THE CHEVALLEY–GRAS FORMULA OVER GLOBAL FIELDS

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Abstract. In this article we give an adelic proof of the Chevalley–Gras formula for global fields, which itself is a generalization of the ambiguous class number formula. The idea is to reduce the formula to the Hasse norm theorem and to the local and global reciprocity laws. We also give an adelic proof of the Chevalley–Gras formula for the class group of divisors of degree 0 in the function field case, which extends a result of Rosen.

1. Introduction

Let K/k be a cyclic extension of number fields with Galois group G. Let $\mathfrak{m}$ be a modulus of k, which gives rise to a modulus $\mathfrak{m}_K$ of K. The ray class group $\mathrm{Cl}_K^{\mathfrak{m}_K}$ modulo $\mathfrak{m}_K$ admits a G-module structure. The Chevalley–Gras formula describes an explicit relationship between the generalized ambiguous ray class number $|(\mathrm{Cl}_K^{\mathfrak{m}_K}/\mathcal{C})^G|$ and $|\mathrm{Cl}_k^{\mathfrak{m}}/N(\mathcal{C})|$, where $\mathcal{C}\subset \mathrm{Cl}_K^{\mathfrak{m}_K}$ is any G-submodule and N is the norm map from K to k. In the case when the submodule $\mathcal{C}$ and the modulus $\mathfrak{m}$ are trivial, the formula then relates the class numbers $|\mathrm{Cl}_K^G|$ and $|\mathrm{Cl}_k|$, which is the classical ambiguous class number formula due to Chevalley [3, p. 406]. A proof of Chevalley's formula can be found in Gras' book [7, Lemma 6.1.2 and Remark 6.2.3] or in Lang's book [10, Chapter 13, §4, Lemma 4.1]. Lemmermeyer [11] gives an elementary proof which follows closely the approach taken by Lang, but avoiding the machinery of cohomologies. The existing proofs of Chevalley's formula are reduced to a result of the Herbrand quotient of global units. In [5, Théorème 4.3], Gras gave a formula for narrow class groups with arbitrary $\mathcal{C}$. In [6, Théorème 2.7] (also see the English translation [8, Section 2]), he proved this formula for ray class groups. His proof is based on Chevalley's formula. Recently, a generalization of Chevalley's class number formula to dihedral extensions has been investigated by Caputo and Nuccio [2].

In this article we give an adelic proof of the Chevalley–Gras formula over global fields. More precisely, using the adelic language, we reduce the formula to the Hasse norm theorem and to the local and global index theorems, which is shorter and more conceptual.

In the function field case, the class group of divisors of degree 0 deserves a special attention. The ambiguous class number formula (the case $\mathcal{C}=0$) for functions fields was obtained by Rosen [14]. We also give an adelic proof of the formula with an arbitrary G-submodule $\mathcal{C}$.

In the last section we add an elementary exposition of a cohomological variant for S-ray class groups, for the sake of completeness. This formulation is valid for an arbitrary Galois extension K/k, and is essentially equivalent to Chevalley's original formula in the cyclic case, thanks to the theorem on the Herbrand quotient of global units.

2. The Chevalley–Gras formula

In this section, we recall the definition of the S-ray class group and prove the Chevalley–Gras formula. We then give some special cases of this formula for future convenience. In Example 2.6, we use this formula to reprove a classical result of Rédei on the 4-rank of the narrow class group of quadratic fields as this approach does not seem to appear in the literature.

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2.1. **Notation and S-ray class groups.** Let F be a global field, that is, F is either a number field (a finite extension of $\mathbb{Q}$) or a global function field (a finite extension of $\mathbb{F}_p(t)$ for some prime p). Let V_F denote the set of all places of F, and let $V_{F,\infty}$ (resp. $V_{F,f}$) denote the subset of archimedean (resp. finite) places. (So in the function field case, $V_{F,\infty} = \emptyset$.) For each place $w \in V_F$, the completion of F at w is denoted by F_w. Let $\mathcal{O}_w$ denote the ring of integers of F_w if w is finite. The canonical embedding from F to the completion F_w is also denoted by w.

The letter S always denotes a non-empty finite set of places of F containing $V_{F,\infty}$. Denote by $\mathcal{O}_{F,S}$ the ring of S-integers of F, which consists of all elements $a \in F$ such that $a \in \mathcal{O}_w$ for all $w \notin S$. An S-modulus $\mathfrak{m}$ is a formal product $\mathfrak{m}_\infty \cdot \mathfrak{m}_f$, where $\mathfrak{m}_f$ is a nonzero integral ideal of $\mathcal{O}_{F,S}$ and $\mathfrak{m}_\infty$ is a formal product of some real places if F is a number field and $\mathfrak{m}_\infty$ is always 1 otherwise. Let $S(\mathfrak{m}) := \{w \in V_F; w|\mathfrak{m}\}$ be the support of $\mathfrak{m}$. Let I_F be the free abelian group generated by $V_{F,f}$, and $I_F^{S(\mathfrak{m})}$ the subgroup generated by $V_{F,f} \smallsetminus S(\mathfrak{m})$. The ideal $\mathfrak{m}_f$ corresponds to an effective divisor in I_F whose support is disjoint from S (in $V_{F,f}$). Let

$$F^{\mathfrak{m}} := \{x \in F^\times | x \equiv 1 \bmod \mathfrak{m}_f \text{ and } w(x) > 0 \text{ for each real place } w \mid \mathfrak{m}_\infty\}.$$

Let $i : F^\times \to I_F$ be the natural map defined by $a \mapsto \sum_{w \in V_{F,f}} \operatorname{ord}_w(a) w$. Note that $i(F^{\mathfrak{m}}) \subset I_F^{S(\mathfrak{m})}$. The *ray class group of F modulo* $\mathfrak{m}$ is defined as

$$\mathrm{Cl}_F^{\mathfrak{m}} := I_F^{S(\mathfrak{m})}/i(F^{\mathfrak{m}}).$$

The *S-ray class group of F modulo* $\mathfrak{m}$ is defined as

$$\mathrm{Cl}_{F,S}^{\mathfrak{m}} := \mathrm{Cl}_F^{\mathfrak{m}} / \langle \text{image of } S \cap V_{F,f} \rangle. \tag{2.1}$$

Since S is non-empty, $\mathrm{Cl}_{F,S}^{\mathfrak{m}}$ is finite.

Alternatively, let $I_F^{S(\mathfrak{m}) \cup S} \subset I_F^{S(\mathfrak{m})}$ be the subgroup generated by $V_{F,f} \smallsetminus (S \cup S(\mathfrak{m}))$. Then we have a projection $\mathrm{pr} : I_F^{S(\mathfrak{m})} \to I_F^{S(\mathfrak{m})}/\langle S \cap V_{F,f} \rangle \simeq I_F^{S(\mathfrak{m}) \cup S}$. Composing with i, we obtain a map $i_S : F^{\mathfrak{m}} \to I_F^{S(\mathfrak{m}) \cup S}$, which maps a to $\mathrm{div}^S(a) := \sum_{w \notin S} \operatorname{ord}_w(a) w$. Then one can define $\mathrm{Cl}_{F,S}^{\mathfrak{m}}$ by

$$I_F^{S(\mathfrak{m}) \cup S}/i_S(F^{\mathfrak{m}}),$$

and this agrees with the definition (2.1). The group I_F^S also can be naturally identified with the ideal group of $\mathcal{O}_{F,S}$. Under this identification, the map $i_S : F^{\mathfrak{m}} \to I_F^{S(\mathfrak{m}) \cup S}$ sends a to the principal ideal $a\mathcal{O}_{F,S}$. Put $P_F^{\mathfrak{m},S} := i_S(F^{\mathfrak{m}})$, the subgroup of principal S-ideals modulo $\mathfrak{m}$, and then we have $\mathrm{Cl}_{F,S}^{\mathfrak{m}} = I_F^{S(\mathfrak{m}) \cup S}/P_F^{\mathfrak{m},S}$. In the case where $\mathfrak{m} = 1$, this is the *S-ideal class group of F* and is denoted by $\mathrm{Cl}_{F,S}$.

