General Direction-of-Arrival Estimation: A Signal Subspace Approach

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A high resolution algorithm is presented for resolving multiple incoherent and coherent plane waves that are incident on an array of sensors. The incident sources may be a mixture of narrowband and broadband sources, and, the geometry of the array is unrestricted. This algorithm makes use of a fundamental property possessed by those eigenvectors of the array spectral density matrix that are associated with eigenvalues that are larger than the sensor noise level. Specifically, it is shown that these eigenvectors may each be represented as linear combinations of the steering vectors identifying the incident plane waves. This property is then used to solve the important special cases of: 1) incoherent sources incident on a general array, and 2) coherent sources incident on an equispaced linear array. Simulation results are then presented to illustrate the high resolution performance achieved with this new approach relative to that obtained with MUSIC and spatial smoothed MUSIC in which the CSS focusing method is employed.

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I. INTRODUCTION

A number of array signal processing algorithms have been proposed to solve the classical direction-of-arrival problem. Two of the more widely employed of these techniques are beam forming and a general class of procedures that utilize the eigen-characterization of the array spatial correlation matrix. These include so-called noise-eigenvector-based methods as developed by Pisarenko [17], Liggett [16], Schmidt [19, 20], Johnson and Degraaf [14], Wax, Shan, and Kailath [26], Kumaresan and Tufts [15], and Cadzow [4]. Noise eigenvector methods are attractive due to their high resolution, asymptotically unbiased estimate capability in the case of uncorrelated or partially correlated sources. This class of estimation methods was first proposed for narrowband sources and makes use of the (noise) eigenvectors corresponding to the smallest eigenvalues of the spatial covariance matrix. These methods have been shown to provide higher resolution direction-of-arrival estimates than that achieved by conventional beamforming, Capon's MLM, and AR methods [12].

In practical direction-of-arrival problems, significant difficulties arise when the signals are highly correlated (e.g., multipath propagation and smart jammers). To overcome the deleterious effects due to coherent sources, a number of approaches have been proposed for narrowband signal processing. These include a computationally demanding modification of the MUSIC algorithm as proposed by Schmidt [20, 21], the decorrelating methods of Gabriel [13], and Widrow, Duvall, Gooch, and Newman [28], and the spatial smoothed MUSIC method as introduced by Evans, Johnson and Sun [11] and further developed by Shan, Wax, and Kailath [22]. For differing reasons, each of these noise-eigenvector-based methods provide a less than satisfactory resolution of the coherency problem. For example, spatial smoothing leads to an effective decrease in aperture size and therefore a decrease in resolution capability. The signal-eigenvector-based methods of Cadzow [7] and Reddi [18] provide another vehicle for countering coherent source difficulties. Cadzow's method is applicable to general array geometries while the Reddi approach is restricted to equispaced linear arrays. Methods which are not based on eigenvectors have also been proposed for resolving the coherent narrowband array problem. For example, maximum likelihood and Di method [9] which uses the Vandermonde structure of the steering matrix to effect a coherent detection capability.

The superiority of the noise subspace approach for noncoherent narrowband sources made it a logical candidate for use in broadband signal direction-of-arrival estimation problems. A temporal-spatial approach was proposed by Coker and Ferrara [8] and by Bienvenu [1] under the assumption that the spectral density matrix of the source signals is known. Wax, Shan, and Kailath divided the broadband frequency into nonoverlapping narrowbands, and then narrowband noise subspace processing was performed on each band [27]. An averaging procedure was then made to combine the multiple individual narrowband frequency results to effect a single direction-of-arrival estimate. This approach is computationally demanding since it involves an eigenvalue-eigenvector decomposition of the spectral density matrix at each frequency band being analyzed. A rational noise subspace theory was introduced by Su and Morf as a generalization of the one-dimensional signal subspace approach [23]. It is often the case that in broadband signal transmission, the individual narrowband signal-to-noise ratios (SNRs) are not sufficient for a resolution fo closely spaced sources. Thus, Wang and Kaveh showed that by using frequency domain averaging, high resolution direction-of-arrival estimation for multiple broadband sources was achievable [24, 25]. Their method is based on the eigendecomposition of a frequency domain combination of modified spectral density matrix estimates for a number of narrowbands in the temporal frequency domain. They recognized the main difficulty in developing coherent signal subspace processing for broadband sources arises from the fact that the signal subspace at one frequency is different from that at another frequency. Their basic idea was to construct a single signal subspace by using a transformation matrix which translates the signal subspaces for all the frequency bands into a common one.

A signal eigenvector method for direction-of-arrival estimation that circumvents the difficulties arising from narrowband coherent signals was recently proposed by Cadzow [7]. This method is generally superior to noise-eigenvector-based methods in terms of resolution, bias, and variance of direction-of-arrival estimates for coherent sources. In this paper, the signal eigenvector method is extended to treat the case in which the incident plane waves may be a combination of broadband and narrowband sources that are coherent and (or) incoherent. This method in conjunction with the Wang-Kaveh focusing concept is found to provide effective direction-of-arrival estimates.

In Section II, it is shown that a spectral domain characterization of the sensor signals is more appropriate than a time domain characterization for general nonarrowband sources. An eigenanalysis of the spectral density matrix of the array is undertaken in Sections III and IV. Section V then examines noise-eigenvector-based methods and establishes their vulnerability to coherent sources. The signal eigenvector method is then presented in Section VI and it is shown to be effective for both incoherent and coherent sources. Sections VII and VIII provide a computationally effective implementation of the signal eigenvector method for the special case of equispaced linear arrays. Up to this point, it has been assumed that the underlying array and noise spectral density matrices are given. The more practical case in which only sampled values of the sensor signals are available for making direction-of-arrival estimates is treated in Section IX. In Section X, the Wang-Kaveh CSS focusing method is incorporated to effectively combine the direction-of-arrival information from several frequencies. Finally, simulation results are given in Section XI that illustrate the improved direction-of-arrival estimates that are achieved by the signal eigenvector algorithm method relative to the spatial smoothed MUSIC and the Wang-Kaveh methods.

II. SIGNAL MODEL FORMULATION

To solve the direction-of-arrival problem, the underlying plane wave directions are obtained by modeling the delay pattern that appears across the sensors of the array. Various modeling techniques that address the so-called narrowband array direction-of-arrival problem exist in which the signals present are modeled as being essentially sinusoidal [13, 18]. For narrowband methods, propagating delays are generally characterized as producing phase shifts among the sensor outputs. On the other hand, broadband (nonnarrowband) signals propagation delays are often estimated by locating peaks in crosscovariance functions among the sensors [22].

The distinction between signals which can be considered as narrowband and those which may not is very subtle. For example, the classification of a signal as being narrowband is dependent on both the bandwidth of the signal and the physical size of the array. Thus, a given signal may be narrowband for one array and yet broadband for another array. Since the algorithm to be used in the direction-of-arrival estimation is often linked to its bandwidth characterization, this distinction is fundamental. Narrowband estimation techniques are generally associated with spatially small arrays, while broadband or time difference of arrival (TDOA) methods are more often associated with spatially large arrays. In addition, it is to be noted that applications exist (e.g., sonar) where both narrowband and broadband signals are simultaneously present.

In the analysis to follow, it is assumed that there is given an array of M omnidirectional sensors (i.e., unity gain and zero phase shift in all directions) with the individual sensors being located at the points $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_M$ in real three space. Furthermore, any plane wave incident on the array is assumed to be characterized by the two properties: 1) the primary energy in any incident plane wave is contained within a frequency band of width $2\omega_c$ centered at ω_0, and 2) the medium's propagation velocity is equal to the constant

c in this frequency band. Under these two assumptions, it follows that the individual sensor signals arising from a single incident plane wave are time-shifted versions of one another. Specifically, the signal received at the mth sensor from a single incident plane wave is represented as

$$\tilde{x}_m(t) = f(t+\tau(m))\cos(\omega_0(t+\tau(m))+\phi) \quad \text{for} \quad 1 \le m \le M. \tag{1}$$

In this relationship, $f(t)$ is the baseband source (envelope) signal whose bandwidth ω_c corresponds to half the incident plane wave's bandwidth, ϕ is the associated phase angle, and $\tau(m) = \mathbf{k}'\mathbf{z}_m/c$ corresponds to the delay of sensor m relative to the origin of real three space. The prime symbol (′) denotes vector transposition. The real 3×1 unit vector $\mathbf{k}$ designates the direction of the incident plane wave.

Since signal processing operations are best carried out at baseband frequencies, this is achieved by performing the standard frequency down shifting operation on the received sensor signals. This frequency downshifting operation entails multiplying each of the received sensor signals by the sinusoidal signals $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$ and then lowpass filtering these M product signal pairs. The bandwidth of the lowpass filters employed should be selected to at least equal the bandwidth of the source signal $f(t)$. Under this restriction, the outputs of the two lowpass filters are given by the *in-phase* component $f(t+\tau(m))\cos(\omega_0\tau(m)+\phi)$ and the *quadrature* component $f(t+\tau(m))\sin(\omega_0\tau(m)+\phi)$ for $1 \le m \le M$. It is convenient to represent this signal pair as the complex time function

$$x_m(t) = f(t+\tau(m))e^{j(\omega_0\tau(m)+\phi)} \qquad \text{for} \quad 1 \le m \le M \tag{2}$$

in which the real and imaginary parts are recognized as being the in-phase and quadrature components, respectively. To treat the narrowband and broadband source cases in a unified manner, it is beneficial to analyze these sensor signals in the frequency domain. This formally entails taking the Fourier transform of these time domain signals and results in

$$X_m(\omega) = F(\omega)e^{j\phi}e^{j(\omega+\omega_0)\tau(m)} \qquad \text{for} \quad 1 \le m \le M \tag{3}$$

where $F(\omega)$ denotes the Fourier transform of the source signal $f(t)$.

