$$\rho\frac{\partial^2\vec{u}}{\partial t^2}-(\lambda+2\mu)\vec{\nabla}(\vec{\nabla}\cdot\vec{u})+\mu\vec{\nabla}\times(\vec{\nabla}\times\vec{u})=\vec{F} \tag{1}$$

usando $\qquad \vec{\nabla}^2\vec{A}-\vec{\nabla}(\vec{\nabla}\cdot\vec{A})=-\vec{\nabla}\times(\vec{\nabla}\times\vec{A})$

$$\begin{aligned}\rho\tfrac{\partial^2\vec{u}}{\partial t^2}-(\lambda+2\mu)\vec{\nabla}(\vec{\nabla}\cdot\vec{u})+\mu(\vec{\nabla}(\vec{\nabla}\cdot\vec{u})-\vec{\nabla}^2\vec{u})=\vec{F}\\ \rho\tfrac{\partial^2\vec{u}}{\partial t^2}=(\lambda+\mu)\vec{\nabla}(\vec{\nabla}\cdot\vec{u})+\mu\vec{\nabla}^2\vec{u}+\vec{F}\end{aligned}$$

Tensorialmente

$$\boxed{\rho\frac{\partial^2 u_i}{\partial t^2}=(\lambda+\mu)\frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j}+\mu\frac{\partial^2 u_i}{\partial x_j\partial x_j}+F_i}$$

Si $F_i=\delta_{ij}\delta(t)\delta(x)$ se obtiene que la solución es $u_i=G_{ij}$. Buscamos una solución ortogonal con simetría esférica. Para ello descompondremos el problema con 2 operadores ortogonales (M^pyM^s).

Si $\vec{v}$ es un vector arbitrario $\vec{v}=\vec{v}(\vec{x},t)$

$$\begin{aligned}M^p\vec{v} &= \vec{\nabla}(\vec{\nabla}\cdot\vec{v})\\ M^s\vec{v} &= \vec{\nabla}^2\vec{v}-\vec{\nabla}(\vec{\nabla}\cdot\vec{v}) = -\vec{\nabla}\times(\vec{\nabla}\times\vec{v})\end{aligned}$$

donde

$$\begin{aligned}M^p(M^s\vec{v}) &= M^s(M^p\vec{v}) = 0 \quad \text{(ortogonalidad)}\\ M^p(M^p\vec{v}) &= M^p(\vec{\nabla}^2\vec{v})\\ M^s(M^s\vec{v}) &= M^s(\vec{\nabla}^2\vec{v})\end{aligned}$$

Reescribiendo (1) con $\boxed{\alpha^2=\frac{\lambda+2\mu}{\rho} \quad \text{y} \quad \beta^2=\frac{\mu}{\rho}}$

$$\rho\tfrac{\partial^2\vec{u}}{\partial t^2}-(\lambda+2\mu)M^p\vec{u}-\mu M^s\vec{u}=\vec{F}$$

$$\frac{\partial^2\vec{u}}{\partial t^2}=\frac{\vec{F}}{\rho}+\alpha^2M^p\vec{u}+\beta^2M^s\vec{u} \tag{2}$$

Podemos escribir $\vec{u}(\vec{x},t)=M^p\vec{A}^p(\vec{x},t)+M^s\vec{A}^s(\vec{x},t)$

donde al reemplazar en (2) y como M^p y M^s no dependen de t

$$\left(M^p\tfrac{\partial^2\vec{A}^p}{\partial t^2}+M^s\tfrac{\partial^2\vec{A}^s}{\partial t^2}\right)=\tfrac{\vec{F}}{\rho}+\alpha^2M^p(M^p\vec{A}^p+M^s\vec{A}^s)+\beta^2M^s(M^p\vec{A}^p+M^s\vec{A}^s)$$

como

$$\begin{aligned}M^k(M^k\vec{A}^k) &= M^k(\vec{\nabla}^2\vec{A}^k) \qquad \text{con} \quad k=\{s,p\}\\ M^p(M^s\vec{A}) &= M^s(M^p\vec{A}) = 0\end{aligned}$$

