)$ and modify the weight function.
3. The numerical data supports the above conjecture.

Two equivalent conjectures

Conjecture

For any $k \geq 0$

$$\lim_{X\to\infty} \frac{\#\{l \leq X : l \equiv 1 \bmod 8, \mathcal{T}_2(l) \cong \mathbb{Z}/2^{k+1}\mathbb{Z}\}}{\#\{l \leq X : l \equiv 1 \bmod 8\}} = \frac{1}{2^{k+1}};$$

$$\lim_{X\to\infty} \frac{\#\{l \leq X : l \equiv -1 \bmod 8, \mathcal{T}_2(l) \cong \mathbb{Z}/2^{k+2}\mathbb{Z}\}}{\#\{l \leq X : l \equiv -1 \bmod 8\}} = \frac{1}{2^{k+1}}.$$

Conjecture

Let $a_l + b_l\sqrt{l}$ be the fundamental unit of $\mathbb{Q}(\sqrt{l})$. For each $k \geq 0$,

$$\lim_{X\to\infty} \frac{\#\{l \text{ prime}:\ l \leq X,\ l \equiv 1 \bmod 8,\ v_2(a_l) = k+2\}}{\#\{l \text{ prime}:\ l \leq X,\ l \equiv 1 \bmod 8\}} = \frac{1}{2^{k+1}};$$

$$\lim_{X\to\infty} \frac{\#\{l \text{ prime}:\ l \leq X,\ l \equiv -1 \bmod 8,\ v_2(a_l) = k+3\}}{\#\{l \text{ prime}:\ l \leq X,\ l \equiv -1 \bmod 8\}} = \frac{1}{2^{k+1}}.$$

Shanks-Sime-Washington's Conjecture

Let ℓ be a prime, χ_ℓ be the quadratic character associated to $\mathbb{Q}(\sqrt{\ell})$, and $L_2(s,\chi_\ell)$ be the associated 2-adic L-function.

Conjecture (Shanks-Sime-Washington)

For each integer $k \geq 0$,

$$\lim_{X\to\infty} \frac{\#\{\ell \leq X : \ell \equiv 9 \bmod 16 \textit{ and } v_2(L_2(0,\chi_\ell)) = k+2\}}{\#\{\ell \leq X : \textit{ and } \ell \equiv 9 \bmod 16\}} = \frac{1}{2^{k+1}}.$$

$$\lim_{X\to\infty} \frac{\#\{\ell \leq X : \ell \equiv 9 \bmod 16 \textit{ and } v_2(L_2(1,\chi_\ell)) = k+2\}}{\#\{\ell \leq X : \textit{ and } \ell \equiv 9 \bmod 16\}} = \frac{1}{2^{k+1}}.$$

The above conjecture is a special case of ours due to the p-adic class number formula.

Questions and remarks.

Thank you all.