Quantum angular momentum / A.L 2019-08-27.

We calculate the commutation relations between l_z and x_i, p_i. ($x_1 = x$, $x_2 = y$, $x_3 = z$ etc.)

$$[l_z, x] = [xp_y - yp_x, x] = \Big/ \begin{matrix} \text{since } [u_1 + u_2, v] \\ = [u_1, v] + [u_2, v] \end{matrix} \Big/ = [xp_y, x] - [yp_x, x].$$

$$= \Big/ \text{since } [ab, c] = [a,c]b + a[b,c] \Big/ = \underbrace{\cancel{[x,x]}}_{0} p_y + x\underbrace{\cancel{[p_y, x]}}_{0} - \underbrace{\cancel{[y,x]}}_{0} p_x - y\underbrace{\cancel{[p_x, x]}}_{-i\hbar}.$$

The only non-zero term is $-y\overbrace{[p_x, x]}^{-i\hbar} = -y(-i\hbar) = i\hbar y$

$\therefore$ $[l_z, x] = i\hbar y$. Similarly we get $[l_z, y] = -i\hbar x$

and $[l_z, z] = 0$.

In the same way we can show $\left\{ \begin{matrix} [l_z, p_x] = p_y, \\ [l_z, p_y] = -p_x, \\ [l_z, p_z] = 0. \end{matrix} \right.$

If we would calculate the relations for l_y and l_x also it would be clear that we can use the Levi-Civita symbol here as well:

$$[l_i, x_j] = i\hbar \epsilon_{ijk} x_k$$

$$[l_i, p_j] = i\hbar \epsilon_{ijk} p_k.$$

We can now calculate the commutators between l_x, l_y, l_z by using the previous relations.

$$[l_x, l_y] = [y p_z - z p_y, l_y] = [y p_z, l_y] - [z p_y, l_y]$$

$$= \cancelto{0}{[y, l_y]} p_z + y[p_z, l_y] - [z, l_y] p_y - z \cancelto{0}{[p_y, l_y]} =$$

$$= y[p_z, l_y] - [z, l_y] p_y = \Big/ \begin{array}{l} [z, l_y] = -x i\hbar \\ [p_z, l_y] = -p_x i\hbar \end{array} \Big/$$

$$= i\hbar(\overbrace{x p_y - y p_x}^{l_z}) = \underline{\underline{i\hbar l_z}}.$$