$$\left(M^p\frac{\partial^2\vec{A}^p}{\partial t^2}+M^s\frac{\partial^2\vec{A}^s}{\partial t^2}\right)=\frac{\vec{F}}{\rho}+\alpha^2M^p(\vec{\nabla}^2\vec{A}^p)+\beta^2M^s(\vec{\nabla}^2\vec{A}^s)$$
$$M^p\left(\alpha^2\vec{\nabla}^2\vec{A}^p-\frac{\partial^2\vec{A}^p}{\partial t^2}\right)+M^s\left(\beta^2\vec{\nabla}^2\vec{A}^s-\frac{\partial^2\vec{A}^s}{\partial t^2}\right)+\frac{\vec{F}}{\rho}=0$$

Además, $\vec{F}=\vec{f}(t)\delta(x)$; usando la relación $\nabla^2\left(\frac{-1}{4\pi r}\right)=\delta(x)$ donde $r=|\vec{x}|=\sqrt{x^2+y^2+z^2}$, $\vec{F}=\vec{f}(t)\vec{\nabla}^2\left(\frac{-1}{4\pi r}\right)=\vec{\nabla}^2\left(\frac{-\vec{f}(t)}{4\pi r}\right)$.
Así
$M^p\left(\alpha^2\vec{\nabla}^2\vec{A}^p-\frac{\partial^2\vec{A}^p}{\partial t^2}\right)+M^s\left(\beta^2\vec{\nabla}^2\vec{A}^s-\frac{\partial^2\vec{A}^s}{\partial t^2}\right)=\frac{\vec{\nabla}^2}{\rho}\left(\frac{f(t)}{4\pi r}\right)$

como
$$\begin{aligned}M^p\vec{v}+M^s\vec{v}&=\vec{\nabla}(\vec{\nabla}\cdot\vec{\nabla})+\nabla^2\vec{v}-\vec{\nabla}(\vec{\nabla}\cdot\vec{\nabla})\\(M^p+M^s)\vec{v}&=\nabla^2\vec{v}\end{aligned}$$

$\rho M^p\left(\alpha^2\vec{\nabla}^2\vec{A}^p-\frac{\partial^2\vec{A}^p}{\partial t^2}\right)+\rho M^s\left(\beta^2\vec{\nabla}^2\vec{A}^s-\frac{\partial^2\vec{A}^s}{\partial t^2}\right)=M^p\left(\frac{f(t)}{4\pi r}\right)+M^s\left(\frac{f(t)}{4\pi r}\right)$
Así, resolviendo para $\vec{A}$ que es análogo a $\vec{A}^s$ y $\vec{A}^p$ con $c^2=\{\alpha^2,\beta^2\}$).

$$c^2\vec{\nabla}^2\vec{A}-\frac{\partial^2\vec{A}}{\partial t^2}=\frac{\vec{f}(t)}{4\pi r\rho}\Big/\cdot r$$

Asumiendo simetría esférica para $\vec{A}=\vec{A}(r,t)$

$\vec{\nabla}^2\vec{A}=\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\vec{A})\to r\vec{\nabla}^2\vec{A}=\frac{\partial^2}{\partial r^2}(r\vec{A})$
$c^2\frac{\partial^2}{\partial r^2}(\vec{A}r)-r\frac{\partial^2\vec{A}}{\partial t^2}=\frac{\vec{f}(t)}{4\pi\rho};\quad(\vec{f}(t)=\ddot{\vec{p}}(t))$
$c^2\frac{\partial^2}{\partial r^2}(r\vec{A})-\frac{\partial^2}{\partial t^2}(r\vec{A})=\frac{\vec{f}(t)}{4\pi\rho}=\frac{\ddot{\vec{p}}(t)}{4\pi\rho}$

Ec. de onda inhomogénea para $r\vec{A}$

$$r\vec{A}=\underbrace{\vec{U}_1\left(t-\frac{r}{c}\right)+\vec{U}_2\left(t+\frac{r}{c}\right)}_{\text{Sol. Homogénea}}-\underbrace{\frac{\vec{p}(t)}{4\pi\rho}}_{\text{Sol.Particular}}$$

