Derivation of the Equation of Motion of A Rigid-link Robot Manipulator

A Iterative Newton-Euler Dynamic Formulation

The formulation of joint motion and torque of a rigid-link robot manipulator is composed of two iterations. The first (outward) iteration derives recursively the propagation of angular velocities and accelerations (kinematics) from joint 0 to n, with the Newton-Euler equation applied to each link connecting adjacent joints. The second (inward) iteration derives recursively the force and torque interactions (dynamics) from joint $n+1$ to 1. Here, we adopt the conventions and notations used in **[29] from the references of the main text**.

The outward and inward Newton-Euler equations are summarized as follows:

Outward iterations $i : 0 \to n$

$$\begin{aligned}
{}^{i+1}\omega_{i+1} &= {}^{i+1}_{i}R\,{}^{i}\omega_i + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \\
{}^{i+1}\dot{\omega}_{i+1} &= {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\,{}^{i}\omega_i \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} \\
{}^{i+1}\dot{v}_{i+1} &= {}^{i+1}_{i}R\,({}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times ({}^{i}\omega_i \times {}^{i}P_{i+1}) + {}^{i}\dot{v}_i) \\
{}^{i+1}\dot{v}_{c_{i+1}} &= {}^{i+1}\dot{\omega}_{i+1} \times {}^{i+1}P_{c_{i+1}} + {}^{i+1}\omega_{i+1} \times ({}^{i+1}\omega_{i+1} \times {}^{i+1}P_{c_{i+1}}) + {}^{i+1}\dot{v}_{i+1} \\
{}^{i+1}F_{i+1} &= m_{i+1}{}^{i+1}\,\dot{v}_{c_{i+1}} \\
{}^{i+1}N_{i+1} &= {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{C_{i+1}}I_{i+1}\,{}^{i+1}\omega_{i+1}
\end{aligned} \tag{1}$$

Inward iterations $i : n+1 \to 1$

$$\begin{aligned}
{}^{i}f_i &= {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} + {}^{i}F_i \\
{}^{i}n_i &= {}^{i}N_i + {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{c_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} \\
\tau_i &= {}^{i}n_i^T\,{}^{i}\hat{Z}_i
\end{aligned} \tag{2}$$

B State Space Equation

Denote

$$
\Theta = \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_n \end{bmatrix}
$$
$$
\tau = \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix}
$$
$$
f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}
$$

In general, the Newton-Euler equations (1)-(2) yield a closed-form equation of motion that can be written in the form of

$$
\tau = M(\Theta)\ddot{\Theta} + V(\dot{\Theta}, \Theta) + G(\Theta) + B(\Theta)f \tag{3}
$$

where $M(\Theta)$ is the mass matrix, $V(\dot{\Theta}, \Theta)$ is the Coriolis terms and $G(\Theta)$ is the gravity terms and $B(\Theta)$ is the force matrix.

Given this closed-form equation of motion (3), we can solve for the acceleration to get

$$
\begin{aligned}
M(\Theta)\ddot{\Theta} &= \tau - V(\Theta, \dot{\Theta}) - G(\Theta) + B(\Theta)f \\
\ddot{\Theta} &= M(\Theta)^{-1}\Big(\tau - V(\Theta, \dot{\Theta}) - G(\Theta) - B(\Theta)f\Big)
\end{aligned} \tag{4}
$$

Then, we can discretize the continuous equation of motion by applying numerical integration scheme such as forward Euler method, as follows:

$$
\begin{aligned}
\dot{\Theta}(t+h) &= \dot{\Theta}(t) + \ddot{\Theta}(t)h \\
\Theta(t+h) &= \Theta(t) + \dot{\Theta}(t)h + \frac{1}{2}\ddot{\Theta}(t)h^2
\end{aligned} \tag{5}
$$

where h is the time-step size.

