Hashing

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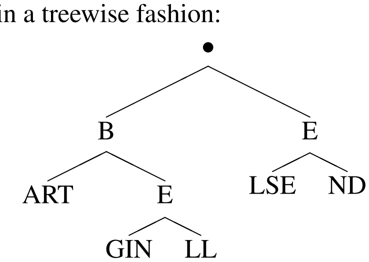
Hashing, hash functions, hash tables, come into play when a compiler, and in particular its parser, needs to store names (of identifiers) and further information about the object of that name.

1 Introduction

A compiler, and in particular its parser, needs to store variables and information about them. The data structure for this is in effect addressed by the name of the variable, rather than by any numerical index. This sort of storage is sometimes called 'associative'. The design of a data structure for this is a good example of trade-offs between efficiency and expediency.

- If variables are stored in the order in which they are encountered, storage is very fast, but searching will take time linear in the number of items stored.
- If the list if kept sorted, searching for an item will take logarithmic time. However, insertion is now more expensive, taking linear time because all elements following have to be copied up.
- A naively implemented linked list would give both insertion and retrieval time linearly in the number of stored items. In the insertion phase we avoid the copying, but finding the place to insert is harder, and in fact we can not use bisection here.
- Items can be stored in a treewise fashion:

  The cost of inserting and retrieving is then linear in the length of the string, at least for as far as is necessary to disambiguate from other strings.

These are then the issues we need to address:

- What is the cost of inserting a new item?
- What is the cost of finding and retrieving an item?
- What is the cost of deleting an item?

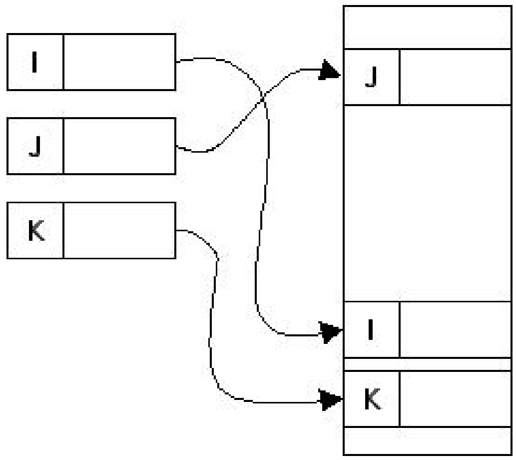


Figure 1: A hash function without conflicts

2 Hash functions

A hash function is function that maps non-numeric keys into a range of integers, interpreted as the locations in a table. It would be nice if this function was injective, to avoid mapping two keys to the same location but surprisingly hard, as is clear from the ‘birthday paradox’: it takes only 23 random picks from a 365-entry table to have even chances of a collision. If we know all keys in advance, it is possible to design a function that maps them uniquely into a table of precisely the right size, but this is unrealistic, since the number of possible keys in a programming language is very large, indeed unbounded.

A ‘hash function’ is then a function that maps keys in some space to a range of integers $0 \ldots M-1$. A good hash function has the following properties:

- The hash value is fully determined by the data being hashed. For instance, it should not have a ‘memory’.
- The hash function uses as much as possible of the input data. Program variables often have names such as `ikey`, so emphasis on the first letter is a bad idea.
- The hash function ”uniformly” distributes the data across the entire set of possible hash values.
- The hash function generates very different hash values for similar strings. Variables like `key1`, `key2`, et cetera should not be mapped into a cluster.

Figure **??** illustrates a hash function without conflicts.

Let us assume that we have found a way of mapping the names onto a large integer space, for instance by interpreting the bit pattern of the name as an integer. A simple hash function would be

$$h(K) = K \bmod M, \tag{1}$$

where M is the size of the hash table.

Certain values of M are less desirable. For instance, if M is even, say $M = 2M'$, then the statement $r = K \bmod M$ (say $K = nM + r$ for some n) implies

$$\begin{aligned} K = 2K' \quad &\Rightarrow \quad r = 2(nM' - K') \\ K = 2K' + 1 \quad &\Rightarrow \quad r = 2(nM' - K') + 1 \end{aligned}$$

so the key is even, iff the original number is. This places an undue influence on the last digit. If M is a multiple of 3, we find for numbers stored decimally or in bytes that keys that are a permutation of each other get mapped to numbers that differ by a multiple of 3, since both $10^n \bmod 3 = 1$ and $2^8 \bmod 1 = 1$.

2.1 Multiplication and division strategies

A good strategy is to take M prime, and such that $r^k \neq \pm a \bmod M$, where r the radix of the number system, and a, k small. (There is a list of suitable primes on `http://planetmath.org/encyclopedia/GoodHashTablePrimes.html`.)