$\vec{U}_2=\vec{0}$ pues las ondas salen desde el origen!
y para evitar singularidades en $r\to 0$

$0\cdot\vec{A}=\vec{0}=\vec{U}_1(t)=\frac{\vec{p}(t)}{4\pi\rho}\to\vec{U}_1\left(t-\frac{r}{c}\right)=\frac{\vec{p}\left(t-\frac{r}{c}\right)}{4\pi\rho}$
Así, $\vec{A}(r,t)=\frac{\vec{p}\left(t-\frac{r}{c}\right)-\vec{p}(t)}{4\pi\rho r}$
$\vec{A}^p=\frac{\vec{p}\left(t-\frac{r}{\alpha}\right)-\vec{p}(t)}{4\pi\rho r};\vec{A}^s=\frac{\vec{p}\left(t-\frac{r}{\beta}\right)-\vec{p}(t)}{4\pi\rho r}$

donde $\vec{u} = M^p\vec{A}^p + M^s\vec{A}^s$
$M^p\vec{v} = \vec{\nabla}(\vec{\nabla}\cdot\vec{v}); M^s\vec{v} = \vec{\nabla}^2\vec{v} - \vec{\nabla}(\vec{\nabla}\cdot\vec{v})$
$\vec{u} = \vec{\nabla}\left(\vec{\nabla}\cdot\left(\frac{\vec{p}\left(t-\frac{r}{\alpha}\right)-\vec{p}(t)}{4\pi\rho r}\right)\right) + \vec{\nabla}^2\left(\frac{\vec{p}\left(t-\frac{r}{\beta}\right)-\vec{p}(t)}{4\pi\rho r}\right) - \vec{\nabla}\left(\vec{\nabla}\cdot\left(\frac{\vec{p}\left(t-\frac{r}{\beta}\right)-\vec{p}(t)}{4\pi\rho r}\right)\right)$
$\vec{u} = -\vec{\nabla}^2\left(\frac{\vec{p}(t)}{4\pi\rho r}\right) + (\vec{\nabla}^2 - \vec{\nabla}(\vec{\nabla}\cdot))\left(\frac{\vec{p}\left(t-\frac{r}{\beta}\right)}{4\pi\rho r}\right) - \vec{\nabla}\left(\vec{\nabla}\cdot\left(\frac{\vec{p}\left(t-\frac{r}{\alpha}\right)}{4\pi\rho r}\right)\right)$
donde recordamos $r = |\vec{x}| = \sqrt{x^2+y^2+z^2}$.

Reescribiendo en forma tensorial

$u_i = -\delta_{ij}\nabla^2\left(\frac{P_j(t)}{4\pi\rho r}\right) + \left(\delta_{ij}\nabla^2 - \frac{\partial^2}{\partial x_i\partial x_j}\right)\left(\frac{P_j\left(t-\frac{r}{\beta}\right)}{4\pi\rho r}\right) + \frac{\partial^2}{\partial x_i\partial x_j}\left(\frac{P_j\left(t-\frac{r}{\alpha}\right)}{4\pi\rho r}\right)$
Donde $\delta_{ij}\nabla^2\left(\frac{P_j(t)}{4\pi\rho r}\right) = P_j(t)\frac{\partial^2}{\partial x_j\partial x_j}\left(\frac{1}{4\pi\rho r}\right)$
$= \frac{P_j(t)}{4\pi\rho}\left(\frac{\partial^2}{\partial x_j\partial x_j}\left(\frac{1}{r}\right)\right)$
$= \frac{P_j(t)}{4\pi\rho}\left(\frac{3\gamma_j\gamma_j-\delta_{ij}}{r^3}\right) = 0 \quad ; \quad r > 0$

donde $\gamma_j\gamma_j = 1 = \gamma_x^2+\gamma_y^2+\gamma_z^2 = \frac{x^2+y^2+z^2}{r^2}$
$\delta_{jj} = 1+1+1$ (convenio de Einstein)