Although modeling relationships (2) and (3) are equivalent, the former is often used when analyzing narrowband plane waves. A plane wave is said to be narrowband if the approximation $f(t+\tau(m)) = f(t)$ for $1 \le m \le M$ is suitably accurate for all values of time t. It can be shown that this narrowband approximation is accurate provided that $\omega_c \ll c/d$ where d designates the maximum distance between any two array sensors. In such cases, one may replace the terms $f(t+\tau(m))$ by $f(t)$ in relationship (2) and then use these simplified equations to form effective estimates of the $\tau(m)$ delays. These delay estimates can in turn be used to estimate the direction-of-arrival of the plane wave. In this paper, however, this restrictive narrowband assumption is not made. To treat the general case in which the incident plane waves can be a combination of narrowband and broadband plane waves, it is prudent to use the spectral characterization (3).

With these ideas in mind, let us now examine the situation in which there are N incident plane waves traveling in directions $\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_N$ with associated source signals $f_1(t), f_2(t), \ldots, f_N(t)$. Appealing to the principle of superposition, the mth array sensor signal is specified by the linear combination

$$X_m(\omega) = \sum_{n=1}^{N} F_n(\omega)e^{j\phi_n}e^{j((\omega+\omega_0)\tau_n(m))} \qquad \text{for} \quad 1 \le m \le M \tag{4}$$

where $F_n(\omega)$ denotes the Fourier transform of the nth source signal $f_n(t)$ while the parameter $\tau_n(m)$ corresponds to the delay which the nth plane wave is received at sensor m relative to the origin of three space and is given by

$$\tau_n(m) = \mathbf{k}_n'\mathbf{z}_m/c \qquad \text{for} \quad 1 \le m \le M \quad \text{and} \quad 1 \le n \le N. \tag{5}$$

These delay parameters are fundamental to the direction-of-arrival problem since they characterize the combined array geometry-incident plane wave direction-of-arrival information.

In the analysis to follow, a vector space approach is taken for analyzing the direction-of-arrival problem and forming an algorithm for its solution. With this objective in mind, let us express relationships (4) as the $M \times 1$ spectral *array signal vector*

$$\mathbf{x}(\omega) = [X_1(\omega), X_2(\omega), \ldots, X_m(\omega)]'. \tag{6}$$

The lower case letter $\mathbf{x}(\omega)$ (instead of $\mathbf{X}(\omega)$ is here used to designate a vector quantity in keeping with standard vector notation. From relationship (4) it is seen that this sensor signals are linear combinations of the spectral source signal terms $F_n(\omega)e^{j\phi_n}$. It is then convenient to express this linear combination as

$$\mathbf{x}(\omega) = S(\omega)\mathbf{f}(\omega). \tag{7}$$

In this representation, $\mathbf{f}(\omega)$ is the $N \times 1$ spectral source vector as given by

$$\mathbf{f}(\omega) = [F_1(\omega)e^{j\phi_1}, F_2(\omega)e^{j\phi_2}, \ldots, F_N(\omega)e^{j\phi_N}]'. \tag{8}$$

The $M \times N$ *composite steering matrix* $S(\omega)$ appearing in representation (7) plays a prominent role in the developments to follow and its columns are composed of the $M \times 1$ *steering vectors*

$$\mathbf{s}(\omega, \mathbf{k}_n) = [e^{j(\omega+\omega_0)\tau_n(1)}, e^{j(\omega+\omega_0)\tau_n(2)}, \ldots, e^{j(\omega+\omega_0)\tau_n(M)}]' \quad \text{for} \quad 1 \leq n \leq N. \quad (9)$$

The nth steering vector is associated with the nth incident plane wave and is a function of the delay terms $\tau_n(m)$ which from expression (5) are seen to be dependent on both the geometry of the array and the plane waves direction vector $\mathbf{k}_n$. It is for this reason that the steering vector is expressed as an explicit function of the direction vector of the plane wave.

Relationship (7) provides an accurate model of the sensor signal of the array under the restriction that the two assumptions made earlier in this section are valid. It must be noted that in most practical applications, the measured sensor signals are corrupted by environmental noise and instrumentation error. If this corruption enters in an additive fashion, the array signal vector is modeled as

$$\mathbf{x}(\omega) = S(\omega)\mathbf{f}(\omega) + \eta(\omega) \quad (10)$$

in which the mth element of the $M \times 1$ *noise vector* $\eta(\omega)$ designates the Fourier transform of the mth sensor noise signal $\eta_m(t)$. This signal model is similar to that obtained by Wax, et al. [26] where a discrete frequency characterization was made. The direction-of-arrival problem is concerned with using the noise-corrupted sensor signals (10) to effect an estimate of the time delays $\tau_n(m)$. These time delays are in turn used to estimate the plane waves direction vectors $\mathbf{k}_n$.

III. SECOND-ORDER STATISTICS OF ARRAY SIGNAL VECTOR

The second-order statistics of the array's spectral signal vector provide a particularly insightful and useful tool for estimating the number of incident plane waves and their direction-of-arrivals. It is assumed that the plane wave signals and additive noises are zero mean wide-sense stationary complex-valued random processes which are pairwise uncorrelated. The spectral density matrix $P_x(\omega)$ is formally specified by

$$P_x(\omega) = E\{\mathbf{x}(\omega)\mathbf{x}(\omega)^*\}. \quad (11)$$

It can be shown that this spectral density matrix is equal to the Fourier transform of the array correlation matrix $R_x(\tau) = E\{\mathbf{x}(t+\tau)\mathbf{x}(t)^*\}$ where $\mathbf{x}(t)$ designates the $M \times 1$ temporal snapshot vector whose components are given in expression (2).

Upon substitution of expression (10) into relationship (11), the spectral density matrix is found to be

$$\begin{aligned} P_x(\omega) &= S(\omega)E\{\mathbf{f}(\omega)\mathbf{f}(\omega)^*\}S(\omega)^* + E\{\eta(\omega)\eta(\omega)^*\} \\ &= S(\omega)P_f(\omega)S(\omega)^* + \sigma^2(\omega)P_\eta(\omega). \quad (12) \end{aligned}$$

The $N \times N$ *source spectral density matrix* $P_f(\omega)$ appearing in this expression is given by

$$P_f(\omega) = E\{\mathbf{f}(\omega)\mathbf{f}(\omega)^*\}. \quad (13)$$

This spectral density matrix characterizes the second-order statistical relationship between the N incident plane wave source signals taken as pairs at frequency ω. The N source signals are said to be pairwise *noncoherent* at frequency ω if the rank of $P_f(\omega)$ equals N. If $P_f(\omega)$ should have a less than full rank, this indicates that a subset of the source signals are pairwise *coherent* (e.g., multipath effects) at frequency ω. The $M \times M$ *noise spectral density matrix* $\sigma^2 P_\eta(\omega)$ appearing in expression (12) is specified by

$$\sigma^2(\omega)P_\eta(\omega) = E\{\eta(\omega)\eta(\omega)^*\}. \quad (14)$$

In what is to follow, it is assumed that matrix $P_\eta(\omega)$ is known and has been normalized so that its trace equals M but that the noise power level $\sigma^2(\omega)$ is unknown. The general fundamental direction-of-arrival problem can now be formulated.

Direction-of-Arrival Problem: Let the array spectral density matrix be specified by $P_x(\omega) = S(\omega)P_f(\omega)S(\omega)^* + \sigma^2(\omega)P_\eta(\omega)$. Given the spectral density matrices $P_x(\omega)$ and $P_\eta(\omega)$, develop a procedure for recovering the constituent entities N, $\sigma^2(\omega)$, $S(\omega)$, and $P_f(\omega)$. It is assumed that the number of incident plane waves N is unknown but that the plane wave direction-of-arrivals are such as to ensure that the $M \times N$ composite steering matrix $S(\omega)$ has full rank N.

In solving this direction-of-arrival problem, the approach to be taken entails first determining the noise level $\sigma^2(\omega)$ and the composite steering matrix $S(\omega)$. Once these entities have been determined, the source spectral density matrix $P_f(\omega)$ is obtained using the relationship

$$\begin{aligned} P_f(\omega) = {} & [S(\omega)^*S(\omega)]^{-1}S(\omega)^* \\ & \times [P_x(\omega) - \sigma^2(\omega)P_\eta(\omega)]S(\omega)[S(\omega)^*S(\omega)]^{-1} \end{aligned} \quad (15)$$

as derived from expression (12). We now develop an effective procedure for solving the direction-of-arrival problem. As suggested earlier, vector space techniques play an important role in this investigation.

