Hard-margin SVM primal form $\min_{w,b} \frac{1}{2}\|w\|_2^2$ ①

s.t. $y_i(w^T x_i + b) \geq 1, \forall i$

Dual form derive

According to Lagrangian : $\min_x f(x)$ ② s.t. $g_i(x) \leq 0$

Lagrangian function $L(x, \{u_i\}) = f(x) + \sum_i u_i g_i(x)$

from ①, ② : $f(x) = \frac{1}{2}\|w\|_2^2$, $g_i(x) = 1 - y_i(w^T x_i + b)$

hence, Lagrangian function $L(w, b, \{u_i\}) = \frac{1}{2}\|w\|_2^2 + \sum_i u_i(1 - y_i(w^T x_i + b))$ ③

when $g_i(x) = 1 - y_i(w^T x_i + b) > 0$, then $\max_{u_i}(L(w, b, \{u_i\})) = \frac{1}{2}\|w\|_2^2 + \infty = \infty$ ④

when $g_i(x) = 1 - y_i(w^T x_i + b) \leq 0$, then $\max_{u_i}(L(w, b, \{u_i\})) = \frac{1}{2}\|w\|_2^2 + 0 = \frac{1}{2}\|w\|_2^2$ ⑤

combine ④, ⑤ : $\min_{w,b} \max_{u_i} L(w, b, \{u_i\}) = \min_{w,b} \{\infty, \frac{1}{2}\|w\|_2^2\} = \min_{w,b} \frac{1}{2}\|w\|_2^2$

hence, $\min_{w,b} \frac{1}{2}\|w\|_2^2$ s.t $y_i(w^T x_i + b) \geq 1$ same as $\min_{w,b} \max_{u_i} L(w, b, u_i)$ s.t $u_i \geq 0$ ⑥

derivative in ⑥ : $\frac{\partial L(w, b, u_i)}{\partial w} = w - \sum_i u_i y_i x_i = 0 \Rightarrow w = \sum_i u_i y_i x_i$ ⑦

$\frac{\partial L(w, b, u_i)}{\partial b} = \sum_i u_i y_i = 0$ ⑧

combine ③ ⑥ ⑦ ⑧ : $\min_{w,b} \max_{u_i} L(w, b, u_i) = \max_{u_i}\left(\frac{1}{2}(\sum_i u_i y_i x_i)^2 + \sum_i u_i(1 - y_i((\sum_i u_i y_i x_i^T) x_i + b))\right)$

$= \max_{u_i}\left(\frac{1}{2}(\sum_i u_i y_i x_i^T)(\sum_i u_i y_i x_i) + \sum_i u_i - (\sum_i u_i y_i x_i^T)(\sum_i u_i y_i x_i) - \sum_i u_i y_i b\right)$

$= \max_{u_i}\left(-\frac{1}{2}(\sum_i u_i y_i x_i^T)(\sum_i u_i y_i x_i) + \sum_i u_i\right) = \min_{u_i}\left(\frac{1}{2}(\sum_i u_i y_i x_i^T)(\sum_i u_i y_i x_i) - \sum_i u_i\right)$

hence, hard-margin svm dual formation is $\min_{u_i} \left(\frac{1}{2} (\sum_i u_i y_i x_i^T)(\sum_i u_i y_i x_i) - \sum_i u_i \right)$

s.t $u_i \geq 0$

$\sum_i u_i y_i = 0$