Así, $u_i = \left(\delta_{ij}\nabla^2 - \frac{\partial^2}{\partial x_i\partial x_j}\right)\left(\frac{P_j\left(t-\frac{r}{\beta}\right)}{4\pi\rho r}\right) + \frac{\partial^2}{\partial x_i\partial x_j}\left(\frac{P_j\left(t-\frac{r}{\alpha}\right)}{4\pi\rho r}\right)$
$u_i = (\delta_{ij}-\gamma_i\gamma_j)\frac{\ddot{P}_j\left(t-\frac{r}{\beta}\right)}{4\pi\rho\beta^2 r} + \frac{\gamma_i\gamma_j}{4\pi\rho r\alpha^2}\ddot{P}_j\left(t-\frac{r}{\alpha}\right) + \frac{P_j\left(t-\frac{r}{\alpha}\right)-P_j\left(t-\frac{r}{\beta}\right)}{4\pi\rho}\frac{\partial^2}{\partial x_i\partial x_j}\left(\frac{1}{r}\right)$

donde $\frac{\partial^2}{\partial x_i\partial x_j}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x_i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x_j}\right)$

$= \frac{2}{r^3}\frac{\partial r}{\partial x_i}\frac{\partial r}{\partial x_j} - \frac{1}{r^2}\frac{\partial}{\partial x_i}\left(\frac{\partial r}{\partial x_j}\right)$
$= \frac{2}{r^3}\gamma_i\gamma_j - \frac{1}{r^2}\left(\frac{\partial}{\partial x_i}\left(\frac{x_j}{r}\right)\right)$
$= \frac{2}{r^3}\gamma_i\gamma_j - \frac{1}{r^2}\left(\frac{\partial x_j}{\partial x_i}\left(\frac{1}{r}\right) - \frac{x_j}{r^2}\frac{\partial r}{\partial x_i}\right)$
$= \frac{2}{r^3}\gamma_i\gamma_j - \frac{1}{r^3}\delta_{ij} + \frac{1}{r^3}\gamma_j\gamma_i$
$= \frac{3\gamma_i\gamma_j-\delta_{ij}}{r^3}$
$u_i = (\delta_{ij}-\gamma_i\gamma_j)\frac{1}{4\pi\rho\beta^2 r}\ddot{P}_j\left(t-\frac{r}{\beta}\right) + \frac{\gamma_i\gamma_j}{4\pi\rho\alpha^2 r}\ddot{P}_j\left(t-\frac{r}{\alpha}\right) + \left(\frac{3\gamma_i\gamma_j-\delta_{ij}}{4\pi\rho r^3}\right)\left(P_j\left(t-\frac{r}{\alpha}\right) - P_j\left(t-\frac{r}{\beta}\right)\right)$
Para hallar G_{ij} basta considerar

$$u_i = G_{ij} \quad \leftrightarrow \quad f_i = \delta_{ij}\delta(t)\delta(\vec{r})$$
$$f_i = \delta_{ij}\delta(t)$$

Así $P_i(t) = \delta_{ij} R(t)$
$R(t) = \begin{cases} t & ; \quad t > 0 \\ 0 & ; \quad t < 0 \end{cases} \rightarrow \ddot{P}_i(t) = \delta(t)$

Finalmente tenemos lo siguiente,

$$G_{ij} = -\tfrac{(-\delta_{ij}+\gamma_i\gamma_j)}{4\pi\rho\beta^2 r}\delta\left(t - \tfrac{r}{\beta}\right) + \tfrac{\gamma_i\gamma_j}{4\pi\rho\alpha^2 r}\delta\left(t - \tfrac{r}{\alpha}\right) +$$

$$\left(\tfrac{3\gamma_i\gamma_j - \delta_{ij}}{4\pi\rho r^3}\right) \underbrace{\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}} \tau\delta(1-\tau)d\tau}_{t} \quad \rightarrow \boxed{\text{Usando la representación integral de } R(t)}$$