Equation (**??**) requires you to perform a division. The same equation based on multiplication would use an integer $A \approx w/M$, where w is the maxint value. Then $1/M = A/w$, which is simply A with an imaginary decimal point to its left. Observing that

$$K \bmod M = M(K/M \bmod 1)$$

we define

$$h(K) = \lfloor M \left(\left(\frac{A}{w} K \right) \bmod 1 \right) \rfloor.$$

As an example of the value of using a prime table size, consider hashing the Bible, which consists of 42,829 unique words, into an open hash table with 30,241 elements (a prime number). In the end, 76.6 percent of the slots were used and that the average chain was 1.85 words in length (with a maximum of 6). The same file run into a hash table of 30,240 elements (evenly divisible by integers 2 through 9) fills only 60.7 percent of the slots and the average chain is 2.33 words long (maximum: 10).

2.2 String addition strategies

One could Derive a hash key by adding or XORing together all bytes in a string.

```
h = <some value>
for (i=0; i<len(var); i++)
  h = h + <byte i of string>;
```

This runs into the problem that anagrams map into the same key, and nearby strings into nearby keys. This could be remedied by introducing a table of random numbers:

```
h = <some value>
for (i=0; i<len(var); i++)
  h = Rand( h XOR <byte i of string> );
```

Exercise 1. This algorithm only gives a one-byte key. How would you derive longer keys? Give pseudo-code for the algorithm.

2.3 Examples

Here are a couple of published hash functions:

```
/* UNIX ELF hash
 * Published hash algorithm used in the UNIX ELF format for object files
 */
unsigned long hash(char *name)
{
    unsigned long h = 0, g;

    while ( *name ) {
        h = ( h << 4 ) + *name++;
        if ( g = h & 0xF0000000 )
          h ^= g >> 24;
        h &= ~g;
    }

}
```

This hash key is then reduced to an index in the hash table by

```
#define HASHSIZE 997
    static int M = HASHSIZE;
    return h % M;
```

Another hash function:

```
/* djb2
 * This algorithm was first reported by Dan Bernstein
 * many years ago in comp.lang.c
 */
unsigned long hash(unsigned char *str)
{
        unsigned long hash = 5381;
        int c;
        while (c = *str++) hash = ((hash << 5) + hash) + c;
        return hash;
}
```

Note the use of bit shifts to implement multiplication.

3 Collisions

The above techniques for generating randomly spread out addresses are generally sufficient. The problem to worry about is how to handle collisions, that is, if $h(k_1) = h(k_2)$ for different keys k_1, k_2. We will investigate several techniques for dealing with this.

For all of the strategies below, any performance analysis is statistical in nature. The average expected behaviour is often excellent, but the worst case behaviour is always very bad. In

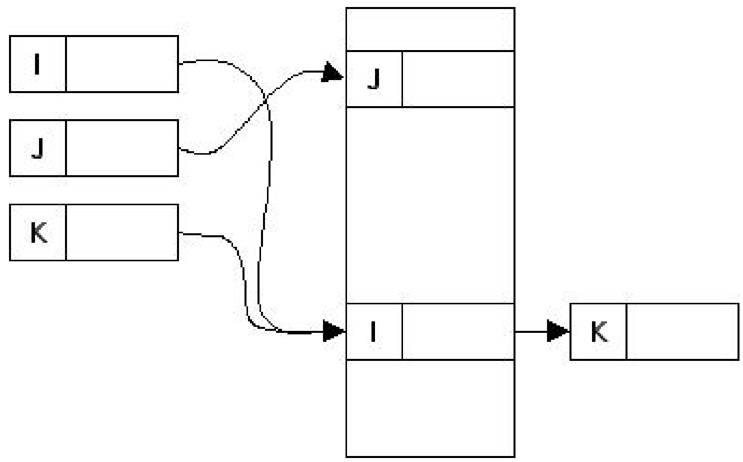


Figure 2: Separate chaining as a solution for hash conflicts

the worst case, all hash addresses map to the same location, and search time is propertional to the number of elements in the table.

The question is now how to find the storage locations for the elements with conflicting hash keys. We will look at one strategy that allocates space outside the hash table ('open hash table'), and two that resolve the conflict by finding different locations in the table ('closed hash table').

3.1 Separate chaining

A simple solution to hash conflicts is the create a linked list from each table entry, as shown in figure **??**. This way of implementing a hash table is called 'separate chaining' or 'open hashing'. One problem with this approach is that we need to maintain two different kinds of storage: static in the table and dynamic for the linked lists.

The linked lists can be created by `malloc` (and released by `free`) calls. However, these are very expensive compared to using a freespace pointer in a list. To amortize the cost, one could allocate a block of cells, and dole them out gradually. The problem with this strategy is dealing with deletions. If a cell is deleted, rerouting the pointers is easy, but additionally we have to keep track of which cells in the allocated chunk are free. This makes the open hash table somewhat complicated to implement.

