Class groups in Kummer towers

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The Kummer towers

Let $K_{n,m} = \mathbb{Q}(\sqrt[\ell^n]{p}, \mu_{2\ell^m})$ where ℓ and p are two primes.

$$\begin{tikzcd}
K_{0,\infty} := \mathbb{Q}(\mu_{2\ell^\infty}) \arrow[r, dash, "\mathbb{Z}_\ell"] & K_{\infty,\infty} := \cup_{n,m} K_{n,m} \\
K_{0,1} = \mathbb{Q}(\mu_{2\ell}) \arrow[u, dash, "\mathbb{Z}_\ell"] \arrow[r, dash, "non-galois"'] \arrow[ur, dash, "\mathbb{Z}_\ell \rtimes \mathbb{Z}_\ell"', red] & K_{\infty,1} := \mathbb{Q}(\sqrt[\ell^\infty]{p}, \mu_{2\ell}) \arrow[u, dash, "\mathbb{Z}_\ell"'] \\
\mathbb{Q} \arrow[u, dash] \arrow[r, dash, "non-galois"'] & K_{\infty,0} := \mathbb{Q}(\sqrt[\ell^\infty]{p}) \arrow[u, dash]
\end{tikzcd}$$

Warm up: class numbers in cyclotomic towers

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Theorem (Iwasawa, Ferrono-Washington)

- *If* $\ell \nmid h_{0,1}$*, then* $\ell \nmid h_{0,m}$ *for any* $m \geq 1$*.*
- *If* $\ell \mid h_{0,1}$*, then there exists* $\lambda \in \mathbb{Z}_{>0}$ *and* $\nu \in \mathbb{Z}_{\geq 0}$ *such that* $\mathrm{ord}_\ell(h_{0,m}) = \lambda m + \nu$ *for sufficiently large* m*.*

Example: $\ell = 2$

So $K_{n,m} = \mathbb{Q}(\sqrt[2^n]{p}, \mu_{2^{m+1}})$.
Let us look at the class numbers of the non-Galois tower $K_{n,0} = \mathbb{Q}(\sqrt[2^n]{p})$.

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Gauss genus theory

We have $h_{1,0} := h(\mathbb{Q}(\sqrt{p}))$ is odd.

Remark

Let h^+ denote the narrow class numbers, then

$$\begin{cases} 2 \mid\mid h^+(\mathbb{Q}(\sqrt{p})), & \text{if } p \equiv 3 \bmod 4, \\ 2 \nmid h^+(\mathbb{Q}(\sqrt{p})), & \text{if } p \equiv 1 \bmod 4. \end{cases}$$

In 1980s, Parry showed that $A(\mathbb{Q}(\sqrt[4]{p})) := \mathrm{Cl}(\mathbb{Q}(\sqrt[4]{p}))[2^\infty]$ is cyclic.

(i) (Parry)If $p = 2$ or $p \equiv 3, 5 \bmod 8$, then $2 \nmid h_{2,0}$.

(ii) (Parry) If $p \equiv 1, 7 \bmod 8$, then $2 \mid h_{2,0}$.

Results on class groups of $K_{2,0} = \mathbb{Q}(\sqrt[4]{p})$

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(iii) (Parry)If $p \equiv 7 \bmod 16$, then $2 \parallel h_{2,0}$.

(v) (Lemmermeyer 2010) If $p \equiv 9 \bmod 16$, then $2 \parallel h_{2,0}$.

(vi) (L. 2019) If $p \equiv 15 \bmod 16$, then $4 \mid h_{2,0}$.

Remark

We do not have congruence conditions to describe the 2-divisibility when $p \equiv 1, 15 \bmod 16$.

Our Observation(2018)

Recall $h_{n,0} = h(\mathbb{Q}(\sqrt[2^n]{p}))$ where p is a prime.