IV. EIGENANALYSIS OF ARRAY SPECTRAL DENSITY MATRIX

A solution to the fundamental direction-of-arrival problem posed in the last section is readily obtained by employing the concept of generalized eigenvectors. In particular, we first make a generalized eigendecomposition of the matrix pair $(P_x(\omega), P_\eta(\omega))$ as formally defined by

$$P_x(\omega)\mathbf{e}_m(\omega) = \lambda_m(\omega)P_\eta(\omega)\mathbf{e}_m(\omega) \qquad \text{for} \quad 1 \le m \le M \tag{16}$$

in which it is noted that the spectral density matrices $P_x(\omega)$ and $P_\eta(\omega)$ are positive definite. Without loss of generality, the eigenvalues are here ordered in the monotonically nonincreasing fashion $\lambda_m(\omega) \ge \lambda_{m+1}(\omega)$ for $1 \le m \le M$. The distribution of these eigenvalues plays an important role in the solution procedure to be developed. The following Lemma provides insight as to why this is so.

LEMMA 1. *Let the $M \times N$ composite steering matrix $S(\omega)$ have rank N, the $N \times N$ source spectral density matrix $P_f(\omega)$ have rank K where $K \le N$, and, the $M \times M$ noise spectral density matrix $P_\eta(\omega)$ have rank M. Furthermore, let the number of incident plane waves N be less than the number of sensors M. It then follows that the eigenvalues as specified in relationship* (16) *are distributed as*

$$\lambda_1(\omega) \ge \cdots \ge \lambda_K(\omega) > \lambda_{K+1}(\omega) = \cdots = \lambda_M(\omega) = \sigma^2(\omega). \tag{17}$$

PROOF. A Proof of this Lemma is obtained by noting from expression (16) that any generalized eigenvector associated with the smallest generalized eigenvalue $\sigma^2(\omega)$ is also a standard eigenvector associated with the zero eigenvalue of matrix $S(\omega)P_f(\omega)S(\omega)^*$, that is

$$\begin{aligned} P_x(\omega)\mathbf{e}_m(\omega) &= \sigma^2(\omega)P_\eta(\omega)\mathbf{e}_m(\omega) \\ &\leftrightarrow S(\omega)P_f(\omega)S(\omega)^*\mathbf{e}_m(\omega) = \mathbf{0} \end{aligned} \tag{18}$$

Thus, a generalized eigenvector of $P_x(\omega)$ associated with eigenvalue $\sigma^2(\omega)$ lies in the null space of $S(\omega)P_f(\omega)S(\omega)^*$. It follows from the rank restrictions placed on the matrices $S(\omega)$ and $P_f(\omega)$ that this null space has dimension $M - K$ and Lemma 1 therefore follows.

Signal and Noise Matrices

As previously indicated, eigenvector-based approaches play a prominent role in many contemporary array processing algorithms. In keeping with this importance, it is to be noted that the generalized eigenvector set

$$\{\mathbf{e}_1(\omega), \mathbf{e}_2(\omega), \ldots, \mathbf{e}_K(\omega), \mathbf{e}_{K+1}(\omega), \ldots, \mathbf{e}_M(\omega)\} \tag{19}$$

constitutes a basis for C^M. The K left most generalized eigenvectors appearing in this basis are seen to correspond to those eigenvalues which are larger than the sensor noise power $\sigma^2(\omega)$. These generalized eigenvectors are hereafter referred to as signal eigenvectors. These signal eigenvectors are used to form the columns of the $M \times K$ *signal matrix* $E_s(\omega)$ where

$$E_s(\omega) = [\mathbf{e}_1(\omega) \vdots \mathbf{e}_2(\omega) \vdots \ldots \vdots \mathbf{e}_K(\omega)]. \tag{20}$$

In a similar fashion, the $M - K$ right-most generalized eigenvectors appearing in the above basis are seen to correspond to the smallest eigenvalue $\sigma^2(\omega)$ which is equal to the sensor noise power. As such, these generalized eigenvectors are commonly referred to as noise eigenvectors. Furthermore, these noise eigenvectors form the columns of the $M \times (M - K)$ *noise matrix* $E_\eta(\omega)$ as given by

$$E_\eta(\omega) = [\mathbf{e}_{K+1}(\omega) \vdots \mathbf{e}_{K+2}(\omega) \vdots \ldots \vdots \mathbf{e}_M(\omega)]. \tag{21}$$

It is to be noted that the signal and noise matrices are $P_\eta(\omega)$-orthogonal in the sense that $E_\eta(\omega)^* P_\eta(\omega) E_s(\omega) = 0$.

V. NOISE-EIGENVECTOR-BASED METHODS

By definition, N incident plane waves are said to be *incoherent* at frequency ω if their associated $N \times N$ source spectral density matrix $P_f(\omega)$ is diagonal and full rank. If this full rank matrix is nondiagonal, the incident plane waves are said to be partially incoherent. On the other hand, the incident plane waves are said to be *coherent* if $P_f(\omega)$ has a less than full rank. The term coherent implies that a subset of the N source signals at frequency ω are linearly related to one another. Coherent sources can give rise to poor performance when noise eigenvector based algorithms are used to solve the fundamental direction-of-arrival problem. The following Theorem indicates why this is the case.

THEOREM 1. *Let $N \le M - 1$ plane waves impinge on an array composed of M sensors in which the associated $M \times N$ composite steering matrix $S(\omega)$ has rank N where $N < M$, the $N \times N$ source spectral density matrix $P_f(\omega)$ has rank K where $K \le N$, and, the $M \times M$ noise spectral density matrix $P_\eta(\omega)$ has rank M. The noise eigenvectors associated with the $M - K$ noise eigenvalues*

$$\lambda_{K+1}(\omega) = \lambda_{K+2}(\omega) = \cdots = \lambda_M(\omega) = \sigma^2(\omega) \tag{22}$$

of eigenrelationship (16) *take on two distinctive characteristics depending on whether the source spectral density matrix $P_f(\omega)$ has a full or less than full rank.*
Noncoherent Sources (Rank $[P_f(\omega)] = N$). *The noise eigenvectors associated with the noise eigenvalue $\sigma^2(\omega)$ form a basis for the null space of matrix $S(\omega)^*$. Furthermore, the steering vectors $\mathbf{s}(\omega, \mathbf{k}_n)$ as specified in expression* (9) *each lie in the null space of the conjugated transposed noise matrix* (21), *that is*

$$E_\eta(\omega)^* \mathbf{s}(\omega, \mathbf{k}_n) = \mathbf{0} \qquad \textit{for} \quad 1 \le n \le N \tag{23}$$

Coherent Sources (Rank $[P_f(\omega)] = K < N$). *The noise eigenvectors associated with the noise eigenvalues $\sigma^2(\omega)$ of multiplicity $M-K$ generally do not lie in the null space of $S(\omega)^*$, that is, $E_\eta(\omega)^* \mathbf{s}(\omega,\mathbf{k}_n) \neq \mathbf{0}$ typically holds.*

The noise eigenvalue multiplicity (22) was established in Lemma 1 and followed from the fact that any noise eigenvalue associated eigenvector must lie in the null space of $S(\omega)P_f(\omega)S(\omega)^*$ as denoted by $\mathcal{N}[S(\omega)P_f(\omega)S(\omega)^*]$. The dimension of this null space (i.e., its nullity) is equal to the sum of dimensions of the mutually orthogonal subspaces $\mathcal{N}[S(\omega)^*]$ and $\mathcal{N}[S(\omega)^*]^\perp \cap \mathcal{N}[P_f(\omega)S(\omega)^*]$.[1] Since $S(\omega)$ has full rank, it follows that the dimensions of $\mathcal{N}[S(\omega)^*]$ and $\mathcal{N}[S(\omega)^*]^\perp \cap \mathcal{N}[P_f(\omega)S(\omega)^*]$ are $M-N$ and $N-K$, respectively, and relationship (22) directly follows. To establish (23), it is noted that when $P_f(\omega)$ has full rank, the subspace $\mathcal{N}[S(\omega)^*]^\perp \cap \mathcal{N}[P_f(\omega)S(\omega)^*]$ is empty. We therefore concluded that every eigenvector associated with the noise eigenvalue $\sigma^2(\omega)$ lies in $\mathcal{N}[S(\omega)^*]$ and relationship (23) therefore follows. Finally, when $P_f(\omega)$ has a less than full rank, the subspace $\mathcal{N}[S(\omega)^*]^\perp \cap \mathcal{N}[P_f(\omega)S(\omega)^*]$ is not empty. It therefore follows that a noise eigenvector need not be exclusively contained in the subspace $\mathcal{N}[S(\omega)^*]$.

