

Hamilton Olympiad 2018 Solutions

- H1.** The positive integers m and n satisfy the equation $20m + 18n = 2018$. How many possible values of m are there?

Comment

There are many approaches to this question. It is straightforward to show that, given a solution $m = M$ and $n = N$, the pair $m = (M - 9k)$ and $n = (N + 10k)$ is also a solution for any integer value of k . This allows one to ‘spot’ a solution and generate a set of solutions from that one.

However, if we use this approach, it is important to show that there aren't any other solutions which are not in this set. (Indeed, in any question where we are trying to find all possible solutions, finding them is only half the battle – a key step is to show that we have found them *all*.)

The first of these solutions shows a way to complete the argument introduced above; the second demonstrates a slightly different approach.

Solution 1

Let M and N be a solution to the given equation, so $20M + 18N = 2018$.

For any integer k , the pair $m = (M - 9k)$ and $n = (N + 10k)$ is also a solution, since

$$20(M - 9k) + 18(N + 10k) = 20M - 180k + 18N + 180k = 2018.$$

It is easy to verify (and not too difficult to spot) that $m = 100, n = 1$ is a solution.

This quickly leads us to the following twelve solutions, which are all of the form $m = 100 - 9k, n = 1 + 10k$ for $k = 0, 1, 2, \dots, 11$:

m	100	91	82	73	64	55	46	37	28	19	10	1
n	1	11	21	31	41	51	61	71	81	91	101	111

We will now show that *any* solution to the given equation *must* be of the form $m = 100 - 9k, n = 1 + 10k$.

Let (M, N) be a pair of integers for which $20M + 18N = 2018$.

Since $20 \times 100 + 18 \times 1 = 2018$, we can subtract to give

$$20(100 - M) + 18(1 - N) = 0.$$

Then

$$10(100 - M) = 9(N - 1). \tag{*}$$

Since 10 and 9 have no common factor greater than 1, $(100 - M)$ must be a multiple of 9, so we can write $100 - M = 9k$ for some integer k . This leads to $M = 100 - 9k$.

Then we have, from (*), $90k = 9(N - 1)$, which leads to $N = 1 + 10k$.

So we have now shown that any pair of integers (M, N) which are a solution to the equation must be of the form $(100 - 9k, 1 + 10k)$ for some integer k .

For $100 - 9k$ to be positive we need $k \leq 11$ and for $1 + 10k$ to be positive we need $k \geq 0$, so the twelve solutions listed above are the only twelve.

Solution 2

If $20m + 18n = 2018$ then

$$10m + 9n = 1009$$

$$10m = 1009 - 9n$$

$$10m = 1010 - (9n + 1)$$

$$m = 101 - \frac{9n + 1}{10}.$$

Since m is an integer, $9n + 1$ must be a multiple of 10, which means that $9n$ must be one less than a multiple of 10. For this to be true, the final digit of $9n$ must be 9, which happens exactly when the final digit of n is 1.

Every value of n with final digit 1 will produce a corresponding value of m . No other value of n is possible (and hence no other possible values of m possible either).

So now we just need to count the number of positive values of n with final digit 1 which also give a positive value of m .

Since m is positive, $\frac{9n + 1}{10}$ can be at most 100, with equality when $n = 111$.

Hence $0 < n \leq 111$. There are 12 values of n in this range which have final digit 1, and so there are 12 possible values of m .

[*Note that we have not found any of the solutions to the given equation; we did not need to do this in order to answer the question.*]

- H2.** How many nine-digit integers of the form ‘ $pqrpqrpqr$ ’ are multiples of 24?
(Note that p , q and r need not be different.)

Comment

When attempting to solve questions like this one, it is natural to turn to divisibility tests (which are often learned right at the start of secondary school). One key property of these tests is that they work in both directions: for example, all numbers which are divisible by 8 have their final three digits divisible by 8, and conversely all numbers with final three digits divisible by 8 are themselves divisible by 8. In the first solution below, this property is essential – without it we would not be able to guarantee that we have found all results.

Solution 1

In order to be a multiple of 24, any number must be both a multiple of 3 and a multiple of 8. Conversely, since 8 and 3 are *coprime* (i.e. they have no common factor greater than 1), any number which is a multiple of both 3 and 8 must be a multiple of 24. It is therefore sufficient for us to count all the numbers of the required form which are multiples of both 3 and 8.

