

Solutions to the Olympiad Hamilton Paper

H1. The five-digit integer ‘ $a679b$ ’ is a multiple of 72.

What are the values of a and b ?

Solution

Since $72 = 8 \times 9$ any number that is a multiple of 72 is both a multiple of 8 and a multiple of 9. Conversely, because 8 and 9 have no common factor greater than 1 and $8 \times 9 = 72$, any number that is both a multiple of 8 and a multiple of 9 is a multiple of 72.

A number is a multiple of 8 if, and only if, its last three digits form a multiple of 8. Hence ‘ $79b$ ’ is a multiple of 8. Since 800 is a multiple of 8, the only number of the required form is 792, and thus $b = 2$.

A number is a multiple of 9 if, and only if, the sum of its digits is a multiple of 9. Hence $a + 6 + 7 + 9 + b$ is a multiple of 9. Because b is 2, it follows that $a + 24$ is a multiple of 9, and hence $a = 3$.

Hence $a = 3$ and $b = 2$.

H2. In Vegetable Village it costs 75 pence to buy 2 potatoes, 3 carrots and 2 leeks at the Outdoor Market, whereas it costs 88 pence to buy 2 potatoes, 3 carrots and 2 leeks at the Indoor Market.

To buy a potato, a carrot and a leek costs 6 pence more at the Indoor Market than it does at the Outdoor Market.

What is the difference, in pence, between the cost of buying a carrot at the Outdoor Market and the cost of buying a carrot at the Indoor Market?

Solution

Suppose that a potato costs p pence more at the Indoor Market than at the Outdoor Market, that a carrot costs c pence more, and that a leek costs k pence more. Note that some or all of p , c and k may be negative.

Then from the first sentence we obtain

$$\begin{aligned} 2p + 3c + 2k &= 88 - 75 \\ &= 13, \end{aligned} \tag{1}$$

and from the second sentence we have

$$p + c + k = 6,$$

so that

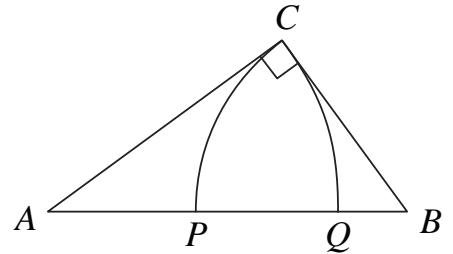
$$2p + 2c + 2k = 12. \tag{2}$$

Subtracting equation (2) from equation (1), we get $c = 1$.

Hence the cost of buying a carrot at the Indoor Market is one penny more than buying a carrot at the Outdoor Market.

- H3.** The diagram shows two circular arcs CP and CQ in a right-angled triangle ABC , where $\angle BCA = 90^\circ$. The centres of the arcs are A and B .

Prove that $\frac{1}{2}PQ^2 = AP \times BQ$.



Solution

Let BC , CA and AB have lengths a , b and c respectively. Now triangle ABC is right-angled, so applying Pythagoras' Theorem, we get

$$a^2 + b^2 = c^2.$$

Since A and B are the centres of the arcs, we have $BP = a$ and $AQ = b$, and hence $AP = c - a$ and $BQ = c - b$. Therefore

$$\begin{aligned} AP \times BQ &= (c - a)(c - b) \\ &= c^2 - ac - bc + ab. \end{aligned}$$

Furthermore,

$$\begin{aligned} PQ &= AB - AP - BQ \\ &= c - (c - a) - (c - b) \\ &= a + b - c. \end{aligned}$$

Hence

$$\begin{aligned} PQ^2 &= (a + b - c)^2 \\ &= a^2 + b^2 + c^2 + 2ab - 2ac - 2bc. \end{aligned}$$

So, from $a^2 + b^2 = c^2$, we get

$$\begin{aligned} PQ^2 &= 2c^2 + 2ab - 2ac - 2bc \\ &= 2 \times AP \times BQ. \end{aligned}$$

Thus $\frac{1}{2}PQ^2 = AP \times BQ$, as required.

- H4.** The points A , B and C are the centres of three faces of a cuboid that meet at a vertex. The lengths of the sides of the triangle ABC are 4, 5 and 6.

What is the volume of the cuboid?

Solution

Let the cuboid have centre O and dimensions $2x \times 2y \times 2z$, as shown. Joining O to each of A , B and C forms three right-angled triangles with sides x , y and 4; y , z and 5; and z , x and 6. Hence, using Pythagoras' Theorem, we get