Substituting (4) into (5), we yield a discrete state space equation as follows:

$$
\begin{bmatrix} \dot{\Theta}_{t+1} \\ \Theta_{t+1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ hI & I \end{bmatrix} \begin{bmatrix} \dot{\Theta}_t \\ \Theta_t \end{bmatrix} + \begin{bmatrix} hI \\ \frac{1}{2}h^2 I \end{bmatrix} M(\Theta_t)^{-1}\Big(\tau_t - V(\dot{\Theta}_t, \Theta_t) - G(\Theta_t) - B(\Theta_t)f_t\Big) \tag{6}
$$

with time samples indexed by t.

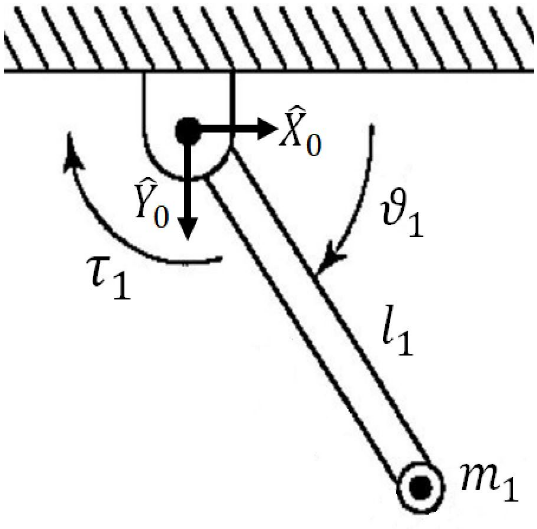

Figure A: One-link planar robot manipulator with point mass at distal ends of links.

C One-Link Planar Robot Manipulator

Here, we compute the closed-form dynamic equations for the one-link planar robot manipulator shown in Fig. A (**Fig. 2 from the main text**). In this case, $n = 1$. For simplicity, we assume that the mass distribution is simple, i.e., all mass exists as a point mass at the distal end of the link.

First, we need to determine the values of the various quantities that appears in the outward and inward Newton-Euler equations (1)-(2).

The vectors that locate the center of mass and the tip of the link are

$$\begin{aligned} {}^1P_{C_1} &= l_1\hat{X}_1 \\ {}^1P_2 &= l_1\hat{X}_1 \end{aligned} \tag{7}$$

The rotation between successive link frames is given by

$$\begin{aligned} {}^{i}_{i+1}R &= \begin{bmatrix} c_{i+1} & -s_{i+1} & 0 \\ s_{i+1} & c_{i+1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ {}^{i+1}_{i}R &= \begin{bmatrix} c_{i+1} & s_{i+1} & 0 \\ -s_{i+1} & c_{i+1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{8}$$

The base of the robot is not rotating; hence, we have

$$
\begin{aligned}
\omega_0 &= 0 \\
\dot{\omega}_0 &= 0
\end{aligned} \tag{9}
$$

To include gravity forces, we consider

$$
{}^0\dot{v}_0 = -g\hat{Y}_0 \tag{10}
$$

Due to the point-mass assumption, the inertia tensor with respect to the center of mass of the link is a zero matrix:

$$
{}^{C_1}I_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{11}
$$

There are only one force vector acting on the end-effector, so we have

$$
\begin{aligned}
{}^0f_2 &= \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \\
{}^0n_2 &= 0
\end{aligned} \tag{12}
$$

with respect to the origin at joint 0.

Substituting (7)-(8) and (9)-(11) into (1), the outward iteration from joint 0 to 1 is evaluated, as follows:

$$
\begin{aligned}
{}^1\omega_1 &= \dot{\theta}_1\,\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \\
{}^1\dot{\omega}_1 &= \ddot{\theta}_1\,\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} \\
{}^1\dot{v}_1 &= \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} = \begin{bmatrix} -gs_1 \\ -gc_1 \\ 0 \end{bmatrix} \\
{}^1\dot{v}_{c_1} &= \begin{bmatrix} 0 \\ l_1\ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -l_1\dot{\theta}_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -gs_1 \\ -gc_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -l_1\dot{\theta}_1 - gs_1 \\ l_1\ddot{\theta}_1 - gc_1 \\ 0 \end{bmatrix} \\
{}^1F_1 &= \begin{bmatrix} -m_1l_1\dot{\theta}_1 - m_1gs_1 \\ m_1l_1\ddot{\theta}_1 - m_1gc_1 \\ 0 \end{bmatrix} \\
{}^1N_1 &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned} \tag{13}
$$