Expression (23) provides a general procedure for solving the fundamental direction-of-arrival problem when the source spectral density matrix $P_f(\omega)$ is full rank. This entails first determining the $M-N$ generalized eigenvectors associated with the smallest (noise) eigenvalue $\sigma^2(\omega)$ of eigenexpression (16). The $M \times (M-N)$ noise matrix $E_\eta(\omega)$ is next formed. Finally, the direction-of-arrival function

$$d(\mathbf{k}) = 1/\mathbf{s}(\omega,\mathbf{k})^* E_\eta(\omega) W E_\eta(\omega)^* \mathbf{s}(\omega,\mathbf{k}) \qquad (24)$$

is evaluated at all direction vectors $\mathbf{k}$ lying on the unit sphere in the real three space. The $(M-N)\times(M-N)$ matrix W here appearing is required to be positive definite. In accordance with relationship (23), this function becomes infinite at those direction vectors corresponding to the directions of incident plane waves. Thus, peaks in the function $d(\mathbf{k})$ are used to identify incident plane waves. It is to be noted that this general direction-of-arrival function corresponds to the MUSIC algorithm when $W = I$ [20, 21].

Unfortunately, relationship (23) is generally invalid if a subset of the N incident plane waves are coherent at frequency ω. Noise-eigenvector-based methods therefore typically provide relatively inferior performance in such cases. When the impinging plane waves are highly, though not perfectly correlated, it is also found that noise-eigenvector-methods display an undesirable sensitivity to data inaccuracies. Fortunately, the signal-eigenvector-approach to be described in the Section VI forms a particularly useful vehicle for solving the direction-of-arrival problem independent of whether the incident plane waves are incoherent, partially coherent, or perfectly coherent.

[1] The subspace $\mathcal{N}[S(\omega)^*]^\perp$ designates the set of all vectors in C^M which are orthogonal to every vector in $\mathcal{N}[S(\omega)^*]$.

VI. SIGNAL EIGENVECTOR METHOD

From the preceding discussion, it is apparent that the generalized eigenvectors associated with the noise eigenvalues of the matrix pair $(P_x(\omega), P_\eta(\omega))$ provide an unreliable means for detecting and identifying coherent sources. The question naturally arises as to whether those generalized eigenvectors associated with the larger eigenvalues can be used more effectively for this purpose. As now formally shown, the answer is in the affirmative.

THEOREM 2. *Let $N \leq M-1$ plane waves impinge on an array composed of M sensors in which the $M \times N$ composite steering matrix $S(\omega)$ has rank N, the $N \times N$ source spectral density matrix $P_f(\omega)$ has rank K where $K \leq N$, and, the $M \times M$ noise spectral density matrix $P_\eta(\omega)$ has rank M. The signal eigenvectors associated with the K signal eigenvalues $\lambda_1(\omega) \geq \lambda_2(\omega) \geq \cdots \geq \lambda_K(\omega) > \sigma^2(\omega)$ of eigenrelationship (16) each satisfy a linear relationship of the form*

$$P_\eta(\omega)\mathbf{e}_k(\omega) = \sum_{n=1}^{N} \alpha_k(n)\mathbf{s}(\omega,\mathbf{k}_n), \qquad \text{for} \quad 1 \leq k \leq K \qquad (25)$$

where the $\mathbf{s}(\omega,\mathbf{k}_n)$ are the $M \times 1$ steering vectors as specified in expression (9).

PROOF. A Proof of this Theorem follows from (16) whereby the generalized eigenvectors associated with the signal eigenvalues are identified with the standard eigenvectors associated with positive eigenvalues of matrix $[P_\eta(\omega)]^{-1}S(\omega)P_f(\omega)S(\omega)^*$. Linear relationship (25) follows directly from this observation. This provides a useful characterization for the array processing problem in which the geometry of the array is unrestricted and the impinging sources may be coherent. When the array is linear equispaced, however, this characterization simplifies to that obtained by Reddi [18]. Reddi used the Vandermonde structure of the composite steering matrix in arriving at his results. That approach is not applicable to general array geometries.

A Solution Procedure for Incoherent Sources Incident on General Array

A variety of approaches for solving the direction-of-arrival problem are suggested by Theorem 2. These methods entail in some manner the ability to solve the system of K nonlinear equations (25) for the direction vectors $\mathbf{k}_n$ for $1 \leq n \leq N$. For instance, nonlinear programming methods could be employed for this purpose. This approach has been taken elsewhere to achieve an effective iterative solution procedure [5, 6]. Unfortunately, this iterative method can be computationally demanding and it may be therefore not useful in certain applications requiring fast tracking capabilities. When the incident sources are incoherent, or, the array is equispaced linear, it is possible to obtain computationally efficient solution procedures whose performance have been found to exceed that of MUSIC and other methods. In this section, we examine the case of incoherent incident sources.

The solution procedure to be developed for the incoherent incident source case is predicated upon examining all possible subarrays consisting of p sensor where $p \leq M$. Each such subarray may be identified with an element of the integer set

$$N^{(p)} = \{\mathbf{n} = (n_1, n_2, \ldots, n_p) : 1 \leq n_1 < n_2 < \cdots < n_p \leq M\}. \tag{26}$$

Namely, every vector $\mathbf{n} \in N^{(p)}$ is to be identified with a specific subarray consisting of the sensor elements $(n_1, n_2, \ldots, n_p)$. It is apparent that the number of p sensor subarrays is specified by $M!/(M-p)!\,p!$.

Associated with each p sensor subarray as identified by the vector $\mathbf{n} \in N^{(p)}$ let us introduce the $p \times M$ matrix whose (m,n)th entry is specified by

$$I_{\mathbf{n}}^{(p)}(m,n) = \delta(n - n_m), \qquad \text{for} \quad 1 \leq m \leq p, \qquad \text{for} \quad 1 \leq n \leq M. \tag{27}$$

The mth row of this matrix is seen to be composed of $M-1$ zeros and a single one which appears in column n_m. Its primary function will be to extract those components of the modified signal eigenvectors $P_\eta(\omega)\mathbf{e}_k(\omega)$ for $1 \leq k \leq K$ which correspond to the specific subarray as identified by $\mathbf{n}$. With these preliminary developments completed, the following fundamental Theorem is offered.

THEOREM 3. *Let $N \leq M-1$ noncoherent plane waves impinge on an array composed of M sensors. Furthermore, let the associated generalized eigenvector decomposition* (16) *be used to form the $M \times N$ signal matrix $E_s(\omega)$ as specified by expression* (20)*. It then follows that for $p \geq N+1$, there always exists a vector $\mathbf{n} \in N^{(p)}$ such that the $p \times N$ matrix $I_{\mathbf{n}}^{(p)} P_\eta(\omega) E_s(\omega)$ has rank N. If $\mathbf{a}_{\mathbf{n}}(\omega)$ designates any nontrivial $p \times 1$ vector satisfying*

$$E_s(\omega)^* P_\eta(\omega)^* I_{\mathbf{n}}^{(p)'} \mathbf{a}_{\mathbf{n}}(\omega) = \mathbf{0} \tag{28}$$

it follows that

$$f_{\mathbf{n}}(\omega, \mathbf{k}) = |\mathbf{a}_{\mathbf{n}}(\omega)^* I_{\mathbf{n}}^{(p)} \mathbf{s}(\omega, \mathbf{k})| = 0, \qquad \text{for} \quad \mathbf{k} = \mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_N \tag{29}$$

where $\mathbf{s}(\omega, \mathbf{k})$ is the general $M \times 1$ steering vector (9).

PROOF. To prove this Theorem, it is first observed that since $P_\eta(\omega)E_s(\omega)$ has rank N, $P_\eta(\omega)E_s(\omega)$ must possess N rows which are linearly independent. Thus, there exists an $\mathbf{n} \in N^{(p)}$ such that $I_{\mathbf{n}}^{(p)} E_s(\omega)$ has rank N. The existence of a vector $\mathbf{a}_{\mathbf{n}}(\omega)$ satisfying (28) is assured since $p > N$. The vector $I_{\mathbf{n}}^{(p)'} \mathbf{a}_{\mathbf{n}}(\omega)$ therefore lies in the null space of $E_s(\omega)^* P_\eta(\omega)^*$. Since the range spaces of the matrices $P_\eta(\omega)E_s(\omega)$ and $S(\omega)$ are equal, however, relationship (28) directly follows. When the sources are restricted to be partially coherent and the array is linear equispaced, Reddi's results and Theorem 3 simplify to that proposed by Kumaresan and Tufts [15].

It is noted that similar to the standard MUSIC algorithm, the solution procedure as described in Theorem 3 requires that the incident sources not be coherent. It has been shown that the proposed signal eigenvector method often provides superior direction-of-arrival estimates relative to MUSIC algorithm for narrowband sources [5]. This observation and the fact that the MUSIC algorithm is used so extensively in array processing demonstrates the importance of Theorem 3. Furthermore, a method for efficiently solving the mixed noncoherent-coherent narrowband problem has been developed [6]. This more general solution approach employs a specific orthogonal decomposition of the vector space C^M and its derivation is too long to be given here. Fortunately, its adoption for specific array geometries is quite straightforward. This point is illustrated in Section VII where the important case of equispaced linear arrays with coherent sources is considered in detail. Similar straightforward implementations may be made for circular and rectangular arrays.