Observation(2018):

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- $$\sqrt[2^{n+1}]{p} \xrightarrow{\mathbf{N}} -\sqrt[2^n]{p} \xrightarrow{\mathbf{N}} \sqrt[2^{n-1}]{p} \cdots$$
- The norm functoriality, i.e. the diagram commutes.

$$\begin{tikzcd}
M^\times \arrow[r, "\phi_M"] \arrow[d, "\mathbf{N}_{M/L}"] & \mathrm{Gal}(M^{ab}/M) \arrow[d, "res"] \\
L^\times \arrow[r, "\phi_L"] & \mathrm{Gal}(L^{ab}/k)
\end{tikzcd}$$

Here M/L are extension of p-adic fields and ϕ_M is the local reciprocity map.

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(i) $\mathrm{Gal}(K_{i+1}/K_i) \cong \mathbb{Z}/\ell\mathbb{Z}$ for $0 \leq i < n$;

(ii) there exists a system $\{a_i \in K_i(\mu_\ell)\}_{0 \leq i < n}$ such that $K_{i+1}(\mu_\ell) = K_i(\mu_\ell, \sqrt[\ell]{a_i})$ and $a_i \equiv \mathbf{N}_{K_{i+1}(\mu_\ell)/K_i(\mu_\ell)}(a_{i+1}) \bmod K_i(\mu_\ell)^{\times \ell}$.

ℓ-Norm-Compatible Tower

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An infinite tower $K_0 \subset K_1 \subset K_2 \subset \cdots$ is called *ℓ-norm-compatible* if $K_0 \subset K_1 \subset \cdots \subset K_n$ is ℓ-norm-compatible for each $n \geq 2$.

Definition of Norm-Compatible

$$\begin{array}{c} K_3 \\ \Big| \, \mathbb{Z}/\ell \\ K_2 \\ \Big| \, \mathbb{Z}/\ell \\ K_1 \\ \Big| \, \mathbb{Z}/\ell \\ K_0 \end{array}$$

Definition of Norm-Compatible

$$\begin{array}{ccc}
K_3 & \quad & {K'}_3 \\
\Big|\,\mathbb{Z}/\ell & & \Big|\,\mathbb{Z}/\ell \\
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\Big|\,\mathbb{Z}/\ell & & \Big|\,\mathbb{Z}/\ell \\
K_0 & & {K'}_0
\end{array}$$

Here $K' = K(\mu_\ell)$.

Definition of ℓ-Norm-Compatible Towers

$$
\begin{array}{ccc}
K_3 & K'_3 = K'_2(\sqrt[\ell]{a_2}) & \\
\Big| \mathbb{Z}/\ell & & \\
K_2 & K'_2 = K'_1(\sqrt[\ell]{a_1}) & a_2 \in K'^{\times}_2/(K'^{\times}_2)^{\ell} \\
\Big| \mathbb{Z}/\ell & & \big\downarrow \mathbf{N}_{K_2/K_1} \\
K_1 & K'_1 = K'_0(\sqrt[\ell]{a_0}) & a_1 \in K'^{\times}_1/(K'^{\times}_1)^{\ell} \\
\Big| \mathbb{Z}/\ell & & \big\downarrow \mathbf{N}_{K_2/K_1} \\
K_0 & K'_0 & a_0 \in K'^{\times}_0/(K'^{\times}_0)^{\ell}
\end{array}
$$

Definition of ℓ-Norm-Compatible Towers

$$
\begin{tikzcd}
K_3 \arrow[d, dash, "\mathbb{Z}/\ell"] \arrow[ddd, dash, bend right, "\ell-NC"'] & K'_3 = K'_2(\sqrt[\ell]{a_2}) & & \\
K_2 \arrow[d, dash, "\mathbb{Z}/\ell"] & K'_2 = K'_1(\sqrt[\ell]{a_1}) & a_2 \arrow[d, mapsto] & \in K'^{\times}_2/(K'^{\times}_2)^{\ell} \arrow[d, "\mathbf{N}_{K_2/K_1}"] \\
K_1 \arrow[d, dash, "\mathbb{Z}/\ell"] & K'_1 = K'_0(\sqrt[\ell]{a_0}) & a_1 \arrow[d, mapsto] & \in K'^{\times}_1/(K'^{\times}_1)^{\ell} \arrow[d, "\mathbf{N}_{K_2/K_1}"] \\
K_0 & K'_0 & a_0 & \in K'^{\times}_0/(K'^{\times}_0)^{\ell}
\end{tikzcd}
$$

Here ℓ-NC$=$ ℓ-norm-compatible.