The digit sum of ‘ $pqrpqrpqr$ ’ is $3(p + q + r)$, which is always a multiple of 3, so all numbers of the form ‘ $pqrpqrpqr$ ’ are divisible by 3 (by the divisibility test for 3).

So now all we need to do is to count all the numbers of the required form which are multiples of 8, since they will also be a multiple of 3 and hence a multiple of 24.

By the divisibility test for 8, we need the three-digit number ‘ pqr ’ to be divisible by 8, and each three-digit multiple of 8 will give us a nine-digit number divisible by 8. So all that remains is to count the three-digit multiples of 8.

The smallest of these is 104 ($= 13 \times 8$) and the largest is 992 ($= 124 \times 8$), so there are 112 three-digit multiples of 8 and therefore 112 numbers of the form ‘ $pqrpqrpqr$ ’ which are multiples of 24.

Solution 2

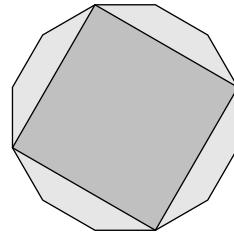
As in Solution 1, we can establish that we need ‘ $pqrpqrpqr$ ’ to be a multiple of both 3 and 8.

Notice that ‘ $pqrpqrpqr$ ’ $= 1001001 \times pqr$. 1001001 is divisible by 3 so every number of the form is ‘ $pqrpqrpqr$ ’ divisible by 3. Since 1001001 is odd, we need the three-digit number ‘ pqr ’ to be divisible by 8.

There are 124 multiples of 8 less than 1000; of these 12 are less than 100, so there are 112 three-digit multiples of 8 and hence 112 numbers with the properties given in the question.

- H3.** The diagram shows a regular dodecagon and a square, whose vertices are also vertices of the dodecagon.

What is the value of the ratio
area of the square : area of the dodecagon?



Comment

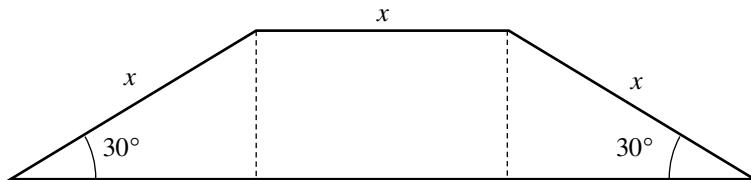
There are several ways to attempt this question. If you know the formula $\frac{1}{2}ab \sin C$ for the area of a triangle, the result can be arrived at very quickly (see Solution 2). The majority of candidates offered a method similar to that described in Solution 1.

Solution 1

Let the side length of the dodecagon be x . Consider one of the four trapezia created by the edge of the square and the edges of the dodecagon.

The interior angle of a dodecagon is 150° , which quickly leads to the fact that the angles at the base of the trapezium are both equal to 30° .

We can then split the trapezium up into a rectangle and two right-angled triangles:



Then, by recognising that each of the triangles in the trapezium is half an equilateral triangle (or by using the well-known values of $\sin 30^\circ$ and $\cos 30^\circ$), we can establish that the height of this trapezium is $x/2$ and the base is $x + x\sqrt{3}$.

The area of the square is therefore $(x + x\sqrt{3})^2 = 2x^2(2 + \sqrt{3})$ and the area of the dodecagon is equal to (the area of the square) + (4 × the area of the trapezium) which is

$$2x^2(2 + \sqrt{3}) + 4 \times \frac{x + (x + x\sqrt{3})}{2} \times \frac{x}{2} = 3x^2(2 + \sqrt{3}).$$

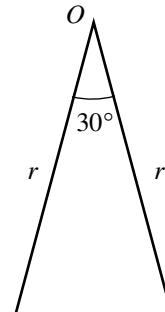
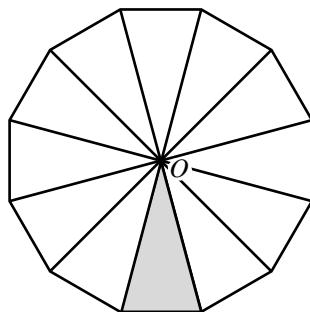
We can divide through by the common factor of $x^2(2 + \sqrt{3})$ to show that the required ratio is 2: 3.

Solution 2

Since the dodecagon is regular, all its diagonals intersect at the same point (its centre). Call this point O . Since the diagonals of the square are two of the diagonals of the dodecagon, they intersect at O . Hence O is also the centre of the square.