Another problem is that, while this strategy is fairly efficient if the number of collisions is low, in order to get the number of collisions low, one typically chooses the table size fairly large, and then this solution is wasteful of storage. It is then better to store all elements in the hash table and maintain links in the table.

> **Exercise 2.** Discuss the value of keeping the lists ordered by key: how does this influence the run time of insertion and retrieval? Programming languages like C have local variables. Does this change your argument?

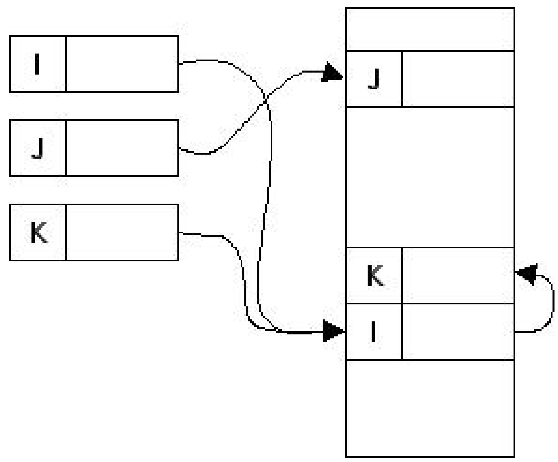


Figure 3: Linear probing as a solution for hash conflicts

3.2 Linear probing

The easiest solution is to store a conflicting element in the location immediately after the computed hash address.

```
struct { ... } node;
node Table[M]; int Free;
/* insert K */
addr = Hash(K);
if (IsEmpty(addr)) Insert(K,addr);
else {
    /* see if already stored */
  test:
    if (Table[addr].key == K) return;
    else {
      addr = Table[addr].link; goto test;}
    /* find free cell */
    Free = addr;
    do { Free--; if (Free<0) Free=M-1; }
    while (!IsEmpty(Free) && Free!=addr)
    if (!IsEmpty(Free)) abort;
    else {
      Insert(K,Free); Table[addr].link = Free;}
}
```

However, this is not the best solution. Suppose that the blocks of size N is occupied, then the free pointer will search $N/2$ locations on average for an address that maps into this block. While this is acceptable, if two blocks coalesce, this makes the search time double. Note that the chance of the cell between two blocks filling up is much larger than the chance of that exact address being generated as hash: each hash in the top block will cause

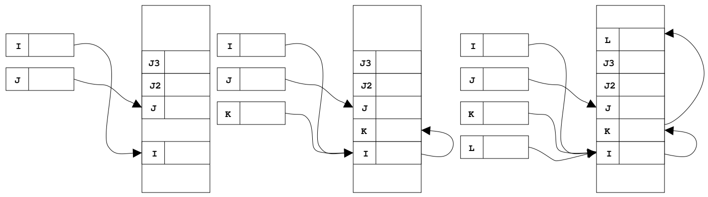


Figure 4: Coalescing of blocks in linear probing

the address to be filled.

This is illustrated in figure **??**. There is a gap of size one between $h(I)$ and a block starting at $h(J)$. When a conflict $h(K) = h(I)$ occurs, the free space pointer fills the gap. A subsequent conflict $h(L) = h(I)$ (or $h(L) = h(K)$) needs the free space pointer to traverse the whole J block to find the next location.

With $\alpha = N/M$ the ratio between occupied cells and total table size, the expected search time with this algorithm is

$$T \approx \begin{cases} \frac{1}{2}\left(1 + \left(\frac{1}{1-\alpha}\right)^2\right) & \text{unsuccessful} \\ \frac{1}{2}\left(1 + \frac{1}{1-\alpha}\right) & \text{successful} \end{cases}$$

It is clear that when α approaches 1, this time will go up unbounded.

The clumping behaviour of this algorithm makes it sensitive to the hash algorithm used. Care has to be taken that successive keys, such as `Ptr1`, `Ptr2`..., do not get mapped to successive hash values $K, K+1, \ldots$.