Properties and Examples:

Cyclic ℓ-extension is ℓ-norm compatible

If K_n/K_0 is a $\mathbb{Z}/\ell^n\mathbb{Z}$-extension, then $K_0 \subset K_1 \subset \cdots \subset K_n$ is ℓ-norm-compatible.
If K_∞/K_0 is a $\mathbb{Z}_\ell$-extension, then $K_0 \subset K_1 \subset K_2 \subset \cdots$ is ℓ-norm-compatible.

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Radical extension is ℓ-norm compatible

Let ℓ be odd and $a \in \mathbb{Z}$.

$$\begin{array}{ccc} K_n = \mathbb{Q}(\mu_\ell, \sqrt[\ell^n]{a}) = K_{n-1}(\sqrt[\ell]{a_{n-1}}) & \qquad & a_{n-1} = \sqrt[\ell^{n-1}]{a} \\ \Big| & & \\ K_{n-1} = \mathbb{Q}(\mu_\ell, \sqrt[\ell^{n-1}]{a}) = K_{n-2}(\sqrt[\ell]{a_{n-2}}) & & a_{n-2} = \sqrt[\ell^{n-2}]{a} \\ \cdots & & \cdots \end{array}$$

Our main results

Assume $K_0 \subset \cdots \subset K_n$ is an ℓ-norm-compatible tower satisfying the following hypothesis:

> **RamHyp**: *Every place of K_0 ramified in K_n/K_0 is totally ramified and at least one prime is ramified.*

We are interested in the A_{K_i}, ℓ-part of the class group of K_i.

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Let S_0 be a set of prime ideals of K_0. Let S_i be the set of prime ideals of K_i above S_0 for each $i \geq 0$.

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Let S_0 be a set of prime ideals of K_0. Let S_i be the set of prime ideals of K_i above S_0 for each $i \geq 0$. Suppose that every prime ideal of S_1 splits completely in K_n/K_1 or ramifies in K_n/K_1.

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Theorem (L-Ouyang-Xu-Zhang 2019)

If $A_{K_1} = \langle \mathrm{cl}(S_1) \rangle$, then $A_{K_i} = \langle \mathrm{cl}(S_i) \rangle$ for any $i \geq 1$.

Corollary

Assume $K_0 \subset \cdots \subset K_n$ is ℓ-norm-compatible and satisfies **RamHyp.**

(1) If A_{K_1} is generated by the ramified primes of K_1, then A_{K_m} is generated by the ramified primes of K_m for each m.

Corollary

Assume $K_0 \subset \cdots \subset K_n$ is ℓ-norm-compatible and satisfies **RamHyp.**
(1) If A_{K_1} is generated by the ramified primes of K_1, then A_{K_m} is generated by the ramified primes of K_m for each m.
(2) If A_{K_1} is trivial, then A_{K_m} is trivial for each m.

Applications: $\ell = 2$

Theorem (L-Ouyang-Xu-Zhang 2019)

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(2) If $p \equiv 5 \bmod 8$, then $h_{n,0}$ and $h_{1,m}$ are odd for $n, m \geq 0$ and $2 \parallel h_{n,m}$ for $n \geq 2$ and $m \geq 1$.

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(2) If $p \equiv 5 \bmod 8$, then $h_{n,0}$ and $h_{1,m}$ are odd for $n, m \geq 0$ and $2 \parallel h_{n,m}$ for $n \geq 2$ and $m \geq 1$.
(3) If $p \equiv 7 \bmod 16$, then $A_{n,0} \cong \mathbb{Z}/2\mathbb{Z}$, $A_{n,1} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ for $n \geq 2$, and $A_{1,m} \cong \mathbb{Z}/2^{m-1}\mathbb{Z}$ for $m \geq 1$.

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Remark

In the above cases, $A_{n,m}$ is generated by the prime ideals above 2.

Recall $K_{n,m} = \mathbb{Q}(\sqrt[\ell^n]{p}, \mu_{2\ell^m})$.
Recall ℓ is called regular if $\ell \nmid h_{0,1} = h(\mathbb{Q}(\mu_\ell))$.

