Quantum angular momentum

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We calculate the commutation relations between l_z and x_i, p_i ($x_1 = x$, $x_2 = y$, $x_3 = z$ etc.)

$$[l_z, x] = [xp_y - yp_x, x] = \Big/ \text{since } [u_1 + u_2, v] = [u_1, v] + [u_2, v] \Big/ = [xp_y, x] - [yp_x, x]$$

$$= \Big/ \text{since } [ab, c] = [a,c]b + a[b,c] \Big/ = \cancel{[x,x]}^{0} p_y + x\cancel{[p_y, x]}^{0} - \cancel{[y,x]}^{0} p_x - y\cancel{[p_x, x]}^{-i\hbar}.$$

Only non-zero term is $-y[p_x, x] = i\hbar y$.

Thus $[l_z, x] = i\hbar y$.

Similarly we get $[l_z, y] = -i\hbar x$ and $[l_z, z] = 0$.

In the same way we can show that

$$\begin{cases} [l_z, p_x] = i\hbar p_y, \\ [l_z, p_y] = i\hbar p_x, \\ [l_z, p_z] = 0. \end{cases}$$

Extending to l_y and l_x we would see that

$$[l_i, x_j] = i\hbar \epsilon_{ijk} x_k,$$

$$[l_i, p_j] = i\hbar \epsilon_{ijk} p_k.$$

Using these relations we can calculate the commutators between l_x, l_y, l_z.

$$[l_x; l_y] = [yp_z - zp_y, l_y] = [yp_z, l_y] - [zp_y, l_y]$$

$$= \overbrace{[y, l_y]}^{0} p_z + y[p_z, l_y] - [z, l_y]p_y - z\overbrace{[p_y, l_y]}^{0}$$

$$= y[p_z, l_y] - [z, l_y]p_y = \Big/ \begin{matrix} [p_z, l_y] = -p_x i\hbar \\ [z, l_y] = -x i\hbar \end{matrix} \Big/$$

$$= i\hbar(\underbrace{xp_y - yp_x}_{l_z}) = \underline{\underline{i\hbar l_z}}.$$