The basic results of Theorem 3 may be incorporated to obtain effective direction-of-arrival estimates. To achieve a desirable degree of data smoothing and to reduce the possibility of spurious peaks, one could use the functionals $f_{N-1}(\omega, \mathbf{k})$, $f_{N-2}(\omega, \mathbf{k}), \ldots, f_M(\omega, \mathbf{k})$ in an integrated fashion (e.g., their sum or product) to affect alternate estimates. For example, one might consider the functional

$$g^{(p)}(\omega, \mathbf{k}) = \sum_{\mathbf{n}} f_{\mathbf{n}}(\omega, \mathbf{k}), \qquad \text{for} \quad p \geq N+1 \tag{30}$$

where the summation is to be carried out over all $\mathbf{n} \in N^{(p)}$ such that $I_{\mathbf{n}}^{(p)} E_s$ has rank N. It is a simple matter to show that

$$g^{(p)}(\omega, \mathbf{k}) = 0, \qquad \text{for} \quad \mathbf{k} = \mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_N. \tag{31}$$

Thus, zeros in this functional serve the role of detecting and identifying incident plane waves.

VII. LINEAR ARRAY AND COHERENT SOURCES

Although Theorem 3 is applicable to partially coherent and incoherent sources, it is possible to use the results of Theorem 2 to treat the important special case of coherent sources incident on an equispaced linear array. To develop a solution for this case, we consider the case of an array which is composed of M equispaced sensors that are located on a line in three space. Without loss of generality, the sensors are taken to lie on the x-axis at locations $\mathbf{z}_m = [(m-1)d,0,0]'$ for $1 \le m \le M$ where d designates the spacing between adjacent sensors. The steering vectors associated with this sensor geometry are from relationship (9) given by

$$\mathbf{s}(\omega,\mathbf{k}_n) = [1, e^{(j(\omega+\omega_0)d\cos(\theta_n)/c)}, \ldots, e^{(j(\omega+\omega_0)d(M-1)\cos(\theta_n)/c)}]', \quad \text{for} \quad 1 \le n \le N. \tag{32}$$

These steering vectors are complex sinusoids whose radian frequencies are $(\omega+\omega_0)d\cos(\theta_n)/c$ where θ_n designates the bearing angle relative to the linear array. From relationship (25), it then follows that the vectors $P_\eta(\omega)\mathbf{e}_n(\omega)$ for $1 \le k \le K$ are each a sum of the complex sinusoids (32). The frequencies of these sinusoids identify the direction-of-arrivals of the incident plane waves. We may therefore use any of a variety of sinusoidal modeling techniques to identify these constituent sinusoid frequencies.

In the analysis to follow, it is convenient to consider the modified signal eigenvectors

$$\mathbf{v}_k(\omega) = P_\eta(\omega)\mathbf{e}_k(\omega), \qquad \text{for} \quad 1 \le k \le K. \tag{33}$$

According to relationship (25), each of these modified signal eigenvectors is composed of a linear combination of the N sinusoidal steering vectors (32). A variety of approaches for solving this direction-of-arrival problem are thereby suggested. For instance, we could simply fit a complex sinusoidal model to each of the K modified signal eigenvectors (33) where it is noted that each of these signal eigenvectors share the same N sinusoidal components (32). In this modeling, it is assumed that the value for N is known or can be deduced from the eigenanalysis. A procedure for so identifying N is subsequently given. With N identified, we may then apply the standard forward-backward linear prediction method to identify the N spacial sinusoids. The forward-backward linear prediction equations resulting from this approach yield the following system of homogeneous linear equations

$$V_k(\omega)\mathbf{a}(\omega) = \mathbf{0}, \qquad \text{for} \quad 1 \le k \le K. \tag{34}$$

In this expression, $\mathbf{a}(\omega)$ is an $(N+1) \times 1$ solution vector of the form

$$\mathbf{a}(\omega) = [1, a_1, \ldots, \mathbf{a}_N]'$$

and $V_k(\omega)$ is a $2(M-N) \times (N+1)$ matrix which is composed of the nth modified signal eigenvector components according to

$$V_k(\omega;i,j) = \begin{cases} v_k(\omega; N+i-j+1), \\ \quad 1 \le i \le M-N; \\ \quad 1 \le j \le N+1 \\ \overline{v}_k(\omega; i+j-M+N-1), \\ \quad M-N+1 \le i \le 2(M-N); \\ \quad 1 \le j \le N+1 \end{cases} \tag{35}$$

for $1 \le k \le K$. In this representation, $\mathbf{v}_k(\omega,k)$ denotes the kth component of the modified signal eigenvector $\mathbf{v}_k(\omega)$ as given in expression (33).

It is symbolically convenient to express the system of equations (34) in the single system of equations format

$$F^{K,N}(\omega)\mathbf{a}(\omega) = \mathbf{0} \tag{36}$$

where $F^{K,N}(\omega)$ is the $2K(M-N) \times (N+1)$ matrix whose first $2(M-N)$ rows are $V_1(\omega)$, whose second $2(M-N)$ rows are $V_2(\omega)$, etc., namely

$$F^{K,N}(\omega) = [V_1(\omega)' \vdots V_2(\omega)' \vdots \ldots V_K(\omega)']'. \tag{37}$$

The superscripts appearing in matrix $F^{K,N}(\omega)$ are used to explicitly emphasize the point that the structure of this matrix depends both upon the number of signal eigenvectors (K) and the number of incident plane waves (N). This notation is employed in order to effectively describe a procedure for selecting the parameter N (i.e., the number of actual incident plane waves) when only an estimate of the eigencharacterization of $P_x(\omega)$ is given. The consequences of using an over-order choice for N are also described in which this notation is helpful.

Under the matrix rank requirements specified in Theorem 2, it follows that the rank of $F^{K,N}(\omega)$ must be given by $\min(2K(M-N),N)$. Thus, the system of linear equations (36) has a unique solution provided that $2K(M-N) \ge N$ or equivalently $N \le 2KM/(1+2K)$. From this inequality it is apparent that if the N incident plane waves are perfectly coherent (i.e., $K=1$) or noncoherent (i.e., $K=N$), then at most $2M/3$ or $M-1$ such plane waves may be identified, respectively. Spatial smoothed MUSIC likewise can achieve these limits [2].

The proposed signal eigenvector algorithm then entails first solving the linear system of equations (36) using any appropriate solution procedure. Clearly any solution procedure seeks to find a nontrivial vector which lies in the null space of matrix $F^{K,N}(\omega)$. It is to be noted that any such solution corresponds to an eigenvector associated with the zero eigenvalue of the $(N+1) \times (N+1)$ matrix $F^{K,N}(\omega)^* F^{K,N}(\omega)$. This zero eigenvalue has a multiplicity of one provided that N has been correctly chosen. Under this uniqueness assumption, it is readily shown that, upon factoring the

z-transform of the resultant $\mathbf{a}(\omega)$ solution coefficients, we obtain

$$A(z) = 1 + \sum_{n=1}^{N} a_n z^{-n} = \prod_{n=1}^{N} [1 - e^{j(\omega+\omega_0)d\cos(\theta_n)/c} z^{-1}]. \tag{38}$$

Thus, the roots of $A(z)$ are seen to identify the spatial frequencies associated with the incident plane waves.

Underestimation of K and Overestimation of N

Before proceeding further, it is worthwhile to consider the effect on the proposed signal eigenvector method when the number of incident plane waves parameter N is selected to be larger than the actual number of incident plane waves while fewer than K signal eigenvectors are used. This is an important practical consideration since in typical direction-of-arrival applications, only estimates of the array spectral density matrix are available and an exact selection for K and N is not possible. It is widely recognized that by using an over-ordered model, a desensitization to the inevitable inaccuracies in the spectral density matrix estimate often results in an improved modeling performance. Furthermore, when the incident sources are spatially close, some of the signal eigenvalues will be only slightly larger than the noise eigenvalue. In order to avoid the possibility of mixing signal and noise eigenvectors that can arise in such situations, only those signal eigenvalues which are clearly larger than the remaining eigenvalues should be used. It is to be noted that noise-eigenvalue-based methods such as MUSIC can not avoid this deleterious mixing. With these objective in mind, the following Theorem is offered.

THEOREM 4. *Consider the problem as formulated in Theorem 2 for the equispaced linear array described in this section. Moreover, let the* $F^{K_0,N_0}(\omega)$ *matrix be constructed as in expressions* (37) *in which* $N_0 \leq N$ *and* $K_0 \leq K$ *are chosen so that* $K_0 \leq N \leq 2K_0(M - N_0)$. *It then follows that the null space of the* $2K_0(M - N_0) \times (N_0 + 1)$ *matrix* $F^{K_0,N_0}(\omega)$ *has dimension* $N_0 + 1 - N$ *and is spanned by any set of linearly independent eigenvectors* $\{\mathbf{u}_1(\omega), \mathbf{u}_2(\omega), \ldots, \mathbf{u}_{N_0+1-N}(\omega)\}$ *which are associated with the zero eigenvalue of matrix* $F^{K_0,N_0}(\omega)^* F^{K_0,N_0}(\omega)$. *Furthermore, all solutions to the system of linear equations* $F^{K_0,N_0}(\omega)\mathbf{a}(\omega) = \mathbf{0}$ *are expressible as*

$$\mathbf{a}(\omega) = \sum_{n=1}^{N_0+1-N} \beta_n(\omega)\mathbf{u}_n(\omega) \tag{39}$$

for arbitrary choices of the $\beta_n(\omega)$ *scalars. The* z*-transform of any such solution takes the form*

$$A(z) = B(z) \prod_{n=1}^{N} [1 - e^{j(\omega+\omega_0)d\cos(\theta_n)/c} z^{-1}] \tag{40}$$

where the θ_n *parameters correspond to the bearing angles of the* N *incident plane waves, and,* $B(z)$ *is a* $N_0 - N$ *order polynomial in* z^{-1} *whose roots generally lie off the unit circle.*

An over estimate of the number of incident plane waves or the use of fewer than the maximum number of signal eigenvectors therefore does not impair our ability to identify the incident plane wave bearing angles. This of course presumes that K_0 and N_0 are selected to satisfy the inequality $K_0 \leq K \leq N \leq N_0 \leq 2K_0(M - N_0)$. The practical consequence of this observation is found to be of significance in real world applications. As a final comment, it is noted that the general solution expression (39) simplifies to the unique solution of relationship (36) when $N_0 = N$.