Let each diagonal have length $2r$.

Then the dodecagon comprises 12 congruent isosceles triangles radiating from O , each with two sides equal to r and vertex angle 30° :



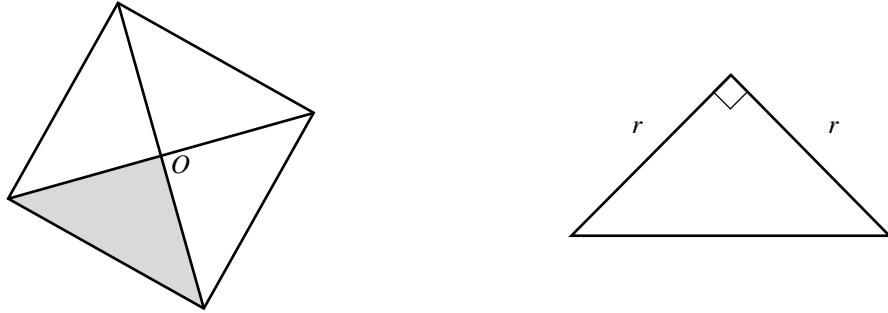
The area of the dodecagon is equal to $12 \times$ the area of one of these triangles, which is equal to

$$12 \times \left[\frac{1}{2} \times r \times r \times \sin 30^\circ \right].$$

Since $\sin 30^\circ = \frac{1}{2}$, this simplifies to

$$\text{Area of dodecagon} = 3r^2.$$

The square can be split into four congruent isosceles right-angled triangles, with perpendicular sides of length r .



The area of the square is equal to $4 \times$ the area of one of these triangles, which is equal to

$$4 \times \left[\frac{1}{2} \times r \times r \right],$$

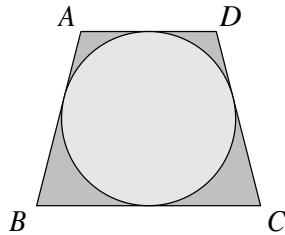
which simplifies to

$$\text{Area of square} = 2r^2.$$

The required ratio is therefore $2r^2 : 3r^2$ which is equal to $2 : 3$.

- H4.** The diagram shows a circle and a trapezium $ABCD$ in which AD is parallel to BC and $AB = DC$. All four sides of $ABCD$ are tangents of the circle. The circle has radius 4 and the area of $ABCD$ is 72.

What is the length of AB ?



Comment

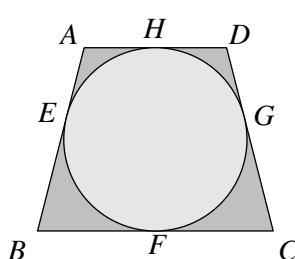
The solution to this question is dependent on the circle theorem which states that the two tangents to a circle from any external point are equal in length. The proof of this is straightforward (join the point to the centre of the circle and find two triangles which are congruent) – if you haven't seen it before you may want to work it through to satisfy yourself that it is indeed true.

Solution

Note that the height of the trapezium is equal to the diameter of the circle = 8.

Since the area of the trapezium is 72, we have $\frac{AD + BC}{2} \times 8 = 72$ and so $AD + BC = 18$.

Let E, F, G and H be the points on AB, BC, CD and DA respectively where the circle is tangent to the sides of the trapezium, as shown in the diagram below.



Since the two tangents to a circle from any external point are equal in length, we know that $AE = AH$.

Similarly, $BE = BF$, $CF = CG$ and $DG = DH$.

Then $AB + DC = (AE + BE) + (CG + DG) = AH + BF + CF + DH = AD + BC$, which we know is equal to 18.

Since $AB = DC$, we have $AB + DC = 2AB = 18$ and hence $AB = 9$.

- H5.** A two-digit number is divided by the sum of its digits. The result is a number between 2.6 and 2.7.

Find all of the possible values of the original two-digit number.

Solution

Let the number be ‘ ab ’ (where $a \neq 0$), which can be written as $10a + b$.

The value we are concerned with is $\frac{10a + b}{a + b}$.

We are given that $2.6 < \frac{10a + b}{a + b} < 2.7$.