For convenience, we also let $\mathcal{O}_w := F_w$ if $w \in V_{F,\infty}$, and for each $w|\mathfrak{m}$ we also write

$$1 + \mathfrak{m}\mathcal{O}_w := \begin{cases} 1 + \mathfrak{m}_f \mathcal{O}_w & \text{if } w \mid \mathfrak{m}_f; \\ (F_w^\times)^2 & \text{if } w \mid \mathfrak{m}_\infty. \end{cases}$$

Let K/k be a finite Galois extension of global fields with Galois group G. Let $(S_K, \mathfrak{m}_K)$ be a pair consisting of a finite set S of places and a modulus $\mathfrak{m}_K$ of K as above. Suppose that both S_K and $\mathfrak{m}_K$ are G-invariant. Then the S-ray class group $\mathrm{Cl}_{K,S_K}^{\mathfrak{m}_K}$ admits an action of G. So for any G-submodule $\mathcal{C}$ of $\mathrm{Cl}_{K,S_K}^{\mathfrak{m}_K}$, one may look for a relationship between $|(\mathrm{Cl}_{K,S_K}^{\mathfrak{m}_K}/\mathcal{C})^G|$ and $|\mathrm{Cl}_{k,S}^{\mathfrak{m}}/N(\mathcal{C})|$, for a suitable pair $(S, \mathfrak{m})$ for k related to $(S_K, \mathfrak{m}_K)$, where N is the norm map from K to k. This question is answered mostly when K/k is cyclic and remains open in general, even for the abelian case.

Suppose $(S, \mathfrak{m})$ is a pair for k. Let S_K be the set of places of K over S and let $\mathfrak{m}_K := \mathfrak{m}_{K,\infty} \cdot \mathfrak{m}_{K,f}$, where $\mathfrak{m}_{K,\infty}$ is the set of real places of K over the support of $\mathfrak{m}_\infty$ and $\mathfrak{m}_{K,f} := \mathfrak{m}_f \mathcal{O}_{K,S}$. Then $(S_K, \mathfrak{m}_K)$ is a G-invariant pair, and we say $(S_K, \mathfrak{m}_K)$ is induced by $(S, \mathfrak{m})$. In this case, we also write $\mathrm{Cl}_{K,S}^{\mathfrak{m}}$ for $\mathrm{Cl}_{K,S_K}^{\mathfrak{m}_K}$ and call it the S-ray class group modulo $\mathfrak{m}$. When $\mathfrak{m} = 1$, we write also $\mathrm{Cl}_{K,S}$ for Cl_{K,S_K} and call it the S-ideal class group of K.

2.2. **The main formula.** Let K/k be a cyclic extension of global fields with group $G = \langle\sigma\rangle$, where σ is a generator. Let $N = N_{K/k}$ be the norm map from K to k. Let $S \supset V_{k,\infty}$ be a finite non-empty set of places of k and $\mathfrak{m}$ an S-modulus. Let $\mathrm{Cl}_{K,S}^{\mathfrak{m}} := \mathrm{Cl}_{K,S_K}^{\mathfrak{m}_K}$ be the S-ray class group modulo $\mathfrak{m}$, where $(S_K, \mathfrak{m}_K)$ is the pair induced by $(S, \mathfrak{m})$.

For $v \in V_{k,f}$, denote by e_v and f_v the ramification index and inertia degree of v in K/k respectively. In the number field case, if v is real and every place $w|v$ of K is complex, we say that v is ramified in K, and put $e_v = 2$ and $f_v = 1$, otherwise, we put $e_v = f_v = 1$. The following theorem which we call the Chevalley–Gras formula over global fields is proved by Gras in the number field case; see [8, Theorem 3.6].

Theorem 2.1. *Let K/k be a cyclic extension of global fields with Galois group G. Let $\mathfrak{m}$ be a modulus of k, and let $S \supset V_{k,\infty}$ be a finite non-empty set of places of k such that $S \cap S(\mathfrak{m}_f) = \emptyset$. Let $\mathcal{C}$ be a G-submodule of the S-ray class group $\mathrm{Cl}_{K,S}^{\mathfrak{m}}$. Let D be any subgroup of $I_K^{S(\mathfrak{m})}$ such that the image of D in $\mathrm{Cl}_{K,S}^{\mathfrak{m}}$ is equal to $\mathcal{C}$. Then*

$$\frac{|(\mathrm{Cl}_{K,S}^{\mathfrak{m}}/\mathcal{C})^G|}{|\mathrm{Cl}_{k,S}^{\mathfrak{m}}/N(\mathcal{C})|} = \frac{\prod\limits_{v \in S \smallsetminus S(\mathfrak{m})} e_v f_v \prod\limits_{v \in S(\mathfrak{m})} [1 + \mathfrak{m}\mathcal{O}_v : N(\prod\limits_{w|v}(1 + \mathfrak{m}\mathcal{O}_w))] \prod\limits_{v \notin S \cup S(\mathfrak{m})} e_v}{[K : k][\Lambda : \Lambda \cap N(K^{\mathfrak{m}})]}.$$

Here $\Lambda = \{x \in k^{\mathfrak{m}} | (x)\mathcal{O}_{k,S} = N(d)\mathcal{O}_{k,S}$ in I_k^S for some $d \in D\}$.

Remark 2.2. (1) The group Λ depends on the choice of D, however, we will see in (2.7) that the index $[\Lambda : \Lambda \cap N(K^{\mathfrak{m}})]$ depends only on $\mathcal{C}$.
(2) The group $N(\prod\limits_{w|v}(1 + \mathfrak{m}\mathcal{O}_w))$ equals $N_{K_w/k_v}(1 + \mathfrak{m}\mathcal{O}_w)$ for any $w \mid v$ as K/k is Galois.

Proof. We first express $\mathrm{Cl}_{K,S}^{\mathfrak{m}}/\mathcal{C}$ in terms of ideles. Let $\mathbb{A}_K^\times$ denote the idele group of K. Let $\mathbb{A}_K^{\mathfrak{m}} = \{(a_w)_w \in \mathbb{A}_K^\times | a_w \in 1 + \mathfrak{m}\mathcal{O}_w$ for each $w \mid \mathfrak{m}\}$. We denote the canonical surjection $\mathbb{A}_K^{\mathfrak{m}} \to I_K^{S(\mathfrak{m})}$ by π. The kernel of π is $U_K^{\mathfrak{m}} := \prod_{w \nmid \mathfrak{m}} \mathcal{O}_w^\times \prod_{w|\mathfrak{m}} 1 + \mathfrak{m}\mathcal{O}_w$. Put

$$U_{K,S}^{\mathfrak{m}} = U_K^{\mathfrak{m}} \prod_{w \in S_K, w \nmid \mathfrak{m}_\infty} K_w^\times = \prod_{w \nmid \mathfrak{m}, w \notin S_K} \mathcal{O}_w^\times \prod_{w|\mathfrak{m}} 1 + \mathfrak{m}\mathcal{O}_w \prod_{w \in S_K, w \nmid \mathfrak{m}_\infty} K_w^\times.$$

Then π induces an isomorphism

$$\mathbb{A}_K^{\mathfrak{m}}/K^{\mathfrak{m}} U_{K,S}^{\mathfrak{m}} \cong \mathrm{Cl}_{K,S}^{\mathfrak{m}}.$$

By the approximation theorem, $\mathbb{A}_K^{\mathfrak{m}} K^\times = \mathbb{A}_K^\times$. Note that $\mathbb{A}_K^{\mathfrak{m}} \cap K^\times U_{K,S}^{\mathfrak{m}} = K^{\mathfrak{m}} U_{K,S}^{\mathfrak{m}}$. So the inclusion $\mathbb{A}_K^{\mathfrak{m}} \subset \mathbb{A}_K^\times$ induces an isomorphism

$$\mathrm{Cl}_{K,S}^{\mathfrak{m}} \cong \mathbb{A}_K^\times / K^\times U_{K,S}^{\mathfrak{m}}.$$

Put $\widetilde{D} = \pi^{-1}(D) K^\times U_{K,S}^{\mathfrak{m}}$. It follows that

$$\mathrm{Cl}_{K,S}^{\mathfrak{m}}/\mathcal{C} \cong \mathbb{A}_K^\times/\widetilde{D}. \tag{2.2}$$

Since $H^1(G, \mathbb{A}_K^\times) = 0$, the fact $G = \langle\sigma\rangle$ is cyclic implies that $(\mathbb{A}_K^\times)^{1-\sigma}$ is the kernel of the norm from $\mathbb{A}_K^\times$ to $\mathbb{A}_k^\times$. So there is an exact commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \widetilde{D} \cap (\mathbb{A}_K^\times)^{1-\sigma} & \longrightarrow & \widetilde{D} & \xrightarrow{N} & N(\widetilde{D}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & (\mathbb{A}_K^\times)^{1-\sigma} & \longrightarrow & \mathbb{A}_K^\times & \xrightarrow{N} & N(\mathbb{A}_K^\times) & \longrightarrow & 1.
\end{array}$$

