Page breaking in TEX

Victor Eijkhout

August 2004

1 Introduction

TEX's page breaking algorithm is simpler than the line breaking one. The reason for this is probably that global optimization of breakpoints, the way it is done in the paragraph algorithm, would take prohibitively much memory, at least, for the amount of memory that computers around 1980 had. The algorithm used is related to the 'best fit' algorithm we discussed in the line breaking chapter.

Theoretically, page breaking is a more complicated problem than line breaking. We will explore the issues in section **??**, but first we will briefly go into the algorithms that TEX actually uses.

2 TEX's page breaking algorithm

The problem of page breaking has two components. One is that of stretching or shrinking available glue (mostly around display math or section headings) to find typographically desirable breakpoints. The other is that of placing 'floating' material, such as tables and figures. These are typically placed at the top or the bottom of a page, on or after the first page where they are referenced. These 'inserts', as they are called in TEX, considerably complicate the page breaking algorithms, as well as the theory.

2.1 Typographical constraints

There are various typographical guidelines for what a page should look like, and TEX has mechanisms that can encourage, if not always enforce, this behaviour.

1. The first line of every page should be at the same distance from the top. This changes if the page starts with a section heading which is a larger type size.
2. The last line should also be at the same distance, this time from the bottom. This is easy to satisfy if all pages only contain text, but it becomes harder if there are figures, headings, and display math on the page. In that case, a 'ragged bottom' can be specified.
3. A page may absolutely not be broken between a section heading and the subsequent paragraph or subsection heading.

4. It is desirable that
   (a) the top of the page does not have the last line of a paragraph started on the preceding page
   (b) the bottom of the page does not have the first line of a paragraph that continues on the next page.

2.2 Mechanisms

The basic goal of page breaking in TeX is to fill up a box of height `\vsize`. The is the goal size of the material without header and footer lines. The box is constructed by adding material to the vertical list until an optimal breakpoint is found. The material before that breakpoint is then put in `\box255`, and the code in `\output`, the 'output routine' is executed. The command to send a box to the output file is `\shipout`.

The typographical rules in the previous section can be realized fairly simply in TeX.

?? The vertical location of the first line on a page is controlled by `\topskip`. If the baseline of the first line is closer to the top than this parameter, glue is inserted to make up for the difference.

?? The vertical location of the last line of a page is controlled by `\maxdepth`. If the last line of the page is deeper than this amount, the reference point of the box is shifted down accordingly.

?? Preventing page breaks between vertical material of different kinds can be done by the proper use of penalties and glue.

?? A break after the first line of a paragraph is prevented by setting the `\clubpenalty`.

?? A break before the last line of a paragraph is prevented by setting the `\widowpenalty`.

2.3 Breakpoints

TeX builds up a current page that is a vertical list of material. It regularly tests whether there is enough material to fill a box of height `\vsize` while incurring a badness less than $10,000$. The breakpoints are similar to those in the line breaking algorithm, mostly occurring at a penalty, or at a glue that follows non-discardable material.

2.4 Some output routines

The very simplest output routine simply takes the vertical list and ships it to the output file:

```
\output={\shipout\box255}
```

Slighly more sophisticated, a header and footer are added:

```
\output={
  \setbox255=\vbox{ <header>
                    \box255
                    <footer>
                  }
  \shipout\box255
  }
```

The following example makes the page one line longer if a widow (break before the last line of a paragraph) occurs. First we save the original `\vsize` and we declare a recognizable value for the `\widowpenalty`:

```
\newif\ifEnlargedPage \widowpenalty=147
\newdimen\oldvsize \oldvsize=\vsize
```

The output routine now works by testing for the widow penalty, and if it is found, increasing the `\vsize` and returning the page material to the list by `\unvbox255`:

```
\output={
    \ifEnlargedPage <output the page>
    \else \ifnum \outputpenalty=\widowpenalty
             \global\EnlargedPagetrue
             \global\advance\vsize\baselineskip
             \unvbox255 \penalty\outputpenalty
          \else  \shipout\box255
    \fi   \fi}
```

Here is the missing bit of code that outputs the enlarged page:

```
    \ifEnlargedPage \shipout\box255
          \global\LargePagefalse
          \global\vsize=\oldvsize
```

2.5 Insertions

Floating material, such as tables and figures, are handled by a mechanism called 'insertions'. Insertions fall in different classes, and insertion material is specified by

```
\insert<class number>{ <material> }
```

If the class number is n, then

- When the output routine is active, `\box`n contains the material of insertion class n.
- `\dimen`n specifies the maximum amount of insertion material that is allowed to be placed on one page. If there is more material, it is split with the remainder carried over to the next page.
- There are further fine points to this algorithm.