Since $a + b$ must be positive (since both a and b are positive), we can multiply through by $(a + b)$ without changing the direction of the inequality signs, to give:

$$2.6(a + b) < 10a + b < 2.7(a + b).$$

The first of these two inequalities leads to

$$1.6b < 7.4a,$$

which can be rearranged to give

$$b < \frac{37}{8}a = 4\frac{5}{8}a. \quad (1)$$

The second inequality leads to

$$7.3a < 1.7b,$$

which can be rearranged to give

$$b > \frac{73}{17}a = 4\frac{5}{17}a. \quad (2)$$

From (2), we can deduce that $b > 4a$, so (since b is a single digit) a can be at most 2.

If $a = 1$, combining (1) and (2) gives $4\frac{5}{17} < b < 4\frac{5}{8}$, which is impossible (since b is a digit and hence an integer).

If $a = 2$, combining (1) and (2) gives $8\frac{10}{17} < b < 9\frac{1}{4}$, which gives $b = 9$.

These values of a and b give the number 29.

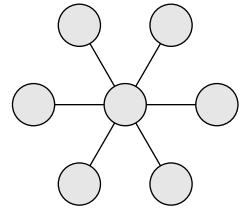
We must check that 29 does have the property we are looking for, which we can do by dividing directly:

$$\frac{29}{11} = 2\frac{7}{11} = 2.6363\dots,$$

which is indeed between 2.6 and 2.7, so the only possible value of the two-digit number is 29.

- H6.** The figure shows seven circles joined by three straight lines.

The numbers 9, 12, 18, 24, 36, 48 and 96 are to be placed into the circles, one in each, so that the product of the three numbers on each of the three lines is the same.



Which of the numbers could go in the centre?

Comment

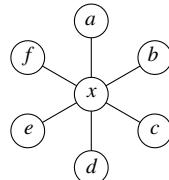
It is possible to argue that, once the centre number has been selected, the other six must be paired ‘largest with smallest’ etc. – but a complete convincing argument can be difficult to assemble. The method below demonstrates an alternative way (which was chosen by the majority of successful candidates) to show the only possible solutions. It is worth remembering that showing that 12 and 96 are the only centre numbers that could possibly work does not fully answer the question – we must show that arrangements are indeed possible (and this is easily done by direct construction).

Solution

Since we are interested in the products of various numbers, it seems natural to find the prime factorisations of the seven numbers we are given:

Number	9	12	18	24	36	48	96
Prime Factorisation	3^2	$2^2 \times 3$	2×3^2	$2^3 \times 3$	$2^2 \times 3^2$	$2^4 \times 3$	$2^5 \times 3$

Let the six numbers around the edge of the figure be a , b , c , d , e and f , and the central number x (as in the diagram below).



Then we need $a \times x \times d = b \times x \times e = c \times x \times f$. Since $x \neq 0$, we can divide by x , giving $ad = be = cf$.

Now consider the number $abcdef$. This must be a cube (since it is equal to, for example, $(ad)^3$). This number can also be written as $\frac{abcdef}{x}$, which is useful since $abcdefx$ is just the product of the seven numbers we have been given. Hence $abcdef = \frac{2^{17} \times 3^{10}}{x}$, and, if we write x as $2^m \times 3^n$, we have $abcdef = 2^{17-m} \times 3^{10-n}$.

Note that, since x must be one of the given numbers, there are only a few possible values for m and n ; in particular $m \leq 5$ and $n \leq 2$.

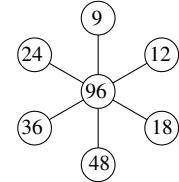
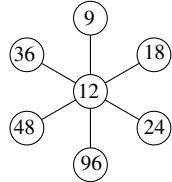
Since $abcdef$ is a cube, 2^{17-m} and 3^{10-n} must both be cubes too, which means that both $17 - m$ and $10 - n$ must be multiples of 3. Hence m could be equal to 2 or 5, and n must be equal to 1.

Hence $(m, n) = (2, 1)$ which gives $x = 12$, or $(m, n) = (5, 1)$ which gives $x = 96$. As mentioned in the comment above, we must now check that each of these cases enables the shape to be filled as required by the question:

When $x = 12$, we have $abcdef = 2^{15} \times 3^9$, so each pair of outside numbers must multiply to $2^5 \times 3^3$.

When $x = 96$, we have $abcdef = 2^{12} \times 3^9$, so each pair of outside numbers must multiply to $2^4 \times 3^3$.

In each case, a small amount of experimentation quickly leads to arrangements that work in each case, such as:



Hence both cases are possible and there are two numbers that could go in the centre space; namely 12 and 96.