The snake lemma gives the short exact sequence

$$1 \to (\mathrm{Cl}_{K,S}^{\mathfrak{m}}/\mathcal{C})^{1-\sigma} \to \mathrm{Cl}_{K,S}^{\mathfrak{m}}/\mathcal{C} \to N(\mathbb{A}_K^\times)/N(\widetilde{D}) \to 1,$$

as one has $(\mathbb{A}_K^\times)^{1-\sigma}/(\mathbb{A}_K^\times)^{1-\sigma}\cap\widetilde{D})\simeq(\mathrm{Cl}_{K,S}^{\mathfrak{m}}/\mathcal{C})^{1-\sigma}$. For any finite G-module M, one has $|M^G|=|M/M^{1-\sigma}|$ by the exact sequence

$$0\to M^G\to M\xrightarrow{1-\sigma}M\to M/M^{1-\sigma}\to 0.$$

Thus we obtain the equality

$$|(\mathrm{Cl}_{K,S}^{\mathfrak{m}}/\mathcal{C})^G|=|N(\mathbb{A}_K^\times)/N(\widetilde{D})|.$$

Recall that Hasse's norm theorem says that $k^\times\cap N(\mathbb{A}_K^\times)=N(K^\times)$. So given an element $N(x)=aN(d)\in N(\mathbb{A}_K^\times)\cap k^\times N(\widetilde{D})$ with $x\in\mathbb{A}_K^\times$, $a\in k^\times$ and $d\in\widetilde{D}$, we have $a=N(y)$ for some $y\in K^\times$. Hence $N(\mathbb{A}_K^\times)\cap k^\times N(\widetilde{D})=N(K^\times)N(\widetilde{D})=N(\widetilde{D})$. Therefore the natural map

$$N(\mathbb{A}_K^\times)/N(\widetilde{D})\to k^\times N(\mathbb{A}_K^\times)/k^\times N(\widetilde{D})$$

is an isomorphism. The global index theorem [9, Chapter IX, §5] says that $|\mathbb{A}_k^\times/k^\times N(\mathbb{A}_K^\times)|=|G|=[K:k]$. This implies that $k^\times N(\mathbb{A}_K^\times)/k^\times N(\widetilde{D})$ is a subgroup of $\mathbb{A}_k^\times/k^\times N(\widetilde{D})$ with index $[K:k]$. Therefore

$$|(\mathrm{Cl}_{K,S}^{\mathfrak{m}}/\mathcal{C})^G|=[K:k]^{-1}|\mathbb{A}_k^\times/k^\times N(\widetilde{D})|. \tag{2.3}$$

To compute $\mathbb{A}_k^\times/k^\times N(\widetilde{D})$, we consider the exact sequence

$$1\to k^\times N(\widetilde{D})U_{k,S}^{\mathfrak{m}}/k^\times N(\widetilde{D})\to\mathbb{A}_k^\times/k^\times N(\widetilde{D})\to\mathbb{A}_k^\times/k^\times U_{k,S}^{\mathfrak{m}}N(\widetilde{D})\to 1. \tag{2.4}$$

We claim that the last term is isomorphic to $\mathrm{Cl}_{k,S}^{\mathfrak{m}}/N(\mathcal{C})$. Under the identification $\mathbb{A}_k^{\mathfrak{m}}/k^{\mathfrak{m}}U_{k,S}^{\mathfrak{m}}\cong\mathrm{Cl}_{k,S}^{\mathfrak{m}}$, we have that $N(\mathcal{C})\subset\mathrm{Cl}_{k,S}^{\mathfrak{m}}$ is the image of $N(\pi^{-1}(D))$ in $\mathrm{Cl}_{k,S}^{\mathfrak{m}}$. Hence

$$\mathrm{Cl}_{k,S}^{\mathfrak{m}}/N(\mathcal{C})\cong\mathbb{A}_k^{\mathfrak{m}}/N(\pi^{-1}(D))k^{\mathfrak{m}}U_{k,S}^{\mathfrak{m}}.$$

Then the inclusion $\mathbb{A}_k^{\mathfrak{m}}\hookrightarrow\mathbb{A}_k^\times$ induces an isomorphism

$$\mathbb{A}_k^{\mathfrak{m}}/N(\pi^{-1}(D))k^{\mathfrak{m}}U_{k,S}^{\mathfrak{m}}\cong\mathbb{A}_k^\times/N(\pi^{-1}(D))k^\times U_{k,S}^{\mathfrak{m}}=\mathbb{A}_k^\times/k^\times U_{k,S}^{\mathfrak{m}}N(\widetilde{D}).$$

The first term of (2.4) can be computed by the exact sequence

$$1\to U_{k,S}^{\mathfrak{m}}\cap k^\times N(\widetilde{D})/N(U_{K,S}^{\mathfrak{m}})\to U_{k,S}^{\mathfrak{m}}/N(U_{K,S}^{\mathfrak{m}})\to k^\times N(\widetilde{D})U_{k,S}^{\mathfrak{m}}/k^\times N(\widetilde{D})\to 1. \tag{2.5}$$

Let G_v and I_v be the decomposition group and inertia group of v respectively. For each place v of k, we choose a place w of K above v. By local class field theory, $H^2(G_v,K_w^\times)=k_v^\times/N_{K_w/k_v}(K_w^\times)\cong G_v$ and $H^2(G_v,\mathcal{O}_w^\times)=\mathcal{O}_v^\times/N_{K_w/k_v}(\mathcal{O}_w^\times)\cong I_v$. Note that $U_{k,S}^{\mathfrak{m}}=(U_{K,S}^{\mathfrak{m}})^G$. It follows from the cyclicity of G and Shapiro's Lemma that

$$\begin{aligned}U_{k,S}^{\mathfrak{m}}/N(U_{K,S}^{\mathfrak{m}})&\cong H^2(G,U_{K,S}^{\mathfrak{m}})\\&\cong\prod_{v\in S\smallsetminus S(\mathfrak{m})}H^2(G_v,K_w^\times)\times\prod_{v\in S(\mathfrak{m})}H^2(G_v,1+\mathfrak{m}\mathcal{O}_w)\times\prod_{v\notin S(\mathfrak{m})\cup S}H^2(G_v,\mathcal{O}_w^\times)\\&\cong\prod_{v\in S\smallsetminus S(\mathfrak{m})}G_v\times\prod_{v\in S(\mathfrak{m})}\frac{1+\mathfrak{m}\mathcal{O}_v}{N_{K_w/k_v}(1+\mathfrak{m}\mathcal{O}_w)}\times\prod_{v\notin S\cup S(\mathfrak{m})}I_v.\end{aligned}$$

This contributes to the numerator of the right hand side term in the theorem. In order to prove the theorem, it suffices to show that the first term of (2.5) is isomorphic to $\Lambda/\Lambda\cap N(K^{\mathfrak{m}})$.

Recall that π is the natural projection $\mathbb{A}_K^{\mathfrak{m}}\to I_K^{S(\mathfrak{m})}$. Write $\bar{D}=\pi^{-1}(D)$ for simplicity. As $U_{k,S}^{\mathfrak{m}}N(\bar{D})\subset\mathbb{A}_k^{\mathfrak{m}}$, it is direct to check that

$$\Lambda=k^{\mathfrak{m}}\cap U_{k,S}^{\mathfrak{m}}N(\bar{D})=k^\times\cap U_{k,S}^{\mathfrak{m}}N(\bar{D}). \tag{2.6}$$

Given $x=uN(\bar{d})\in\Lambda$ with $u\in U_{k,S}^{\mathfrak{m}}$ and $\bar{d}\in\bar{D}$, we define a function f as follows:

$$\begin{aligned}f:\Lambda&\longrightarrow U_{k,S}^{\mathfrak{m}}\cap k^\times N(\bar{D}U_{K,S}^{\mathfrak{m}})/N(U_{K,S}^{\mathfrak{m}})\\x&\longmapsto u \bmod N(U_{K,S}^{\mathfrak{m}})\end{aligned}.$$

We need to show that f is well-defined. Suppose $x = uN(\bar{d}) = u'N(\bar{d}') \in \Lambda$ with $u, u' \in U^{\mathfrak{m}}_{k,S}$ and $\bar{d}, \bar{d}' \in \bar{D}$. Then $u'/u = N(\bar{d}/\bar{d}') \in N(\bar{D}) \cap U^{\mathfrak{m}}_{k,S} \subset N(\mathbb{A}^{\mathfrak{m}}_K) \cap U^{\mathfrak{m}}_{k,S}$. By Lemma 2.3(1), the last group coincides with $N(U^{\mathfrak{m}}_{K,S})$. So f is a well-defined map.