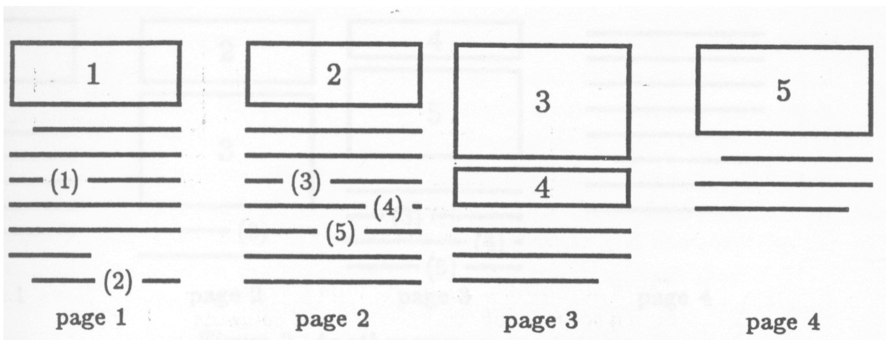
Insertions are thus added, for as far as possible, to the page being formed when the `\insert` command is given. TeX has no way of moving insertions to an earlier page, although moving material to a later page – presumable where more space is available – is possible.

3 Theory of page breaking

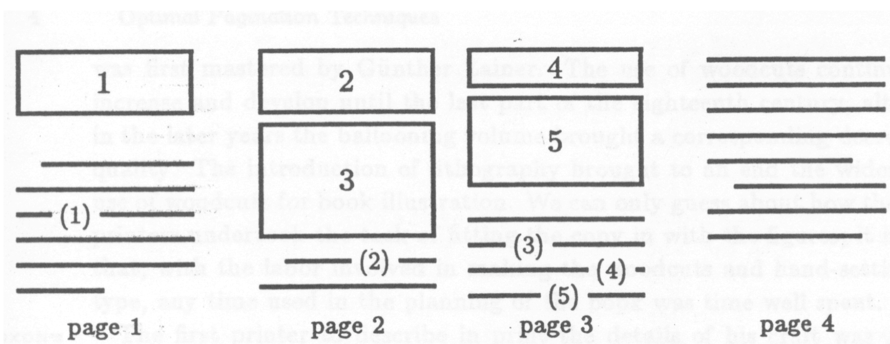
At first glance, the page breaking problem is much like the line breaking problem, except with larger basic blocks, and vertically instead of horizontally. In the line breaking problem, a long list of words is broken into lines, adjusting the margins by stretching or shrinking the interword space. In the page breaking problem, a vertical list of lines, display formulas, and other vertical material, is broken into equal sized pages, using the various amounts of vertical space for taking up the slack.

However, the page breaking problem becomes much harder if we include the possibility of figures and other floating material. In that case, the computed badness (the measure we try to minimize in the breaking process) will include reflect that we want a figure to be close to pages that reference it, or satisfy other ordering rules involving references to it. Maybe surprisingly, even rather simple cost functions make page breaking into an NP-complete problem.

To get an appreciation of the issues, consider this sequence of pages with figures:



References to the figures are here indicated with parenthesized numbers. We see that out of 5 references, 3 are not to a figure on the same page. However, if we move one line from page 1 to 2, and move figure 3 forward a page, we get:



where we see that only one reference is not to the same page. Unfortunately this is a backward reference.

In the remainder of this chapter we will investigate theoretical aspects of functions that try to optimize placement of floating material in a global fashion. It should be noted that this is considerably more sophisticated than what is implemented in TEX. The available algorithm is closer to a 'first fit' strategy.

We will investigate two closely related page breaking algorithms. We show how a particular form of the page breaking problem (the 'MQ' problem) is equivalent to the 2-satisfyability problem, which is known NP-complete. As is usual in the context of proving NP-completeness, we transform our minimization problem 'give the solution with minimum cost' into a decision problem by asking 'is there a solution with a cost $\leq B$' where B is some predetermined bound.

The very similar 'ML' problem, which only differs in the form of the badness function, does have a polynomial time solution.

3.1 The MQ problem is NP-complete

We consider the MQ page breaking problem: Multiply referenced figures, and Quadratic badness measure. Under the simplifying assumptions that each figure takes a whole page, we then have a set $T = \{t_1, \ldots, t_N\}$ of text blocks and a set $F = \{f_1, \ldots, f_N\}$ of figures and a function $W : T \times F$ such that $W(t_i, f_j)$ describes how many times text block i references figure j. We assume a bound $W(t_i, f_j) \leq q(N)$ (where q is a polynomial) dependent on the size of the problem.