Substituting (7)-(8), (12), and (13) into (2), the inward iteration from joint 2 to 1 is evaluated, as follows:

$$\begin{aligned}
{}^1f_1 &= {}^1_2R\,{}^2f_2 + {}^1F_1 \\
&= {}^1_0R\,{}^0f_2 + {}^1F_1 \\
&= \begin{bmatrix} c_1f_x + s_1f_y \\ c_1f_y - s_1f_x \\ f_z \end{bmatrix} + \begin{bmatrix} -m_1l_1\dot{\theta}_1 - m_1gs_1 \\ m_1l_1\ddot{\theta}_1 - m_1gc_1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} c_1f_x + s_1f_y - m_1l_1\dot{\theta}_1 - m_1gs_1 \\ c_1f_y - s_1f_x + m_1l_1\ddot{\theta}_1 - m_1gc_1 \\ f_z \end{bmatrix}
\end{aligned} \tag{14}$$

$$\begin{aligned}
{}^1n_1 &= {}^1N_1 + {}^1_2R\,{}^2n_2 + {}^1P_{c_1} \times {}^1F_1 + {}^1P_2 \times {}^1_2R\,{}^2f_2 + {}^1F_1 \\
&= {}^1N_1 + {}^1_2R\,{}^2n_2 + {}^1P_{c_1} \times {}^1F_1 + {}^1P_2 \times {}^1_0R\,{}^0f_2 \\
&= \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} -m_1l_1\dot{\theta}_1 - m_1gs_1 \\ m_1l_1\ddot{\theta}_1 - m_1gc_1 \\ 0 \end{bmatrix} + \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} c_1f_x + s_1f_y \\ c_1f_y - s_1f_x \\ f_z \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ m_1l_1^2\ddot{\theta}_1 - m_1l_1gc_1 \end{bmatrix} + \begin{bmatrix} 0 \\ -l_1f_z \\ c_1l_1f_y - s_1l_1f_x \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ -l_1f_z \\ c_1l_1f_y - s_1l_1f_x + m_1l_1^2\ddot{\theta}_1 - m_1l_1gc_1 \end{bmatrix}
\end{aligned} \tag{15}$$

Extracting the $\hat{Z}$ components of the ${}^1n_1$ (15), we obtain the joint torque equation:

$$\tau_1 = c_1l_1f_y - s_1l_1f_x + m_1l_1^2\ddot{\theta}_1 - m_1l_1gc_1 \tag{16}$$

To incorporate rotational damping into the system, we include the damping torque $-b\dot{\theta}$ (opposing the direction of angular velocity) with damping coefficient b into (16) to get

$$\tau_1 = c_1l_1f_y - s_1l_1f_x + m_1l_1^2\ddot{\theta}_1 - b\dot{\theta} - m_1l_1gc_1 \tag{17}$$

which gives an expression for the joint torque as a function of the angular acceleration, velocity, position and the tip forces.

Omitting the joint index and consider that there is no joint torque actuation, i.e. $\tau = 0$, the dynamics of the one-link robot manipulator is then described by

$$ml^2\ddot{\theta} = mlg\cos\theta - b\dot{\theta} + l\sin\theta f_x - l\cos\theta f_y \tag{18}$$

which is equivalent to the **equation of motion (75) of the main text**.

The discrete state space equation (6) thus simplifies to

$$\begin{bmatrix}\dot{\theta}_{t+1}\\ \theta_{t+1}\end{bmatrix} = \begin{bmatrix}1 & 0\\ h & 1\end{bmatrix}\begin{bmatrix}\dot{\theta}_{t}\\ \theta_{t}\end{bmatrix} + \begin{bmatrix}h\\ \frac{1}{2}h^2\end{bmatrix}\left(\frac{1}{l}g\cos\theta + \frac{1}{ml}\begin{bmatrix}\sin\theta & -\cos\theta\end{bmatrix}\begin{bmatrix}f_x\\ f_y\end{bmatrix}\right) \tag{19}$$

which is equivalent to the **state model (76) of the main text**.