It is clear that f is a group homomorphism. We show that f is surjective. Let $u = tN(\bar{d}a)$ be an element of $U^{\mathfrak{m}}_{k,S} \cap k^\times N(\bar{D}U^{\mathfrak{m}}_{K,S})$ with $t \in k^\times, \bar{d} \in \bar{D}$ and $a \in U^{\mathfrak{m}}_{K,S}$. Then $t = uN(a)^{-1}N(\bar{d})^{-1}$ with $uN(a)^{-1} \in U^{\mathfrak{m}}_{k,S}, N(\bar{d})^{-1} \in N(\bar{D})$. Note that t is in fact in $k^{\mathfrak{m}}$. This shows $t \in \Lambda$ by (2.6). We have $f(t) = uN(a)^{-1} \bmod N(U^{\mathfrak{m}}_{K,S}) \equiv u \bmod N(U^{\mathfrak{m}}_{K,S})$. This proves the surjectivity.

The kernel of f by definition coincides with $\Lambda \cap N(U^{\mathfrak{m}}_{K,S}\bar{D})$. Lemma 2.3(3) shows that it also equals $\Lambda \cap N(K^{\mathfrak{m}})$. Thus, as desired, f induces an isomorphism

$$\Lambda/\Lambda \cap N(K^{\mathfrak{m}}) \cong U^{\mathfrak{m}}_{k,S} \cap k^\times N(\bar{D}U^{\mathfrak{m}}_{K,S})/N(U^{\mathfrak{m}}_{K,S}). \tag{2.7}$$

Observe that the term $k^\times N(\bar{D}U^{\mathfrak{m}}_{K,S})$ is independent of the choice of D because $k^\times N(K^\times \bar{D}U^{\mathfrak{m}}_{K,S}) = k^\times N(\widetilde{D})$. This finishes the proof of the theorem. ∎

Lemma 2.3. *We have the following equalities:*

(1) $N(\mathbb{A}^{\mathfrak{m}}_K) \cap U^{\mathfrak{m}}_{k,S} = N(U^{\mathfrak{m}}_{K,S})$;
(2) $N(K^\times) \cap N(\mathbb{A}^{\mathfrak{m}}_K) = N(K^{\mathfrak{m}})$;
(3) $\Lambda \cap N(U^{\mathfrak{m}}_{K,S}\bar{D}) = \Lambda \cap N(K^{\mathfrak{m}}) = \Lambda \cap N(\mathbb{A}^{\mathfrak{m}}_K)$.

Proof. Recall $\mathbb{A}^{\mathfrak{m}}_K = \{(a_w)_w \in \mathbb{A}^\times_K | a_w \in 1 + \mathfrak{m}\mathcal{O}_w \text{ for each } w \mid \mathfrak{m}\}$. As mentioned, $N(\prod_{w|v} K^\times_w) = N_{K_w/k_v}(K^\times_w)$ for each w as K/k is Galois. For a place $w \nmid \mathfrak{m}$, it is easy to see that $N_{K_w/k_v}(K^\times_w) \cap \mathcal{O}^\times_v = N_{K_w/k_v}(\mathcal{O}^\times_w)$. We obtain $N(\prod_{w|v} K^\times_w) \cap \mathcal{O}^\times_v = N(\prod_{w|v} \mathcal{O}^\times_w)$. For a place $w \mid \mathfrak{m}$, we have the trivial equality $N_{K_w/k_v}(1+\mathfrak{m}\mathcal{O}_w) \cap 1+\mathfrak{m}\mathcal{O}_v = N_{K_w/k_v}(1+\mathfrak{m}\mathcal{O}_w)$. It follows that $N(\mathbb{A}^{\mathfrak{m}}_K) \cap U^{\mathfrak{m}}_{k,S} = N(U^{\mathfrak{m}}_{K,S})$. This proves (1).

To prove (2), let $K^\times \times \mathbb{A}^{\mathfrak{m}}_K$ denote the direct product of $K^\times$ and $\mathbb{A}^{\mathfrak{m}}_K$. Consider the exact commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & K^\times \cap \mathbb{A}^{\mathfrak{m}}_K \arrow[r, "x\mapsto(x,x)"] \arrow[d, "N"'] & K^\times \times \mathbb{A}^{\mathfrak{m}}_K \arrow[r, "(a,b)\mapsto ab^{-1}"] \arrow[d, "N\times N"'] & K^\times \mathbb{A}^{\mathfrak{m}}_K \arrow[r] \arrow[d, "N"'] & 1 \\
1 \arrow[r] & N(K^\times) \cap N(\mathbb{A}^{\mathfrak{m}}_K) \arrow[r, "x\mapsto(x,x)"] & N(K^\times) \times N(\mathbb{A}^{\mathfrak{m}}_K) \arrow[r, "(a,b)\mapsto ab^{-1}"] & N(K^\times)N(\mathbb{A}^{\mathfrak{m}}_K) \arrow[r] & 1.
\end{tikzcd}$$

Note that $K^\times \cap \mathbb{A}^{\mathfrak{m}}_K$ by definition is $K^{\mathfrak{m}}$. By the approximation theorem, $K^\times \mathbb{A}^{\mathfrak{m}}_K = \mathbb{A}^\times_K$. Since $H^1(G, \mathbb{A}^\times_K) = H^1(G, K^\times) = 0$ and G is cyclic, the snake lemma gives an exact sequence

$$(K^\times)^{1-\sigma} \times (\mathbb{A}^{\mathfrak{m}}_K \cap (\mathbb{A}^\times_K)^{1-\sigma}) \to (\mathbb{A}^\times_K)^{1-\sigma} \to N(K^\times) \cap N(\mathbb{A}^{\mathfrak{m}}_K)/N(K^{\mathfrak{m}}) \to 0.$$

The first arrow is surjective by the weak approximation theorem. Thus the last term is 0. This proves (2).

(3) Hasse's norm theorem says that $k^\times \cap N(\mathbb{A}^\times_K) = N(K^\times)$. Recall that $\Lambda = k^\times \cap U^{\mathfrak{m}}_{k,S}N(\bar{D})$ by (2.6). By (2), we have

$$\Lambda \cap N(U^{\mathfrak{m}}_{K,S}\bar{D}) = k^\times \cap N(U^{\mathfrak{m}}_{K,S}\bar{D}) \subset N(K^\times) \cap N(U^{\mathfrak{m}}_{K,S}\bar{D}) \subset N(K^\times) \cap N(\mathbb{A}^{\mathfrak{m}}_K) = N(K^{\mathfrak{m}}).$$

This proves the inclusion $\Lambda \cap N(U^{\mathfrak{m}}_{K,S}\bar{D}) \subset \Lambda \cap N(K^{\mathfrak{m}})$. To show the other inclusion, note that $U^{\mathfrak{m}}_{k,S}N(\bar{D}) \cap N(\mathbb{A}^{\mathfrak{m}}_K) = N(U^{\mathfrak{m}}_{K,S}\bar{D})$ by (1). Then

$$\Lambda \cap N(K^{\mathfrak{m}}) \subset \Lambda \cap N(\mathbb{A}^{\mathfrak{m}}_K) = k^\times \cap N(U^{\mathfrak{m}}_{K,S}\bar{D}) = \Lambda \cap N(U^{\mathfrak{m}}_{K,S}\bar{D}).$$

The last equality follows from (2.6).

The second equality in (3) follows from

$$\Lambda \cap N(\mathbb{A}^{\mathfrak{m}}_K) \subset \Lambda \cap N(K^\times) \cap N(\mathbb{A}^{\mathfrak{m}}_K) = \Lambda \cap N(K^{\mathfrak{m}}).$$

This completes the proof of the lemma. ∎

Remark 2.4. The idea of our adelic proof of Theorem 2.1 comes from [16], which shows that Chevalley's ambiguous class number formula follows immediately from the Hasse norm theorem, and the local and global norm index theorems. When the extension K/k is abelian, we know that the analogous statements for the local and global norm index theorems hold true; see [9, Chap. IX, Sections 3 and 5]. However, to extend Chevalley's formula to abelian extensions, the assumption that K/k is cyclic is crucial in the argument used in [16].

2.3. **Examples.** We list some special cases of the Chevalley–Gras formula in the number field case.