The MQ problem is now the question whether there is an page ordering of text blocks and figures, that is, a mapping $P : (T \cup F) \to \{1, 2, \ldots, 2N\}$ such that

$$\begin{matrix} P(t_i) < P(t_j) \\ P(f_i) < P(f_j) \end{matrix} \qquad \forall_{1 \leq i < j \leq N}$$

and so that

$$S = \sum_{i,j} W(t_i, f_j)(P(t_i) - P(f_j))^2 \leq B$$

In order to show that this problem is NP-complete, we show that it can be transformed into an instance of the maximum 2-satisfiability problem. This means that solving the one problem is equivalent to solving the other, and since the transformation is done in polynomial time, the two problems have the same complexity.

The maximum 2-satisfiability (MAX 2-SAT problem can be formulated as follows. Let there be given n binary variables $x_1, \ldots, x_n$ and m clauses $\{u_1 \vee v_1, \ldots, u_m \vee v_m\}$, where $u_i = x_j$ or $u_i = \neg x_j$ for some j. Given a bound K, is there a way of setting the x_i variables such that at least K of the clauses are satisfied? This problem is known to be NP-complete.

We will now make a pagination problem that 'translates' an instance of the 2-satisfyability problem. That is, given a configuration of binary variables and clauses and a bound K to satisfy, we will make a pagination problem with bound B such that the one bound is satisfied if and only if the other is. Then, since MAX 2-SAT is NP-complete, it follows that this particular pagination problem is also NP-complete.

3.1.1 Construction of the translation

We make the translation between the two types of problems by constructing the page assignment function P, and the weight function W. There are three aspects we need to take care of, so we let $B = B_1 + B_2 + B_3$, and we will determine the B_i bounds separately.

First of all we set $W(t_i, f_i) = b_1$ sufficiently large that only configuration with $|P(t_i) - P(f_i)| = 1$ will satisfy the bound. (Recall that the badness is a sum of $W(t_i, f_j)(P(t_i) -$

$P(f_j))^2$ terms.) To allow for the pages to be ordered this way, we let $B_1 = Nb_1$. The b_1 quantity will be defined in terms of B_2 and B_3 as

$$b_1 = \lceil (B_2 + B_3)/3 \rceil + 1$$

Supposing that at least one t_i, f_i pair is not adjacent, then it follows from this bound that the badness will be

$$S \geq (N-1)b_1 + 2^2 b_1 = (N+3)b_1 > B$$

where the $N-1$ corresponds to the pairs that are adjacent, and the 2^2 to at least one pair at distance 2.

Since text blocks and are next to each other, the only remaining question is which comes first. This limits the number of cases we need to consider in the remainder of this proof.

Now let parameters n for the number of variables and m for the number of clauses be as described above, then our pagination problem will have N text blocks and figures, where $N = 2n + 2m$. The first $4n$ pages encode the values of the variables x_i, with each consecutive 4 pages corresponding to one variable:

$4i-3$	$4i-2$	$4i-1$	$4i$	
t_{2i-1}	f_{2i-1}	f_{2i}	t_{2i}	if x_i is true
f_{2i-1}	t_{2i-1}	t_{2i}	f_{2i}	if x_i is false

To ensure that these are the only configuration that will satisfy the bound, set $W(t_{2i-1}, f_{2i}) = W(f_{2i-1}, t_{2i}) = b_2$ large enough. Either of the above patterns then contributes $2 \cdot 2^2 b_2 = 8b_2$, while the other possibilities ($t\,f\,t\,f$ and $f\,t\,f\,t$) would contribute $(1^2 + 3^2)b_2 = 10b_2$.

Correspondingly, we allow a bound of $B_2 = 4b_2 \sum (i-j)^2$ where i, j range over the pairs that satisfy $W(t_i, f_j) = b_2$. Defining

$$b_2 = 8(m-k) + 5$$

is sufficient to make violation of this condition outweigh the gain from more clauses being satisfied.