In summary, the signal eigenvector approach for solving the direction-of-arrival problem entails the following solution procedure. Under the assumption that $N \leq M - 1$, the smallest eigenvalue of the generalized eigenexpression (16) is set equal to the sensor noise power $\sigma^2(\omega)$. The signal eigenvectors associated with the signal eigenvalues are then used to identify the N bearing angles θ_n for $1 \leq n \leq N$ by first solving the system of linear equations (36) and then factoring the resultant solution's z-transform according to relationship (38). These bearing angles are in turn used to construct the composite steering matrix $S(\omega)$ according to expression (9). Finally, relationship (15) is used to construct the source spectral density matrix. It is important to note that this solution procedure is valid independent of whether the incident plane waves are incoherent, partially coherent, or coherent. This is to be contrasted with noise-eigenvector-based solution approaches.

VIII. DIRECTION-OF-ARRIVAL ALGORITHM: EQUISPACED LINEAR ARRAYS

From the last section, it is apparent that the signal eigenvectors associated with signal eigenvalues can be used to effectively solve the direction-of-arrival problem. However, one must carefully interpret the eigenanalysis (16) in terms of the number of incident plane waves. Specifically, if the noise eigenvalues $\sigma^2(\omega)$ is found to have multiplicity $M - K$, this indicates either the existence of K incident noncoherent plane waves, or the existence of more than K incident plane waves some of which are perfectly coherent. To determine which of these situations pertains, we now make a simple modification of the approach as described by (34) and (36).

One first chooses a value for the number of plane waves thought to be impinging on the linear array. This number shall be denoted as N as its selection may require some experimentation on the users part. Specifically, from the problem description, it is known that N must lie in the interval $K \leq N \leq M - 1$

where the lower bound K is equal to the number of signal eigenvalues that are larger than $\sigma^2(\omega)$. Most importantly, N must be selected so that the homogeneous linear system of equations

$$F^{K,N}(\omega)\mathbf{a}(\omega) = \mathbf{0} \tag{41}$$

has a nontrivial solution. To begin the solution process, we first set N equal to its smallest possible value K and determine whether there exists a nontrivial solution to expression (41). If such a solution exists, this indicates that K noncoherent plane waves are incident on the array. On the other hand, the lack of a solution implies that there are more than K incident plane waves and a subset of these plane waves are perfectly coherent. In this case, we than set $N = K + 1$ in expression (41) and determine whether the new system of equations $F^{K,N}(\omega)\mathbf{a}(\omega) = \mathbf{0}$ has a nontrivial solution. This process is continued until one of the following two situations is found to hold: 1) the smallest value of N in the interval $K \le N \le M - 1$ is found for which relationship (32) has a nontrivial solution, or 2) there is no value of N satisfying $K \le N \le M - 1$ for which (41) has a nontrivial solution. The first of these situations indicates that there are exactly N incident plane waves of which $N + 1 - K$ are perfectly coherent. On the other hand, the second situation indicates that $N > 2KM/(1 + 2K)$ and a resolution of the N incident plane waves is not possible using the proposed approach.

In testing for the existence of a nontrivial null space of $F^{K,N}(\omega)$ which assures a solution to expression (41), we may either make a singular value decomposition of $F^{K,N}(\omega)$ or an eigenanalysis of $F^{K,N}(\omega)^* F^{K,N}(\omega)$. A zero singular value or a zero eigenvalue indicates the existence of a nontrivial solution, respectively. If a value of N is eventually found for which a nontrivial solution exists, then the z-transform of this solution as specified by relationship (38) is made. Upon factoring this z-transform, we associate an incident plane wave with each distinct root that falls on the unit circle. Thus, the unit circle root locations provide the basis for identifying the incident plane waves spatial frequencies. The major steps of this direction-of-arrival algorithm for locating incident plane waves are summarized in Table I.

TABLE I
Steps Of Direction-of-Arrival Algorithm: Linear Equispaced Array

1	Perform an eigenanalysis of the array spectral density matrix $P_x(\omega)$ and determine $M - K$, the multiplicity of the noise eigenvalue $\sigma^2(\omega)$. This analysis yields the signal eigenvectors $\mathbf{e}^k(\omega)$ for $1 \le k \le K$ corresponding to the K signal (largest) eigenvalues.
2	The number of incident plane waves N is set equal to the smallest integer satisfying $K \le N \le M - 1$ for which expression (41) has a nontrivial solution. If such an integer exists, then there are N incident plane waves of which $N + 1 - K$ are perfectly coherent. If no such integer exists, then there are too many incident plane waves for the signal eigenvector method to be effective.
3	Take the z-transform of the solution arising from step 2 in accordance with (38) and identify the N spacial frequencies with the unit circle locations.

IX. PRACTICAL CONSIDERATIONS

We now return to the general array geometry and address some important practical considerations. Our investigations to this point have been primarily theoretical in nature in that the array spectral density matrix $P_x(\omega)$ has been assumed known. In virtually all practical applications, such a priori information is never available. More typically, there is given a set of sampled array sensor signals

$$x_m(\delta), x_m(2\delta), \ldots, x_m(L\delta), \qquad \text{for} \quad 1 \le m \le M \tag{42}$$

with these samples being made over a time interval of T seconds so that the uniform sampling period is $\delta = T/L$. Using this raw data, it is then desired to estimate the array spectral density matrix $P_x(\omega)$. To affect this estimate, let these array sensor samples be divided into Q equal subintervals each consisting of $N = L/Q$ sample, that is

$$x_m^k(n\delta) = x_m(n\delta + [k-1]N\delta), \qquad \text{for} \quad 1 \le m \le M \quad \text{and} \quad 1 \le k \le Q \tag{43}$$

where the superscript k denotes the kth subinterval.

The N point discrete Fourier transform (DFT) of the segmented data sequences (43) are now computed. These DFTs are symbolically represented as

$$X_m^k(\omega_j) = \mathrm{DFT}[x_m^k(n)], \qquad \text{for} \quad 0 \le j \le N-1 \quad \text{and} \quad 1 \le k \le Q. \tag{44}$$

The digital frequency variable ω_j takes on the discrete values $2\pi j/N$ for $0 \le j \le N-1$. Let $X_k(\omega_j)$ denote the $M \times 1$ Fourier coefficient vector evaluated at frequency ω_j of the kth subinterval, that is,

$$\mathbf{X}_k(\omega_j) = [X_1^k(\omega_j), X_2^k(\omega_j), \ldots, X_M^k(\omega_j)]', \qquad \text{for} \quad 1 \le k \le Q. \tag{45}$$

If T/Q is sufficiently large and the decomposed components $X(\omega_j)$ are uncorrelated, the spatial power spectral density matrix $P_x(\omega_j)$ of jth frequency ω_j is then approximately given by

$$P_x(\omega_j) = \frac{1}{Q}\sum_{k=1}^{Q} \mathbf{X}^k(\omega_j)\mathbf{X}^k(\omega_j)^*. \tag{46}$$

It is to be noted that under the assumption that $X(\omega_j)$ is a zero mean normal random vector, the maximum likelihood estimate of $E\{\mathbf{X}(\omega_j)\mathbf{X}(\omega_j)^*\}$ is given by this expression [10].

In using the signal eigenvector method to estimate direction-of-arrivals of incident plane waves, it is logical that only those frequencies ω_j for which the signal-to-noise (SNR) ratio is suitably large be used. Let $\{\omega^1,\omega^2,\ldots,\omega^J\}$ designate the set of frequencies contained in the set $\omega_j = 2\pi j/N$ for $0 \le j \le N_T - 1$ for which the SNR is deemed sufficiently large. The spectral density matrix estimates to be used are then given by

$$P_x(\omega^j) \approx \frac{1}{Q}\sum_{k=1}^{Q} \mathbf{X}^k(\omega^j)\mathbf{X}^k(\omega^j)^* \qquad (47)$$

in which ω^j is within the bandwidth and J is the number of frequency bands to be analyzed.

In using the spectral density matrix estimates (46), it is found that the associated general eigenanalysis (16) produces a linear system of equations estimates (36) in which matrix $F^{K,N}(\omega)$ has full rank. As such, there does not exist a nontrivial vector $\mathbf{a}(\omega)$ which solves these linear equations. It is then necessary to use some appropriately chosen method for obtaining an approximate solution. For our purposes, the near null space SVD method has provided satisfactory performance (i.e., [4, eq. (7.15)]).