Example 2.5. (1) If $\mathfrak{m} = 1$ and S is the set of infinite places, then $\mathrm{Cl}_{K,S}^{\mathfrak{m}}$ is equal to Cl_K, the class group of K. The theorem says

$$\frac{|(\mathrm{Cl}_K/\mathcal{C})^G|}{|\mathrm{Cl}_k/N(\mathcal{C})|} = \frac{\prod_{v\le\infty} e_v}{[K:k][\Lambda : \Lambda\cap N(K^\times)]}.$$

If we let $\mathcal{C}$ and D be trivial, then $\Lambda = \mathcal{O}_k^\times$, the unit group of $\mathcal{O}_k$. The formula becomes the ambiguous class number formula for the class group

$$\frac{|(\mathrm{Cl}_K)^G|}{|\mathrm{Cl}_k|} = \frac{\prod_{v\le\infty} e_v}{[K:k][\mathcal{O}_k^\times : \mathcal{O}_k^\times\cap N(K^\times)]}.$$

(2) If $\mathfrak{m}$ is the product of all the real places of k and S is the set of infinite places, then $\mathrm{Cl}_{K,S}^{\mathfrak{m}}$ is the narrow class group Cl_K^+ of K. Similarly, $\mathrm{Cl}_{k,S}^{\mathfrak{m}} = \mathrm{Cl}_k^+$. Note that $K^{\mathfrak{m}}$ is equal to K^+, the group of totally positive elements of $K^\times$. For a real place v of k, $[1+\mathfrak{m}\mathcal{O}_v : N_{K_w/k_v}(1+\mathfrak{m}\mathcal{O}_w)] = [\mathbb{R}_{>0} : \mathbb{R}_{>0}] = 1$. So the theorem says

$$\frac{|(\mathrm{Cl}_K^+/\mathcal{C})^G|}{|\mathrm{Cl}_k^+/N(\mathcal{C})|} = \frac{\prod_{v\nmid\infty} e_v}{[K:k][\Lambda : \Lambda\cap N(K^+)]}. \tag{2.8}$$

If we further let $\mathcal{C}$ and D be trivial, then $\Lambda = (\mathcal{O}_k^\times)^+$. The formula becomes the ambiguous class number formula for narrow class groups which is first proved by Chevalley in [3, p. 406].

We now use the formula (2.8) to reprove a classical result of Rédei.

Example 2.6 (4-rank of narrow class groups of quadratic fields)**.** Let K be a quadratic number field with discriminant d. Let $T = \{p_1, \cdots, p_t\}$ be the set of prime numbers ramified in K. Let $G = \mathrm{Gal}(K/\mathbb{Q}) = \langle\sigma\rangle$. For $a \in \mathrm{Cl}_K^+$, $N_{K/\mathbb{Q}}(a) = aa^\sigma = 1$ as $\mathbb{Q}$ has class number 1. This implies that $|\mathrm{Cl}_K^+[2]| := |\{a \in \mathrm{Cl}_K^+ \,|a^2 = 1\}| = |(\mathrm{Cl}_K^+)^G|$. The latter term has cardinality 2^{t-1} by Chevalley's formula (2.8). In other words, the 2-rank of Cl_K^+ is $t-1$. The following $\mathbb{F}_2$-matrix is the Rédei matrix:

$$R := \big(\log(p_i, d)_{p_j}\big)_{1\le i,j\le t} \tag{2.9}$$

Here $\log : \{\pm1\} \twoheadrightarrow \mathbb{F}_2$ is the logarithm map and $(p_i, d)_{p_j}$ is the quadratic Hilbert symbol of p_i and d at the prime p_j. Note that the sum of each row of this matrix is zero by the product formula of Hilbert symbols.

A theorem of Rédei [15, Theorem 3.1] says that 4-rank of Cl_K^+ is $t-1-\mathrm{rank}(R)$.

The matrix R is the transpose of the matrix in [15, Theorem 3.1]. One can check that the logarithm Hilbert symbol $\log(p_i, d)_{p_j}$ coincides with the logarithm Kronecker symbol $\left(\frac{p_j^*}{p_i}\right) \in \mathbb{F}_2$ when $i \ne j$. Here $p^* = (-1)^{p-1/2}p_i$ for p is odd. If $2 \mid d$, 2^* is the number such that $d = \prod_{p|d} p^*$.

The proof in [15] uses the explicit construction of the 2-Hilbert class field. We give a proof by applying (2.8) to $K/\mathbb{Q}$. By definition, the 4-rank of Cl_K^+ is $\mathrm{rank}_4 \mathrm{Cl}_K^+ = \dim_{\mathbb{F}_2} \mathrm{Cl}_K^+[4]/\mathrm{Cl}_K^+[2]$. As we mentioned, $a^\sigma = a^{-1}$ for $a \in \mathrm{Cl}_K^+$. It follows that

$$a \bmod \mathrm{Cl}_K^+[2] \in (\mathrm{Cl}_K^+/\mathrm{Cl}_K^+[2])^G \Leftrightarrow a^\sigma a^{-1} = a^{-2} \in \mathrm{Cl}_K^+[2] \Leftrightarrow a \in \mathrm{Cl}_K^+[4].$$

This gives $\mathrm{Cl}_K^+[4]/\mathrm{Cl}_K^+[2] = (\mathrm{Cl}_K^+/\mathrm{Cl}_K^+[2])^G$. We now use (2.8) to compute the order of this group.

Take $\mathcal{C} = \mathrm{Cl}_K^+[2]$ in (2.8). It is well known that $\mathcal{C}$ is generated by the ramified prime ideals. We add a proof for this fact here for the sake of completeness. Suppose $I \in I_K$ such that its image $\mathrm{cl}(I)$ is in $\mathrm{Cl}_K^+[2] = (\mathrm{Cl}^+(K))^G$. Then $I^\sigma I^{-1}$ is generated by some totally positive element $\alpha \in K^\times$. Since $N(I^\sigma I^{-1}) = (1)$ in $I_{\mathbb{Q}}$, we have $N(\alpha) = \pm 1$. Thus $N(\alpha) = 1$ by the positivity of α. By Hilbert's Theorem 90, $\alpha = \beta^\sigma \beta^{-1}$ for some $\beta \in K^\times$. Note that we can assume β is totally positive. Thus $I\beta^{-1}$ is a G-invariant fractional ideal of K. It follows that $I\beta^{-1} \in \langle D, I_{\mathbb{Q}}\rangle$ (see Lemma 4.3(1)), where $D \subset I_K$ is a subgroup such that $\mathrm{cl}(D) = \mathcal{C} = \mathrm{Cl}_K^+[2]$ in Cl_K^+. Thus $\mathrm{cl}(I) \in \mathrm{cl}(D)$.

Let D be the subgroup of I_K generated by the ramified prime ideals. We have shown that D generates $\mathcal{C}$. The group Λ of (2.8) is then the subgroup of $\mathbb{Q}^\times$ generated by $T = \{p_1, \cdots, p_t\}$. Consider the following map

$$\Lambda \to \prod_{j=1}^{t} \{\pm 1\}, \quad x \mapsto \big((x,d)_{p_j}\big)_j .$$

The kernel is $\Lambda \cap N(K^\times)$ by the properties of Hilbert symbols and Hasse's norm theorem, for details see [12, Lemma 2.8]. Note that $\Lambda \cap N(K^\times) = \Lambda \cap N(K^+)$ as $\Lambda \subset \mathbb{Q}^+$. The image has size 2^r where $r = \mathrm{rank}(R)$ is the rank of the Rédei matrix (2.9). Therefore

$$|\,\mathrm{Cl}_K^+[4]/\,\mathrm{Cl}_K^+[2]| = |(\mathrm{Cl}_K^+/\mathrm{Cl}_K^+[2])^G| = 2^{t-1-r}.$$

3. The case of class groups of divisors of degree 0

We let K/k be a cyclic extension of global function fields with Galois group G. Denote by $\mathbb{F}_{q'}$ and $\mathbb{F}_q$ the constant fields of K and k, respectively. Let $\mathbb{A}_K^0$ be the kernel of the degree map

$$\deg_K : \mathbb{A}_K^\times \to \mathbb{Z}, \quad (x_w)_w \mapsto \sum_w \mathrm{ord}_w(x_w)[k_w : \mathbb{F}_{q'}],$$

where k_w is the residue field of w. Let $U_K = \prod_w \mathcal{O}_w^\times$. The class group of divisors and the class group of divisors of degree 0 of K are defined respectively by

$$\mathrm{Cl}_K = \mathbb{A}_K^\times / U_K K^\times \quad \text{and} \quad \mathrm{Cl}_K^0 = \mathbb{A}_K^0 / U_K K^\times.$$

It is well known that Cl_K^0 is finite. The degree map induces the exact sequence

$$0 \to \mathrm{Cl}_K^0 \to \mathrm{Cl}_K \xrightarrow{\deg_K} \mathbb{Z} \to 0.$$

See [1, Chapter V, Theorem 5] for the surjectivity of $\deg_K$. We define Cl_k, $\mathrm{Cl}_k^0, \deg_k, U_k$ for k in the same way. Let N denote the norm map from K to k. For a prime divisor $w \in \mathbb{A}_K^\times / U_K$ of K, by definition $N(w) = v^{[k_w : k_v]}$ where v is the prime divisor of k below w. This implies $\deg_k(N(\mathbb{A}_K^\times)) = [\mathbb{F}_{q'} : \mathbb{F}_q]\mathbb{Z}$.