Next, the $4m$ remaining pages will encode the clauses in a similar manner:

$4n+4j-3$	$4n+4j-2$	$4n+4j-1$	$4n+4j$	
$t_{2n+2j-1}$	$f_{2n+2j-1}$			if u_j is true
$f_{2n+2j-1}$	$t_{2n+2j-1}$			if u_j is false
		t_{2n+2j}	f_{2n+2j}	if v_j is true
		f_{2n+2j}	t_{2n+2j}	if v_j is false

Further conditions on W ensure that the u_j variables indeed correspond to the proper x_i. For instance

$$W(t_{2n+2j-1}, f_{2i}) = W(t_{2i} = f_{2n+2j-1}) = b_2 \quad \text{if } u_j = x_i$$

This term contributes $2d^2 b_2$ to the badness, where d is twice the difference between the subscripts, in this case $d = (2n + 2j - 2i)$. With a mismatch, a t and f page assignment are reversed, so the contribution becomes $\big((d-1)^2 + (d+1)^2\big) = 2(d^2+1)b_2$.

Proper truth values of the clauses are enforced as follows. Observe that the combination where u_j and v_j are both false is the only one that gives a false result. This corresponds to the pagination

$$f_{2n+2j-1} \quad t_{2n+2j-1} \quad f_{2n+2j} \quad t_{2n+2j}$$

In this configuration f_{2n+2j_1} and t_{2n+2j} are spread the furthest apart, so we penalize that with

$$W(t_{2n+2j}, f_{2n+2j_1}) = 5, \qquad W(t_{2n+2j-1}, f_{2n+2j}) = 3.$$

This gives a contribution of 32 for the three true cases, and 48 for the false result. Correspondingly, to allow for K clauses to be satisfied we allow $B_3 = 48(m - K) + 32K$.

Finally, by defining the polynomial $q(i) = 64(i + 1)^4$, we have $q(N) > b_1 \geq b_2 > 5 > 3$, so W is bounded as required.

3.1.2 NP-completeness

In order to establish NP-completeness of the problem MQ, we need to establish that something is a true instance of MAX 2-SAT iff the translated instance is true for MQ.

Given a truth assignment of the x_is that makes K clauses true, we can now construct a pagination P with a satisfied bound of B.

Conversely, let a pagination with bound B be given, then the corresponding truth assignment makes K clauses true. To see this, inspect each of the details of the translation between the problems, and observe that any violation will cause the bound B to be exceeded.

3.2 The ML problem has a polynomial time solution

The 'ML' problem (Multiply referenced figures, Linear badness function) is identical to MQ, except for the form of the badness function. Having a linear badness function makes it possible to solve this problem by dynamic programming in linear time.

As in MQ, we have text blocks t_i and figures f_j that take up whole pages. We generalize the problem slightly to having different numbers of text and figure blocks:

$$T = \{t_1, \ldots, t_N\}, \qquad F = \{f_1, \ldots, f_M\}$$

The function $W : T \times F$ is such that $W(t_i, f_j) \geq 0$ describes how many times text block i references figure j.

The ML problem is now the question whether, given the above, and given a bound B, there is an page ordering of text blocks and figures, that is, a mapping $P : (T \cup F) \rightarrow \{1, 2, \ldots, M + N\}$ such that

$$\begin{matrix} P(t_i) < P(t_j) \\ P(f_i) < P(f_j) \end{matrix} \qquad \forall_{1 \leq i \leq N, 1 \leq j \leq M}$$

and so that

$$S = \sum_{i,j} W(t_i, f_j) \, |P(t_i) - P(f_j)| \leq B$$

3.2.1 Dynamic programming solution

The key to a dynamic programming solution of ML is to identify subproblems. The subproblem we consider is

Given i text blocks and j figures, what is the least badness of placing these on the first $i+j$ pages. Call this partial badness B_{ij}.

The problem here is the 'dangling' references (t_r, f_s) with $r > i, s \leq j$ or $r \leq i, s > j$. The measure $R_{i,j}$ is defined as the number of dangling references after these blocks and figures have been placed.

A dangling reference is either

A forward reference: A text block refering to a figure not yet placed. The number of forward references from the $i+j$ block is

$$F_{ij} = \sum_{\substack{1\leq r\leq i \\ j<s\leq M}} W(t_r, f_s)$$

A backward reference: A figure that will be reference on a text block not yet placed.) The number of backward references from the $i+j$ block is

$$B_{ij} = \sum_{\substack{i<r\leq N \\ 1\leq s\leq j}} W(t_r, f_s)$$

which makes $R_{ij} = F_{ij} + B_{ij}$.