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Recall ℓ is called regular if $\ell \nmid h_{0,1} = h(\mathbb{Q}(\mu_\ell))$.

Theorem (L-Ouyang-Xu-Zhang 2019)

Let ℓ be an odd regular prime. Assume that p is a prime generating the group $(\mathbb{Z}/\ell^2\mathbb{Z})^\times$ or $p = \ell$. Then $\ell \nmid h_{n,m}$ for any $n, m \geq 0$.

Applications: $\ell = 3$

Theorem (L-Ouyang-Xu-Zhang 2019)

Let p be a prime number, $K_{n,m} = \mathbb{Q}(\sqrt[3^n]{p}, \mu_{3^m})$. (1) If $p = 3$ or $p \equiv 2, 5 \bmod 9$, then $3 \nmid h_{n,m}$ for $n, m \geq 0$.

Theorem (L-Ouyang-Xu-Zhang 2019)

Let p be a prime number, $K_{n,m} = \mathbb{Q}(\sqrt[3^n]{p}, \mu_{3^m})$. (1) If $p = 3$ or $p \equiv 2, 5 \bmod 9$, then $3 \nmid h_{n,m}$ for $n, m \geq 0$.

(2) If $p \equiv 4, 7 \bmod 9$ and $\left(\frac{3}{p}\right)_3 \neq 1$, then $A_{n,m} \cong \mathbb{Z}/3\mathbb{Z}$ and $A_{n,m}$ is generated by the prime ideals above 3 for $n \geq 1$, $m \geq 0$.

Lemma (Stable lemma)

Suppose K_2/K_0 satisfies **RamHyp**.

$$\begin{tikzcd} K_2 \arrow[d, dash] \arrow[dd, dash, bend left, "G\cong\mathbb{Z}/\ell^2\mathbb{Z}"] \arrow[dd, dash, bend right=60] \\ K_1 \arrow[d, dash] \\ K_0 \end{tikzcd}$$

Lemma (Stable lemma)

Suppose K_2/K_0 *satisfies* **RamHyp**.

$$\begin{tikzcd} K_2 \arrow[d, dash] \\ K_1 \arrow[d, dash] \\ K_0 \arrow[u, bend right=60, dash] \arrow[uu, bend left=60, dash, "G\cong\mathbb{Z}/\ell^2\mathbb{Z}"'] \end{tikzcd}$$

Then $|A_{K_0}| = |A_{K_1}|$ *implies that*

$$|A_{K_0}| = |A_{K_1}| = |A_{K_2}|.$$

Remarks on the Stable lemma

Apply to $\mathbb{Z}_\ell$-extension

Let K_∞/K be a $\mathbb{Z}_\ell$-extension and K_n its n-th layer. It is well known there exists n_0 such that K_∞/K_{n_0} satisfies **RamHyp**. Then we recover Fukuda's result on Stable theorems in $\mathbb{Z}_\ell$ extensions.

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Let K_∞/K be a $\mathbb{Z}_\ell$-extension and K_n its n-th layer. It is well known there exists n_0 such that K_∞/K_{n_0} satisfies **RamHyp**. Then we recover Fukuda's result on Stable theorems in $\mathbb{Z}_\ell$ extensions.

Apply to radical extension

Suppose $\mu_{\ell^2} \subset K$. Let $K_n = K(\sqrt[\ell^n]{a})$ where $a \in K$. Then $\mathrm{Gal}(K_{m+2}/K_m) \cong \mathbb{Z}/\ell^2\mathbb{Z}$ for any m. We show that there exists some n_0 such that K_∞/K_{n_0} satisfies **RamHyp**. If $|\mathrm{Cl}_{K_m}[\ell^\infty]| = |\mathrm{Cl}_{K_{m+1}}[\ell^\infty]|$ for some $m \geq n_0$, repeatedly applying the Stable lemma, then one can get $|\mathrm{Cl}_{K_{m+i}}[\ell^\infty]| = |\mathrm{Cl}_{K_m}[\ell^\infty]|$ for any $i \geq 0$.