Let $\mathcal{C}$ be a G-submodule of Cl_K^0. Choose any subgroup D of $\mathbb{A}_K^0$ such that the image of D in Cl_K^0 is equal to $\mathcal{C}$, and put $\Lambda := k^\times \cap N(D)U_k$ in $\mathbb{A}_k^\times$. Note that Λ depends on the choice of D, however, its image in $k^\times \cap N(D)N(K^\times)U_k / N(K^\times)$ depends only on $\mathcal{C}$. In particular, the index $[\Lambda : \Lambda \cap N(K^\times)]$ depends only on $\mathcal{C}$. Let $d(K/k) \in \mathbb{Z}$ denote the positive generator of the ideal $\deg_K(\mathrm{Cl}_K^G)$ of $\mathbb{Z}$.

Theorem 3.1. *With notations as above, one has*

$$|(\mathrm{Cl}_K^0/\mathcal{C})^G| = |\,\mathrm{Cl}_k^0/N(\mathcal{C})| \frac{[\mathbb{F}_{q'} : \mathbb{F}_q] \prod_v e_v}{[K:k][\Lambda : \Lambda \cap N(K^\times)]} d(K/k).$$

Remark 3.2. Putting $\mathcal{C} = 0$ and $D = 0$, we obtain the following formula

$$|(\mathrm{Cl}_K^0)^G| = |\,\mathrm{Cl}_k^0\,| \frac{[\mathbb{F}_{q'} : \mathbb{F}_q] \prod_v e_v}{[K:k] \cdot [\mathbb{F}_q^\times : \mathbb{F}_q^\times \cap N(K^\times)]} d(K/k).$$

When $q' = q$, this recovers the ambiguous class number formula obtained by Rosen (see [14, Theorem 8 and Proposition 2]. It is shown [14, p.164] that the invariant $d(K/k)$ divides another invariant

$\delta(K/k)$ which is easier to compute. Rosen also computed $d(K/k)$ in some special cases; see [14, Theorem 4]. For example, if the cyclic extension K/k is unramified everywhere, then $d(K/k) = [K : k]$, see [14, Corollary to Theorem 4].

Lemma 3.3. *Let σ be a generator of G. For any G-submodule $\mathcal{C} \subset \mathrm{Cl}_K^0$, we have*

$$d(K/k) = |(\mathrm{Cl}_K/\mathcal{C})^{1-\sigma}/(\mathrm{Cl}_K^0/\mathcal{C})^{1-\sigma}|.$$

Proof. This follows from the exact sequences

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & (\mathrm{Cl}_K^0/\mathcal{C})^G \arrow[r] \arrow[d] & (\mathrm{Cl}_K/\mathcal{C})^G \arrow[r] \arrow[d] & d(K/k)\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Cl}_K^0/\mathcal{C} \arrow[r] \arrow[d] & \mathrm{Cl}_K/\mathcal{C} \arrow[r, "\deg_K"] \arrow[d, "1-\sigma"] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\mathrm{Cl}_K^0/\mathcal{C})^{1-\sigma} \arrow[r] \arrow[d] & (\mathrm{Cl}_K/\mathcal{C})^{1-\sigma} \arrow[r] \arrow[d] & \mathbb{Z}/d(K/k)\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

∎

Now we give an adelic proof of Theorem 3.1. The reader will realize that the proof is analogous to that of Theorem 2.1.

Proof. Put $\widetilde{D} = DK^\times U_K$. The facts $H^1(G, \mathbb{A}_K^\times) = \hat{H}^{-1}(G, \mathbb{A}_K^\times) = 0$ and $(\mathbb{A}_K^\times)^{1-\sigma} \subset \mathbb{A}_K^0$ give the exact commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \widetilde{D} \cap (\mathbb{A}_K^\times)^{1-\sigma} & \longrightarrow & \widetilde{D} & \xrightarrow{N} & N(\widetilde{D}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & (\mathbb{A}_K^\times)^{1-\sigma} & \longrightarrow & \mathbb{A}_K^0 & \xrightarrow{N} & N(\mathbb{A}_K^0) & \longrightarrow & 1.
\end{array}$$

As $(\mathbb{A}_K^\times)^{1-\sigma}/\widetilde{D} \cap (\mathbb{A}_K^\times)^{1-\sigma} \cong (\mathrm{Cl}_K/\mathcal{C})^{1-\sigma}$, the snake lemma gives the short exact sequence

$$0 \to (\mathrm{Cl}_K/\mathcal{C})^{1-\sigma} \to \mathrm{Cl}_K^0/\mathcal{C} \to N(\mathbb{A}_K^0)/N(\widetilde{D}) \to 0.$$

We remark that $(\mathrm{Cl}_K/\mathcal{C})^{1-\sigma}$ is finite although $\mathrm{Cl}_K/\mathcal{C}$ is infinite. By the above lemma,

$$\begin{aligned}
|(\mathrm{Cl}_K^0/\mathcal{C})^G| &= |N(\mathbb{A}_K^0)/N(\widetilde{D})| \cdot |(\mathrm{Cl}_K/\mathcal{C})^{1-\sigma}/(\mathrm{Cl}_K^0/\mathcal{C})^{1-\sigma}| \\
&= d(K/k)|N(\mathbb{A}_K^0)/N(\widetilde{D})| \\
&= d(K/k)|N(\mathbb{A}_K^0)k^\times/N(\widetilde{D})k^\times|.
\end{aligned}$$

We prove the last equality as follows. Let $N(x) = N(d)a \in N(\mathbb{A}_K^0) \cap N(\widetilde{D})k^\times$ with $x \in \mathbb{A}_\times^0, d \in \widetilde{D}, a \in k^\times$. Then $a = N(xd^{-1}) \in k^\times \cap N(\mathbb{A}_K^0) \subset k^\times \cap N(\mathbb{A}_K^\times) = k^\times \cap N(K^\times)$ by Hasse's norm theorem. Then the inclusion $N(\mathbb{A}_K^0) \subset N(\mathbb{A}_K^0)k^\times$ induces an isomorphism

$$N(\mathbb{A}_K^0)/N(\widetilde{D}) \cong N(\mathbb{A}_K^0)k^\times/N(\widetilde{D})k^\times.$$

Consider the short exact sequence

$$0 \to N(\mathbb{A}_K^0)k^\times/N(\widetilde{D})k^\times \to \mathbb{A}_k^0/N(\widetilde{D})k^\times \to \mathbb{A}_k^0/N(\mathbb{A}_K^0)k^\times \to 0.$$

Suppose that $k = \mathbb{F}_q(E)$ is the function field of some curve E. We apply the degree map to the Artin reciprocity map $\mathrm{Art} : \mathbb{A}_k^\times / k^\times N(\mathbb{A}_K^\times) \simeq G$ and obtain the following exact commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{A}_k^0/k^\times N(\mathbb{A}_K^0) \arrow[r] \arrow[d, "\varphi_1"', "\cong"] & \mathbb{A}_k^\times/k^\times N(\mathbb{A}_K^\times) \arrow[r, "\deg_k"] \arrow[d, "\mathrm{Art}"', "\cong"] & \mathbb{Z}/[\mathbb{F}_{q'} : \mathbb{F}_q]\mathbb{Z} \arrow[r] \arrow[d, "\varphi_2"', "\cong"] & 0 \\ 0 \arrow[r] & \mathrm{Gal}(K/\mathbb{F}_{q'}(E)) \arrow[r] & G \arrow[r] & \mathrm{Gal}(\mathbb{F}_{q'}/\mathbb{F}_q) \arrow[r] & 0. \end{tikzcd} \tag{3.1}$$

Note that the isomorphism φ_2 is induced by the Frobenius map. The commutativity of the right diagram follows from [1, Chapter VIII, Theorem 10]. By the Corollary of [1, Chapter VIII, Theorem 10], the map φ_1 is surjective and hence is an isomorphism.