3.3 Chaining

The problems with linear probing can be prevented by storing conflicting elements at the start or end of the table.

```
struct { ... } node;
node Table[M]; int Free = M;
/* insert K */
addr = Hash(K);
if (IsEmpty(addr)) Insert(K,addr);
else {
    /* see if already stored */
  test:
    if (Table[addr].key == K) return;
    else {
      addr = Table[addr].link; goto test;}
    /* find free cell */
    do { Free--; }
    while (!IsEmpty(Free)
    if (Free<0) abort;
```

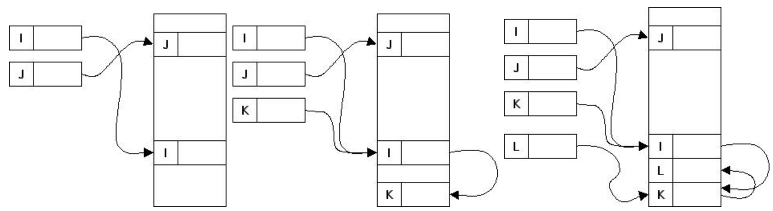


Figure 5: Chaining as a solution for hash conflicts

```
    else {
      Insert(K,Free); Table[addr].link = Free;}
}
```

This algorithm does the same list traversal as linear probing in the case a search is ultimately successful. However, for an unsuccessful search the `Free` pointer will usually be decreased by one, and only occasionally by two or more, when it runs into already occupied positions. Since these are likely to be spread out, having to search more than two steps will be rare.

In this algorithm, occasionally a hash address will be an address that has further links. We say that we have lists coalescing. This increases search time slightly, but not by much, and preventing this increases insertion time, because we would have to move cells.

With $\alpha = N/M$ the fraction of used to total table entries, find that the number of entries searched is

$$T \approx \begin{cases} 1 + (e^{2\alpha} - 1 - 2\alpha)/4 & \text{unsuccessful} \\ 1 + (e^{2\alpha} - 1 - 2\alpha)/8\alpha + \alpha/4 & \text{successful} \end{cases}$$

The hash algorithm of TEX is a variant of this chaining algorithm.

3.4 Other solutions

The solutions to the conflict problem given so far can be called 'linear rehashing'. The following strategies are called 'nonlinear rehashing'.

Random probing Try $(h(m)+p_i) \bmod s$, where p_i is a sequence of random numbers. This requires either reproducible random numbers, or storing these numbers. In order to prevent colliding keys to collide on the next try too, the random number needs to depend on the key.

Add the hash Try $(i \times h(m)) \bmod s$. This requires s to be a prime number; with this approach clumping is prevented.

They have the advantage that occupied locations in the table remain fairly scattered. On the other hand, they require further hash calculations. Also, because of the irregular memory access pattern, the cost of memory operations may become significant here.

3.5 Deletion

A surprising aspect of closed hash table algorithms is that generally it is hard to delete elements. Since most algorithms give coalescing lists, we can not mark a cell empty if its key is to be removed. Instead, we mark a cell 'deleted', which removes the key, but leaves the link intact. There are algorithms that can deal with deletions, but they have a fairly high complexity.

On the other hand, deleting in an open hash table algorithm is simple. The complication there is the freeing of the memory. If the cells are allocated in chunks, the decision to free a chunk becomes complicated.

4 Other applications of hashing

The foremost application of hashing is in compilers. However, there are other uses.

4.1 Truncating searches

In applications such as chess programs, you want to avoid evaluating a configuration twice if it's arrived at two different ways. This can be done by storing evaluations in a table. This table can be addressed by the configuration as a key itself, but these keys are long and span a large space, so searching will probably be expensive. Instead, one can use a hash table.

If two configurations generate the same hash key, they can be, but need not be the same, so further decision might be needed. To avoid this second stage work, a good quality hash function is essential.

(This example comes from `http://www.seanet.com/~brucemo/topics/hashing.htm`.)

4.2 String searching

The question 'does a string of length M appear anywhere in a document of length N' can be answered in $O(NM)$ time by a sequence of string comparisons. However, we can do considerably better, reducing the complexity to $O(N+M)$.

A hash function that adds characters together will give the same hash key for strings that are anagrams of each other. This means that instead of doing a string comparison we can compare hash keys, and only if they are equal resort to a full string comparison. To get the complexity down, we note that if the hash function is of the form

$$h(k) = \left\{\sum_i k[i]\right\} \bmod K,$$

where k is a character string, then (for a text t long enough)

$$h(t[2\ldots n+1]) = h(t[1\ldots n]) + t[n+1] - t[1]$$

(with addition/subtraction modulo K) so we can cheaply update the hash key in $O(1)$ time.

5 Discussion

In a comparison between hash tables and, for instance, tree-based storage, there is no clear preference. Hash tables can be faster, because, until the table fills up, access is $O(1)$. A similar complexity can be obtained for trees, but

- memory access in trees is more chaotic, leading to worse cache or paging behaviour;
- memory is allocated and freed dynamically; circumventing that takes considerable programming effort;
- trees can become unbalanced, and balancing them is tricky to program, and takes time;
- the optimal search time can be made approximately equal, but that is again hard to code.

Closed hash tables have the advantage of simpler storage management, and, until they fill up, no worse performance than open tables.

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