$$x^2 + y^2 = 4^2$$

$$y^2 + z^2 = 5^2$$

$$\text{and } z^2 + x^2 = 6^2.$$

Adding all three equations, we obtain

$$\begin{aligned} 2x^2 + 2y^2 + 2z^2 &= 16 + 25 + 36 \\ &= 77 \end{aligned}$$

so that

$$x^2 + y^2 + z^2 = \frac{77}{2}.$$

Subtracting each of the first three equations in turn, we therefore get

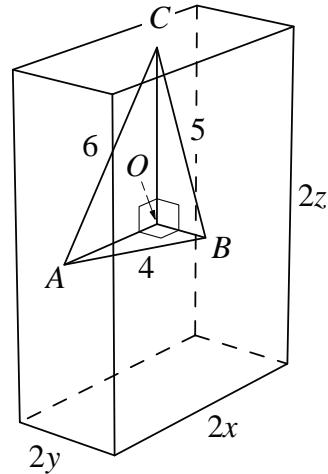
$$z^2 = \frac{77}{2} - 16 = \frac{45}{2}$$

$$x^2 = \frac{77}{2} - 25 = \frac{27}{2}$$

$$\text{and } y^2 = \frac{77}{2} - 36 = \frac{5}{2}.$$

Now $2x$ is equal to $\sqrt{4x^2}$ (because x is positive), with similar results for y and z . So the volume of the cuboid $2x \times 2y \times 2z$ is therefore

$$\begin{aligned} \sqrt{54} \times \sqrt{10} \times \sqrt{90} &= \sqrt{54 \times 10 \times 90} \\ &= \sqrt{6 \times 90 \times 90} \\ &= 90\sqrt{6}. \end{aligned}$$



- H5.** Some boys and girls are standing in a row, in some order, about to be photographed. All of them are facing the photographer. Each girl counts the number of boys to her left, and each boy counts the number of girls to his right.

Let the sum of the numbers counted by the girls be G , and the sum of the numbers counted by the boys be B .

Prove that $G = B$.

Solution

Consider one of the girls. For each boy to her left, draw an arc connecting the two, as indicated in the following figure (which is shown from the photographer's point of view).



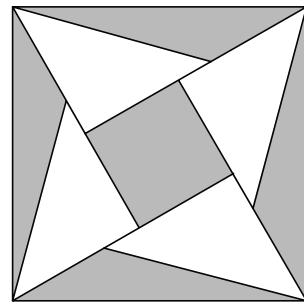
Thus one arc is drawn for each boy that she counts. We repeat this for every girl, and as a result G arcs are drawn altogether.

Now consider adopting the same procedure for the boys. The arcs that would be drawn for the boys are exactly the same as the arcs drawn for the girls. Indeed, in the above figure, the arc corresponds either to a girl with a boy to her left, or to a boy with a girl to his right.

It follows that $G = B$ since both count the total number of arcs.

- H6.** The diagram shows four identical white triangles symmetrically placed on a grey square. Each triangle is isosceles and right-angled. The total area of the visible grey regions is equal to the total area of the white triangles.

What is the smallest angle in each of the (identical) grey triangles in the diagram?



Solution

Choose the unit of measurement so that the two equal sides of each white triangle have length 1. Then the total area of the four white triangles is 2, because they can be assembled to form two squares of side 1.

Therefore the total area of the visible grey regions is also equal to 2, and hence the area of the original grey square is 4. Thus the sides of the grey square have length 2.

Now consider one of the triangles formed by one white triangle and an adjacent grey triangle, such as triangle ABC in Figure 1.

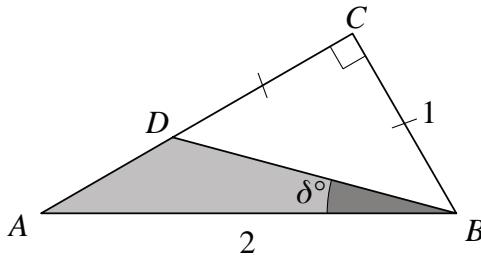


Figure 1

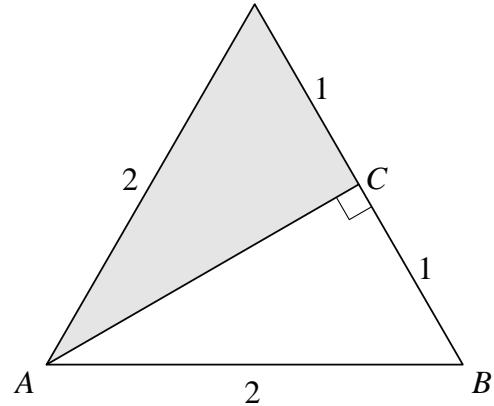


Figure 2

Figure 1 indicates what we know: the white triangle BCD is isosceles and right-angled; the length of AB is 2; and the length of BC is 1. We are asked to find the value of δ° , which we shall find as the difference between $\angle ABC$ and $\angle DBC$.

Each of the equal angles in an isosceles right-angled triangle is 45° (from ‘base angles of an isosceles triangle’ and ‘the sum of the angles in a triangle is 180° ’). Therefore $\angle DBC = 45^\circ$.

In triangle ABC , which is right-angled, the length of the hypotenuse is 2, and another side has length 1. It follows that triangle ABC is half an equilateral triangle, as shown in Figure 2. But each angle in an equilateral triangle is 60° , so that $\angle ABC = 60^\circ$.

Hence $\delta = 60 - 45$, and therefore the smallest angle in each grey triangle is 15° .