X. FOCUSING METHOD

The solution procedure as outlined in the previous sections solves the fundamental direction-of-arrival problem at a single frequency. In order to achieve a significant improvement in estimation performance, it is essential that we suitably combine direction information at all frequencies for which the SNR is reasonably large. With this in mind, it is now assumed that estimates of the steering matrices $S(\omega^j)$ at frequencies ω^j for $1 \le j \le J$ have been separately determined using the approach taken above. Since the range spaces of these steering matrices are generally different for distinct frequencies, we cannot simply take their average to effect an improved bearing angle estimation. Wang and Kaveh have devisied a focusing procedure whose objective is to transform each steering matrix estimate to its image steering matrix at a common frequency ω_0 [25]. One may then average these J transformed steering matrices estimates to obtain an enhanced estimate of the underlying steering matrix at frequency ω_0. This in turn will lead to an enhanced estimate of the direction vectors. The following Theorem as presented by Wang and Kaveh [25] is applicable to general array geometries and conveys the basic concepts of the focusing method.

THEOREM 5. (*See* [25]). *Under the condition that each of the steering matrices $S(\omega^j)$ for $1 \le j \le J$ has rank N, there exist nonsingular $M \times M$ matrices $T(\omega^j)$ for $j = 1,2,\ldots,J$ such that*

$$T(\omega^j)S(\omega^j) = S(\omega_0), \qquad \textit{for} \quad 1 \le j \le J. \qquad (48)$$

It is noted that the matrices $T(\omega^j)$ are not unique. Wang and Kaveh suggest a computationally appealing selection for the $T(\omega^j)$ matrices in the case of uniform spaced linear arrays. In particular, let θ_1 designate a trial bearing angle about which all the underlying true bearing angles lying within a neighborhood of θ_1 are to be obtained. The diagonal matrix

$$T(\omega^j) = \mathrm{diag}[t_{11},t_{22},\ldots,t_{MM}] \qquad (49)$$

where the diagonal elements are specified by $t_{mm} = \exp(jd(\omega_0 - \omega^j)(m-1)\cos\theta_1/c)$ for $1 \le m \le M$ in which d is the spacing between adjacent sensors and c is the propagation velocity. It is assumed that this $T(\omega_j)$ is such that relationship (48) is approximately satisfied for all bearing angles in a neighborhood of θ_1.

The effect of the focusing operation is that of transforming random vector $\mathbf{X}(\omega_j)$ into the random vector $\mathbf{Y}(\omega^j)$ where

$$\mathbf{Y}(\omega^j) = T(\omega^j)\mathbf{X}(\omega^j), \qquad \text{for} \quad 1 \le j \le J. \qquad (50)$$

Let us now analyze the spectral characteristics of the random vector $\mathbf{Y}(\omega^j)$. In particular, the weighted spectral density matrix estimate associated with $\mathbf{Y}(\omega^j)$ is expressed as

$$P_y = \sum_{j=1}^{J} w_j E\{\mathbf{Y}(\omega^j)\mathbf{Y}(\omega^j)^*\} \qquad (51)$$

where the w_j's are normalized weights selected to be proportional to the SNR of the jth frequency band. Inserting expression (50) into this spectral density matrix estimate and taking the expected value gives

$$P_y = \sum_{j=1}^{J} w_j T(\omega^j)E\{\mathbf{X}(\omega^j)\mathbf{X}(\omega^j)^*\}T(\omega^j)^*. \qquad (52)$$

Appealing to expressions (12) and (48), it follows that

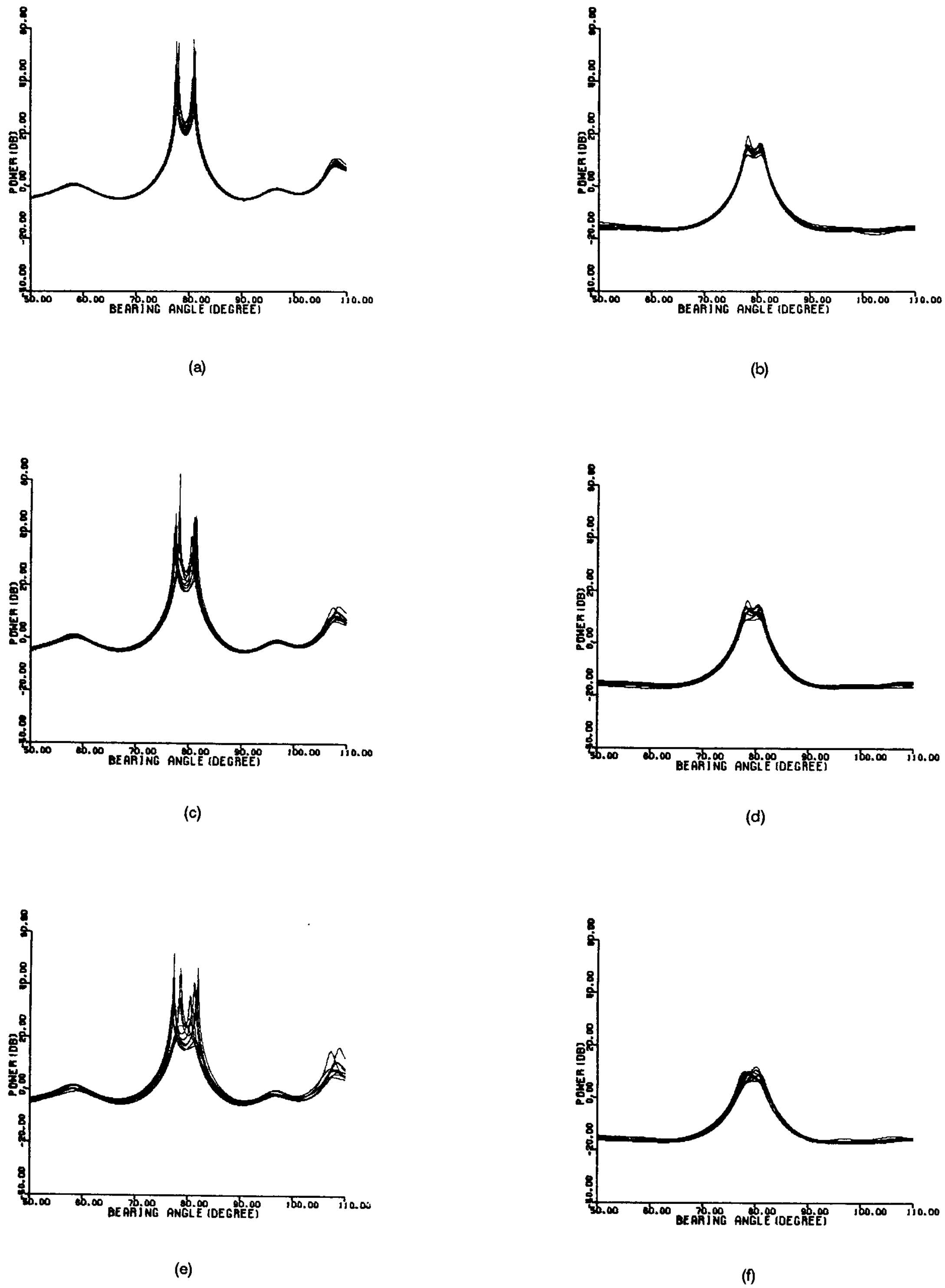
$$P_y = S(\omega_0)P_{cy}S(\omega_0)^* + \sigma^2 P_{\eta y} \qquad (53)$$

where

$$P_{cy} = \sum_{j=1}^{J} w_j P_c(\omega^j) \qquad \text{and}$$
$$P_{\eta y} = \sum_{j=1}^{J} w_j T(\omega^j)P_\eta(\omega^j)T(\omega^j)^*. \qquad (54)$$