It follows from (3.1) that

$$|\mathbb{A}_k^0/k^\times N(\mathbb{A}_K^0)| = \frac{[K : k]}{[\mathbb{F}_{q'} : \mathbb{F}_q]}.$$

To prove the theorem, it remains to show that

$$|\mathbb{A}_k^0/N(\widetilde{D})k^\times| = |\mathrm{Cl}_k^0/N(\mathcal{C})| \frac{\prod_v e_v}{[\Lambda : \Lambda \cap N(K^\times)]}.$$

The argument is the same as that following equation (2.4) in the proof of Theorem 2.1. ∎

4. A cohomological variant for S-ray class groups

Let K/k be a finite Galois extension of global fields with Galois group G. As in Section 2.1, we let $\mathfrak{m}$ be a modulus of k, and S be a non-empty finite set of places of k containing all archimedean places which is disjoint from the support of $\mathfrak{m}_f$. In this Section we shall discuss a cohomological variant of the ambiguous S-ray class number formula of K/k; see Theorem 4.1. This formulation has been generalized to an arbitrary algebraic torus T over k by Gonzalez-Aviles [4] when the modulus $\mathfrak{m}$ is trivial, where the present formula may be viewed as the special case $T = \mathbb{G}_{m,k}$. Furthermore, when K/k is cyclic, we explain that Theorem 4.1 is essentially equivalent to Chevalley's ambiguous class number formula (the case $\mathcal{C} = 0$ in Theorem 2.1), thanks to the theorem on the Herbrand quotient of global S-units. The argument of the proof of Theorem 4.1 is slightly different from Lang's exposition [10, Chapter XIII, Section 4].

We keep the notation of Section 2.1. Recall that we write $\mathrm{Cl}_{K,S}^{\mathfrak{m}}$ for $\mathrm{Cl}_{K,S_K}^{\mathfrak{m}_K}$. Let $E_{K,S}^{\mathfrak{m}}$ be the intersection of the group of S-units of K with $K^{\mathfrak{m}}$. We have the exact sequence

$$1 \to E_{K,S}^{\mathfrak{m}} \to K^{\mathfrak{m}} \xrightarrow{i_S} P_K^{\mathfrak{m},S} \to 1. \tag{4.1}$$

Theorem 4.1. *Let K/k be a finite Galois extension of global fields with Galois group G. Let $\mathfrak{m}$ be a modulus of k, and let $S \supset V_{k,\infty}$ be a finite non-empty set of places of k such that $S \cap S(\mathfrak{m}_f) = \emptyset$. Then*

$$\frac{|(\mathrm{Cl}_{K,S}^{\mathfrak{m}})^G|}{|\mathrm{Cl}_{k,S}^{\mathfrak{m}}|} = \frac{|H^2(G, E_{K,S}^{\mathfrak{m}})|}{|H^1(G, E_{K,S}^{\mathfrak{m}})|} \frac{\prod_{v \notin S \cup S(\mathfrak{m})} e_v \cdot |H^1(G, K^{\mathfrak{m}})|}{|\mathrm{Im}\{H^2(G, E_{K,S}^{\mathfrak{m}}) \to H^2(G, K^{\mathfrak{m}})\}|}. \tag{4.2}$$

When the support $S(\mathfrak{m})$ of $\mathfrak{m}$ is empty, the term $H^1(G, K^{\mathfrak{m}})$ is trivial by Hilbert's Theorem 90 For the general case, we have the following formula.

Proposition 4.2. *Let the notation and the assumptions be the same as in Theorem 4.1. Then*

$$H^1(G, K^{\mathfrak{m}}) \cong \prod_{v \in S(\mathfrak{m})} H^1(G, \prod_{w|v} 1 + \mathfrak{m}\mathcal{O}_w).$$

Furthermore, if the extension K/k is cyclic, then

$$|H^1(G, K^{\mathfrak{m}})| = \prod_{v \in S(\mathfrak{m})} [1 + \mathfrak{m}\mathcal{O}_v : N(\prod_{w|v}(1 + \mathfrak{m}\mathcal{O}_w))].$$

Lemma 4.3. *We have*

(1) $H^1(G, I_K^S) = 0$ *and* $(I_K^S)^G/I_k^S \cong \oplus_{v\notin S}\mathbb{Z}/e_v\mathbb{Z}$;

(2) $(P_K^{\mathfrak{m},S})^G/P_k^{\mathfrak{m},S} \cong \operatorname{Ker}\varphi$, *where* φ *is the natural map* $H^1(G, E_{K,S}^{\mathfrak{m}}) \to H^1(G, K^{\mathfrak{m}})$;

(3) $|H^1(G, P_K^{\mathfrak{m},S})| = |H^1(G, K^{\mathfrak{m}})| \cdot |\operatorname{Ker}\psi| \cdot |\mathrm{Im}\ \varphi|^{-1}$, *where* ψ *is the natural map* $H^2(G, E_{K,S}^{\mathfrak{m}}) \to H^2(G, K^{\mathfrak{m}})$.

Proof. (1) For each place v of k, let G_v be a decomposition group of v, which is uniquely determined up to conjugate. Since G acts transitively on the set of places of K above v, $I_K^S \simeq \oplus_{v\notin S} \oplus_{w|v} \mathbb{Z}w = \oplus_{v\notin S}\mathrm{Ind}_{G_v}^G\mathbb{Z}$. By Shapiro's Lemma, $H^1(G, I_K^S) = \oplus_{v\notin S} H^1(G_v, \mathbb{Z}) = 0$.

For each finite place v of k, let $\mathfrak{p}_v$ be the corresponding prime ideal of $\mathcal{O}_k$, and $\mathfrak{a}_v := \prod_{\mathfrak{P}|\mathfrak{p}_v} \mathfrak{P}$ the prime ideal of $\mathcal{O}_K$ such that $\mathfrak{a}_v^{e_v} = \mathfrak{p}_v\mathcal{O}_K$. It is clear that $(I_K^S)^G$ and I_k^S are free abelian groups generated by $\mathfrak{a}_v$ and $\mathfrak{p}_v$ for all $v \notin S$, respectively. Thus, $(I_K^S)^G/I_k^S \cong \oplus_{v\notin S}\mathbb{Z}/e_v\mathbb{Z}$.

(2) Taking Galois cohomology of the exact sequence (4.1), we get the long exact sequence

$$1 \to (E_{K,S}^{\mathfrak{m}})^G \to k^{\mathfrak{m}} \to (P_K^{\mathfrak{m},S})^G \to H^1(G, E_{K,S}^{\mathfrak{m}}) \xrightarrow{\varphi} H^1(G, K^{\mathfrak{m}}).$$

It follows that $(P_K^{\mathfrak{m},S})^G/P_k^{\mathfrak{m},S} \cong \operatorname{Ker}\varphi$.

(3) Taking Galois cohomology of the exact sequence (4.1), we get the long exact sequence

$$H^1(G, E_{K,S}^{\mathfrak{m}}) \xrightarrow{\varphi} H^1(G, K^{\mathfrak{m}}) \to H^1(G, K^{\mathfrak{m}}) \to H^1(G, P_K^{\mathfrak{m},S}) \to H^2(G, E_{K,S}^{\mathfrak{m}}) \xrightarrow{\psi} H^2(G, K^{\mathfrak{m}})$$

and an exact sequence

$$0 \to \mathrm{Im}\varphi \to H^1(G, K^{\mathfrak{m}}) \to H^1(G, P_K^{\mathfrak{m},S}) \to \ker\psi \to 0.$$

From this the statement (3) follows. ∎

Proof of Theorem 4.1. Consider the following exact sequence of G-modules

$$0 \to P_K^{\mathfrak{m},S} \to I_K^{S(\mathfrak{m})\cup S} \to \mathrm{Cl}_{K,S}^{\mathfrak{m}} \to 0.$$

Taking Galois cohomology, we have the following exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & P_k^{\mathfrak{m},S} \arrow[r] \arrow[d] & I_k^{S(\mathfrak{m})\cup S} \arrow[r] \arrow[d] & \mathrm{Cl}_{k,S}^{\mathfrak{m}} \arrow[r] \arrow[d, "j"] & 0 & \\
0 \arrow[r] & (P_K^{\mathfrak{m},S})^G \arrow[r] & (I_K^{S(\mathfrak{m})\cup S})^G \arrow[r] & (\mathrm{Cl}_{K,S}^{\mathfrak{m}})^G \arrow[r] & H^1(G, P_K^{\mathfrak{m},S}) \arrow[r] & 0
\end{tikzcd}$$