Remarks on the Stable lemma

$$\begin{tikzcd}
K_4 = K_0(\sqrt[\ell^4]{a}) \arrow[dd, bend left=60, no head, "\textit{cyclic}"] \\
K_3 = K_0(\sqrt[\ell^3]{a}) \arrow[dd, bend left=60, no head, "\textit{cyclic}"] \\
K_2 = K_0(\sqrt[\ell^2]{a}) \arrow[dd, bend left=60, no head, "\textit{cyclic}"] \\
K_1 = K_0(\sqrt[\ell]{a}) \\
\mu_{\ell^2} \subset K_0
\end{tikzcd}$$

Proposition

Assume $\ell \neq 2$. Assume that all the primes above ℓ in K_{n_0,m_0} are totally ramified in K_{n_0+1,m_0+1} for some integers $n_0 \geq 0$ and $m_0 \geq 1$. Then

1. All primes above ℓ in K_{n_0,m_0} are totally ramified in $K_{n,m}/K_{n_0,m_0}$ for all $n \geq n_0$ and $m \geq m_0$;
2. If $|A_{n_0,m_0}| = |A_{n_0+1,m_0+1}|$, then $A_{n,m} \cong A_{n_0,m_0}$ for all $n \geq n_0$ and $m \geq m_0$.
3. If $\ell \nmid h_{n_0+1,m_0+1}$, then $\ell \nmid h_{n,m}$ for all $n \geq n_0$ and $m \geq m_0$.

Stable results in Kummer extension

Suppose we have that the ℓ-class groups of the red fields are same.

$$\begin{tikzcd}
K_{n_0,m_0+2} \arrow[r, dash] \arrow[d, dash] & K_{n_0+1,m_0+2} \arrow[r, dash] \arrow[d, dash] & K_{n_0+2,m_0+2} \arrow[d, dash] \\
K_{n_0,m_0+1} \arrow[r, dash] \arrow[d, dash] & {\color{red}K_{n_0+1,m_0+1}} \arrow[r, dash] \arrow[d, dash] & K_{n_0+2,m_0+1} \arrow[d, dash] \\
{\color{red}K_{n_0,m_0}} \arrow[r, dash] & K_{n_0+1,m_0} \arrow[r, dash] & K_{n_0+2,m_0}
\end{tikzcd}$$

The case $K_{n,m} = \mathbb{Q}(\sqrt[\ell^n]{p}, \mu_{2\ell^m})$

Stable results in Kummer extension

By ramification and class field theory, we have

$$\begin{tikzcd}
K_{n_0,m_0+2} \arrow[r, dash] \arrow[d, dash] & K_{n_0+1,m_0+2} \arrow[r, dash] \arrow[d, dash] & K_{n_0+2,m_0+2} \arrow[d, dash] \\
{\color{red}K_{n_0,m_0+1}} \arrow[r, dash] \arrow[d, dash] & {\color{red}K_{n_0+1,m_0+1}} \arrow[r, dash] \arrow[d, dash] & K_{n_0+2,m_0+1} \arrow[d, dash] \\
{\color{red}K_{n_0,m_0}} \arrow[r, dash] & {\color{red}K_{n_0+1,m_0}} \arrow[r, dash] & K_{n_0+2,m_0}
\end{tikzcd}$$

Stable results in Kummer extension

Apply our stable lemma to the two left vertical lines,

$$\begin{array}{ccccc}
\color{red}{K_{n_0,m_0+2}} & — & \color{red}{K_{n_0+1,m_0+2}} & — & K_{n_0+2,m_0+2} \\
| & & | & & | \\
\color{red}{K_{n_0,m_0+1}} & — & \color{red}{K_{n_0+1,m_0+1}} & — & \color{red}{K_{n_0+2,m_0+1}} \\
| & & | & & | \\
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Apply our stable lemma to the two left vertical lines,

$$\begin{array}{ccccc}
K_{n_0,m_0+2} & — & K_{n_0+1,m_0+2} & — & K_{n_0+2,m_0+2} \\
| & & | & & | \\
K_{n_0,m_0+1} & — & K_{n_0+1,m_0+1} & — & K_{n_0+2,m_0+1} \\
| & & | & & | \\
K_{n_0,m_0} & — & K_{n_0+1,m_0} & — & K_{n_0+2,m_0}
\end{array}$$

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Thanks for your attention