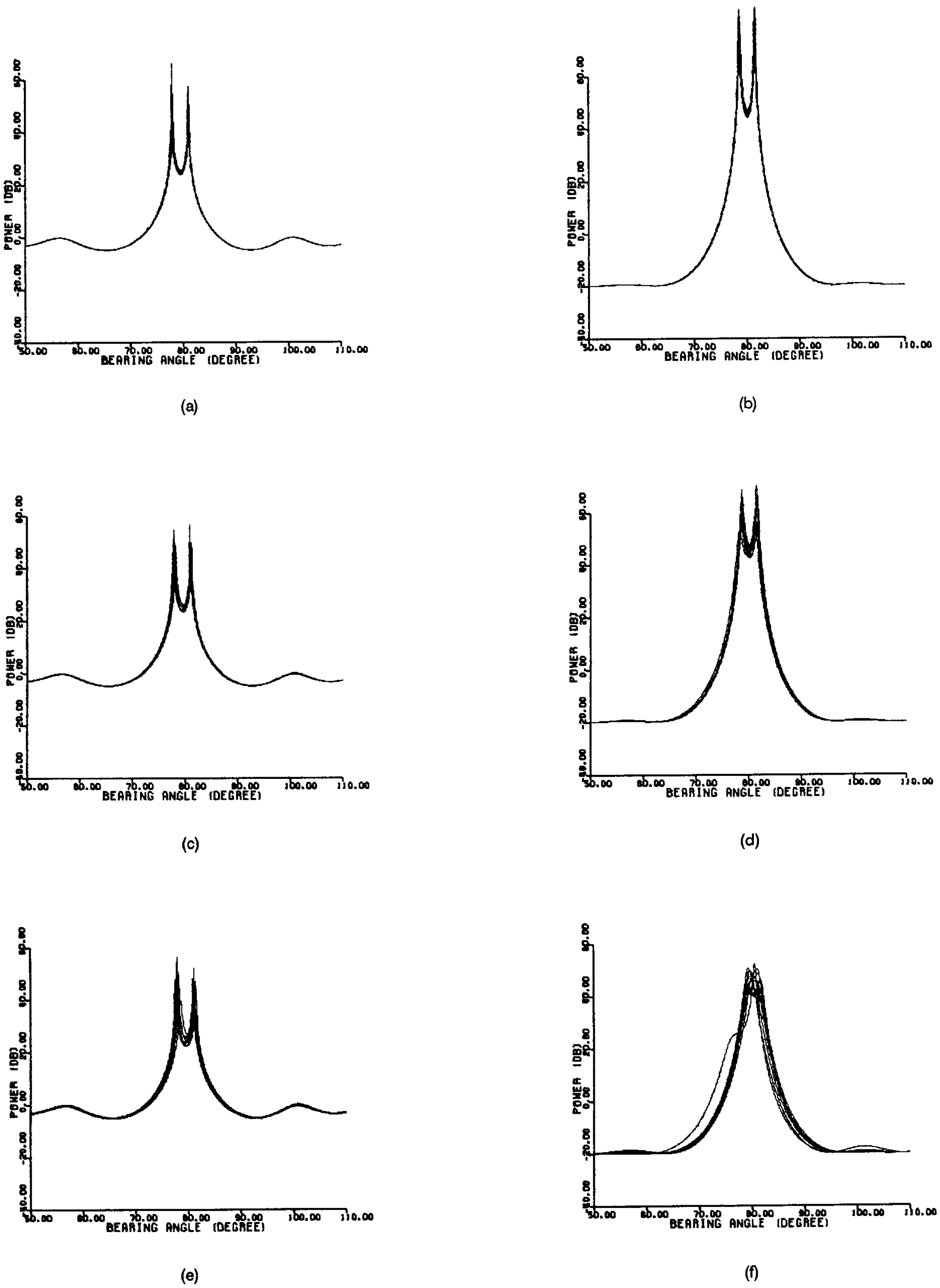
XI. SIMULATION EXPERIMENTS

To test the effectiveness of the proposed signal eigenvector method in resolving closely spaced incident plane waves, two examples are considered. The first example involved two spatially close incoherent sources which are coincident on an array with complex geometry. The second example involved two coherent sources incident on an equispaced linear array. In each



Paper's method MUSIC

Fig. 1. Ten statistically independent superimposed bearing estimates for two incoherent incident plane waves at bearing angles 78° and 81°. Estimates obtained using 11-element two-dimensional nonuniform array.
(a),(b): 0 dB; (c),(d): −5 dB; (e),(f): −10 dB.



Paper's method

Spatial smoothed MUSIC

Fig. 2. Ten statistically independent superimposed bearing estimates for two incoherent incident plane waves at bearing angles 78° and 81°. Estimates obtained using 16-element linear array that had uniform spacing of $d = \lambda_0/2$.
(a),(b): 0 dB; (c),(d): -5 dB; (e),(f): -10 dB.

of these cases, the proposed array processing method is found to provide improved performance relative to that achieved with MUSIC and spatial smoothed MUSIC using the CSS focusing method of Wang and Kaveh.

Example 1

We first examine the case of two band-limited incoherent sources incident on an array with a complex geometry. The array considered is composed of eleven sensors contained in the (x,y) plane at locations

$$\mathbf{z}_1 = [0,0]', \quad \mathbf{z}_2 = [4,4]', \quad \mathbf{z}_3 = [6,8]',$$
$$\mathbf{z}_4 = [11,8]', \quad \mathbf{z}_5 = [9,4]', \quad \mathbf{z}_6 = [5,0]'$$
$$\mathbf{z}_7 = [1,-2]', \quad \mathbf{z}_8 = [16,8]', \quad \mathbf{z}_9 = [14,4]',$$
$$\mathbf{z}_{10} = [10,0]', \quad \mathbf{z}_{11} = [6,-2]'$$

where the unit of spacing corresponds to $2c\pi/3\omega_0$. Two incident plane waves were taken to have bearing angles $\theta_1 = 78°$ and $\theta_2 = 81°$, common center frequency 100 Hz, and the uncorrelated source signals were taken to be zero mean stationary bandlimited Gaussian processes with bandwidth 40 Hz. The sensor noise vector $\eta(t)$ was taken as a complex valued additive bandpass white Gaussian process whose components had identical variances and were statistically independent of the source signals. The array signal model (2) was then employed to generate data to be used in forming the array power spectral density matrix estimates (47) with unity weights. The sensor signals were observed over a $T = 48$ s interval. These signals were then sampled at the rate of 80 samples/s to give a total of $L = 3840$ samples for each of the eleven sensor signals. To approximate the underlying array power spectral density matrix, each data length was decomposed into 60 nonoverlapping contiguous segments each consisting of 64 samples. Using relationship (47), an estimate of the array power spectral density matrix is made with $Q = 33$ equal frequency intervals spaced across the passband.

Using the power spectral density matrix estimate thereby obtained, MUSIC and the algorithm presented here were employed to resolve the two sources. For each method, the Wang-Kaveh focusing method was employed in which the focusing angle was taken to be 79.5°. This experiment was repeated ten times and the resultant ten spectral estimates are shown in superimposed fashion in Figs. 1(a) and (b). This experiment was repeated again at -5 dB and -10 dB and the resultant estimates are depicted in Figs. 1(c)–(f). From these estimates, it is apparent that MUSIC resolves the two sources satisfactorily at 0 dB but not at -5 dB or -10 dB. The method presented here, however, achieved a resolution at each of these three SNR levels.

Example 2

In the next example, the case of two broadband coherent sources incident on a linear array with $M = 16$ uniformly spaced sensors was considered. The sensor spacing was $d = c\pi/\omega_0$ where ω_0 is the center frequency. The impinging sensor signals were specified by

$$x_m(t) = f_1(t + \tau_1(m))\cos[\omega_0(t + \tau_1(m)) + \phi_1] + f_2(t + \tau_2(m))\cos[\omega_0(t + \tau_2(m)) + \phi_2].$$

The source signal $f(t)$ takes on the values plus or minus one at the modulating rate and the delays were taken as

$$\tau_i(m) = d(m-1)\cos(\theta_i)/c, \qquad \text{for} \quad i = 1,2 \quad \text{and} \quad 1 \le m \le M$$

for an equispaced linear array. It is here assumed that $f_1(t) = f_2(t)$ thereby giving rise to perfectly coherent sources. The two spread spectrum signals are taken to have the same center frequency ω_0 and unit amplitudes with bearing angles $\theta_1 = 78°$ and $\theta_2 = 81°$, respectively. The chip rate of the spreading codes and the sampling frequency were chosen so that the center frequency is 0.5π rad and the mainlobe width is 0.5π rad. The number of snapshots was taken as $L = 3840$. Furthermore, the total observation for each sensor was divided into $Q = 60$ segments and each segment was decomposed into $J = 32$ narrowband components.

The spatial smoothed MUSIC algorithm with forward-backward entries used in estimating the power spectral density matrices was then employed with the subarray size being taken as ten (i.e., the number of subarrays is seven). In the method presented here, the parameter selection $K = 1$ and the over-order selection $N_0 = 9$ were used. Using relationship (47), an estimate of the array power spectral density matrix is used to generate $J = 32$ narrowband components across the passband. For each method, the Wang-Kaveh focusing method was employed in which the focusing angle was taken to be 79.5°. This experiment was repeated ten times and the resultant ten spectral estimates are shown in superimposed fashion in Figs. 2(a) and (b). This experiment was repeated again at -5 dB and -10 dB and the resultant estimates are depicted in Figs. 2(c)–(f). From these estimates, it is apparent that spatial smoothed MUSIC resolves the two sources satisfactorily at 0 dB and -5 dB but not at -10 dB. Although the spatial smoothed MUSIC resolves the two sources, it gives rise to much more biased estimates at -5 dB. The method presented here, however, achieved a resolution at each of these three SNR levels with much less biases and variances.

XII. CONCLUSIONS

A signal eigenvector method for direction-of-arrival estimation has been presented which possesses the desirable attributes of providing high resolution performance and being insensitive to coherent sources. This method is applicable for general array geometries and is effective in detecting both narrowband and broadband sources. Application of the signal eigenvector method to the special case of linear arrays with equispaced sensors was fully developed and its performance tested by simulated examples. It has been found to provide improved performance relative to the Wang-Kaveh MUSIC algorithm on many examples treated to date. A computationally efficient approach to implementing the signal eigenvector method for coherent sources is currently under development.

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