We remark that the map j is not injective in general. The Snake Lemma gives the exact sequence

$$0 \to \operatorname{Ker} j \to (P_K^{\mathfrak{m},S})^G/P_k^{\mathfrak{m},S} \to (I_K^{S(\mathfrak{m})\cup S})^G/I_k^{S(\mathfrak{m})\cup S} \to (\mathrm{Cl}_{K,S}^{\mathfrak{m}})^G/\mathrm{Im}\ j \to H^1(G, P_K^{\mathfrak{m},S}) \to 0. \tag{4.3}$$

By Lemma 4.3, we have

$$\begin{aligned}
|(\mathrm{Cl}_{K,S}^{\mathfrak{m}})^G/\mathrm{Cl}_{k,S}^{\mathfrak{m}}| &= |(\mathrm{Cl}_{K,S}^{\mathfrak{m}})^G/\mathrm{Im}\ j| \cdot |\operatorname{Ker} j|^{-1} \\
&= \frac{|H^1(G, K^{\mathfrak{m}})| \cdot |\operatorname{Ker}\psi| \cdot |\mathrm{Im}\ \varphi|^{-1}}{|\operatorname{Ker}\varphi|} \cdot \prod_{v\notin S(\mathfrak{m})\cup S} e_v \\
&= \frac{|H^1(G, K^{\mathfrak{m}})| \cdot |\operatorname{Ker}\psi|}{|H^1(G, E_{K,S}^{\mathfrak{m}})|} \cdot \prod_{v\notin S(\mathfrak{m})\cup S} e_v \\
&= \frac{|H^1(G, K^{\mathfrak{m}})| \cdot |H^2(G, E_{K,S}^{\mathfrak{m}})|}{|H^1(G, E_{K,S}^{\mathfrak{m}})||\mathrm{Im}\ \psi|} \cdot \prod_{v\notin S(\mathfrak{m})\cup S} e_v.
\end{aligned}$$

This completes the proof of Theorem 4.1. ∎

Proof of Proposition 4.2. The facts $H^1(G,\mathbb{A}_K^\times)=H^1(G,K^\times)=0$ and $(\mathbb{A}_K^\times)^G=\mathbb{A}_k^\times$ will be used. Taking Galois cohomology of the short exact sequence

$$1\to K^{\mathfrak{m}}\to K^\times\to K^\times/K^{\mathfrak{m}}\to 1,$$

we get the exact sequence

$$1\to k^\times/k^{\mathfrak{m}}\to (K^\times/K^{\mathfrak{m}})^G\to H^1(G,K^{\mathfrak{m}})\to 1.$$

Recall that $\mathbb{A}_K^{\mathfrak{m}}=\{(a_w)_w\in\mathbb{A}_K^\times | a_w\equiv 1 \bmod \mathfrak{m} \text{ for } w\mid\mathfrak{m}\}$. We have $K^{\mathfrak{m}}=K^\times\cap\mathbb{A}_K^{\mathfrak{m}}$ and $\mathbb{A}_K^\times=K^\times\mathbb{A}_K^{\mathfrak{m}}$ by the weak approximation theorem. This gives a natural isomorphism

$$K^\times/K^{\mathfrak{m}}\cong\mathbb{A}_K^\times/\mathbb{A}_K^{\mathfrak{m}}.$$

Taking Galois cohomology of $1\to\mathbb{A}_K^{\mathfrak{m}}\to\mathbb{A}_K^\times\to\mathbb{A}_K^\times/\mathbb{A}_K^{\mathfrak{m}}\to 1$, we get the exact sequence

$$1\to\mathbb{A}_k^\times/\mathbb{A}_k^{\mathfrak{m}}\to(\mathbb{A}_K^\times/\mathbb{A}_K^{\mathfrak{m}})^G\to H^1(G,\mathbb{A}_K^{\mathfrak{m}})\to 1.$$

This implies that

$$H^1(G,K^{\mathfrak{m}})\cong H^1(G,\mathbb{A}_K^{\mathfrak{m}}).$$

(Note that this isomorphism can also be deduced from taking Galois cohomology of the short exact sequence $1\to K^{\mathfrak{m}}\to\mathbb{A}_K^{\mathfrak{m}}\to\mathbb{A}_K^{\mathfrak{m}}/K^{\mathfrak{m}}=\mathbb{A}_K^\times/K^\times\to 1$ and using the facts that $(\mathbb{A}_K^\times/K^\times)^G=\mathbb{A}_k^\times/k^\times$ and $H^1(G,\mathbb{A}_K^\times/K^\times)=0$.) Let $\mathbb{A}_K^{S(\mathfrak{m})}$ be the subgroup of $\mathbb{A}_K^\times$ such that

$$\mathbb{A}_K^\times=\mathbb{A}_K^{S(\mathfrak{m})}\times\prod_{w|\mathfrak{m}}K_w^\times\quad\text{ as a direct product.}$$

Then

$$\mathbb{A}_K^{\mathfrak{m}}=\mathbb{A}_K^{S(\mathfrak{m})}\times\prod_{w|\mathfrak{m}}1+\mathfrak{m}\mathcal{O}_w\quad\text{ and }\quad H^1(G,\mathbb{A}_K^{S(\mathfrak{m})})=0.$$

Therefore,

$$H^1(G,\mathbb{A}_K^{\mathfrak{m}})\cong\prod_{v|\mathfrak{m}}H^1(G,\prod_{w|v}1+\mathfrak{m}\mathcal{O}_w).$$

Observe that $H^1(G,\prod_{w|v}1+\mathfrak{m}\mathcal{O}_w)\cong H^1(G_v,1+\mathfrak{m}\mathcal{O}_{w_1})$ by Shapiro's Lemma, where w_1 is a place of K over v and $G_v=\mathrm{Gal}(K_{w_1}/k_v)$. This proves the first part of Proposition 4.2.

Assume that K/k is cyclic. Then the Herbrand quotient of the $\mathrm{Gal}(K_w/k_v)$-module $\mathcal{O}_w^\times$ is 1, see [9, Chapter IX, §3, Lemma 4]. Note that $1+\mathfrak{m}\mathcal{O}_w$ has finite index in $\mathcal{O}_w^\times$. We then have

$$|H^1(G_v,1+\mathfrak{m}\mathcal{O}_w)|=|H^2(G_v,1+\mathfrak{m}\mathcal{O}_w)|=[1+\mathfrak{m}\mathcal{O}_v:N_{K_w/k_v}(1+\mathfrak{m}\mathcal{O}_w)].$$

This completes the proof of Proposition 4.2. ∎

Remark 4.4. Suppose that K/k is cyclic. We can identify $H^2(G,M)$ with the Tate cohomology $\widehat{H}^2(G,M)\simeq\widehat{H}^0(G,M)$ by periodicity for any G-module M. The theorem on the Herbrand quotient of global units says (see [9, Chapter IX, §4, Corollary 2])

$$\frac{|H^2(G,E_{K,S})|}{|H^1(G,E_{K,S})|}=\frac{\prod_{v\in S}|G_v|}{[K:k]}.\tag{4.4}$$

Here G_v is the decomposition group of v and $E_{K,S}$ is the S-units of K. Note that $E_{K,S}^{\mathfrak{m}}$ has finite index in $E_{K,S}$. So they have the same Herbrand quotient. Since $H^2(G,E_{K,S}^{\mathfrak{m}})=E_{k,S}^{\mathfrak{m}}/N(E_{K,S}^{\mathfrak{m}})$ and $H^2(G,K^{\mathfrak{m}})=k^{\mathfrak{m}}/N(K^{\mathfrak{m}})$, we have

$$|\mathrm{Im}\ \psi|=|E_{k,S}^{\mathfrak{m}}N(K^{\mathfrak{m}})/N(K^{\mathfrak{m}})|=[E_{k,S}^{\mathfrak{m}}:E_{k,S}^{\mathfrak{m}}\cap N(K^{\mathfrak{m}})].$$

Thus, by Theorem 4.1 and Proposition 4.2, we obtain the ambiguous class number formula for S-ray class groups

$$\frac{|(\mathrm{Cl}^{\mathfrak{m}}_{K,S})^G|}{|\,\mathrm{Cl}^{\mathfrak{m}}_{k,S}\,|}=\frac{\prod_{v\in S} e_v f_v\cdot \prod_{v\in S(\mathfrak{m})}[1+\mathfrak{m}\mathcal{O}_v : N(\prod_{w|v}(1+\mathfrak{m}\mathcal{O}_w))]\cdot \prod_{v\notin S(\mathfrak{m})\cup S} e_v}{[K:k][E^{\mathfrak{m}}_{k,S}:E^{\mathfrak{m}}_{k,S}\cap N(K^{\mathfrak{m}})]}$$

which is the formula in Theorem 2.1 when $\mathcal{C}=0$.

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