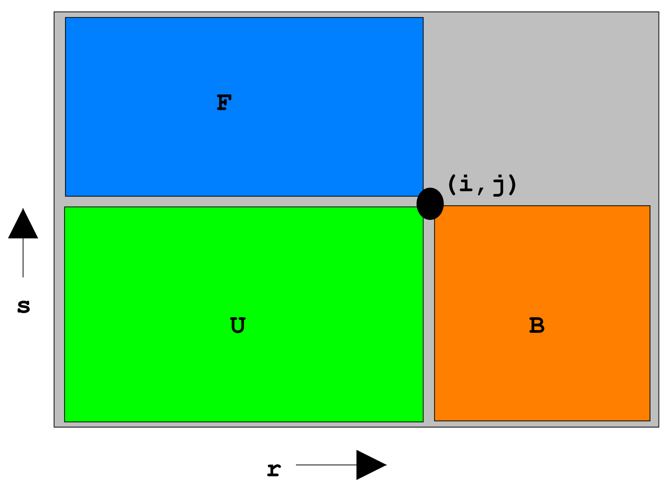


Figure 1: The F_{ij}, B_{ij}, and U_{ij} areas in (r, s) space

For badness calculations involving dangling references, we count only the part to the boundary of the i, j region. Formally:

$$B_{ij} = B_{ij}^{(1)} + B_{ij}^{(2)}$$

where

$$B_{ij}^{(1)} = \sum_{\substack{r\leq i \\ s\leq j}} W(t_r, f_s) \left|P(t_r) - P(f_s)\right|$$

is the part of the badness due to references that are fully resolved within the pages already placed; the part of the badness due to dangling references is

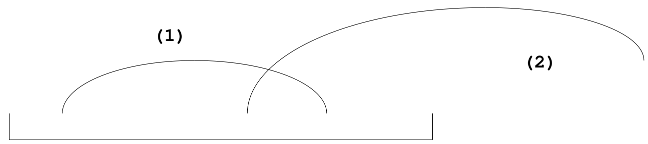


Figure 2: Resolved and dangling references of a block of pages

$$B^{(2)}_{ij} = \sum_{\substack{r>i\\ s\leq j}} W(t_r,f_s)\,\ell(i,j;r,s) + \sum_{\substack{r\leq i\\ s>j}} W(t_r,f_s)\,\ell(i,j;r,s)$$

where

$$\ell(i,j;r,s) = \begin{cases} i+j-P(f_s) & \text{if } r>i \\ i+j-P(t_r) & \text{if } s>j \end{cases}$$

describes the part of the arc between t_r and f_2 that lies in the first $i+j$ pages. These two types of arcs are illustrated in figure **??**.

Figure **??** illustrates how reference arcs change status when we go from $i+j-1$ to $i+j$ pages, say by placing text block t_i:

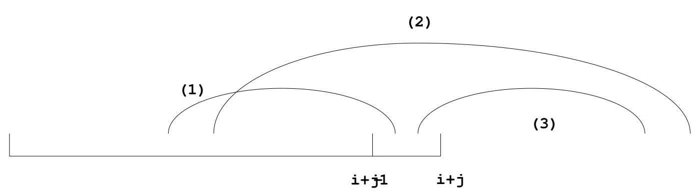


Figure 3: Change in status of resolved and dangling references upon extending a block of pages

(1) References that were unresolved with references originating in t_i move their contribution from the $B^{(2)}$ term to the $B^{(1)}$ term. (Note that a reference to a page one location outside the current block is already fully counted in the badness.)

(2) Dangling references that stay unresolved increase their contribution to the $B^{(2)}$ term by $(\sum_{r\leq i-1,s>j} + \sum_{r>i-1,s\leq j})W(t_r,f_s)$

(3) Arcs that used to fall completely outside the page block, but that are now dangling in the new page block, add a contribution of $\sum_{r=i,s>j} W(t_r,f_s)$ to the $B^{(2)}$ term.

$\sum_{r>i,s\leq j} W(t_r,f_s)$ In sum, $B_{ij} = B_{i-1,j} + R_{ij}$. The same story holds for extending $i+j-1$ pages by placing figure f_j, so we have the recurrence

$$B_{ij} = \min(B_{i-1,j}, B_{i,j-1}) + R_{ij}.$$

We still need to compute the R_{ij} quantities, describing the number of dangling references from placing i text blocks and j figures. Above we saw $R_{ij} = F_{ij} + B_{ij}$. For efficient calculation of these sums, it is convenient to make a table of

$$U_{ij} = \sum_{\substack{1\leq r\leq i\\ 1\leq s\leq j}} W(t_r,f_s)$$

which takes time $O(NM)$, that is, quadratic in the total number of pages. Then

$$R_{ij} = U_{iM} + U_{Nj} - 2U_{ij},$$

as is easily seen from figure **??**.

3.3 Discussion

It is interesting to note that certain details of the NP-completeness proof of MQ rely on the quadratic badness function, rather than on more ‘structural’ properties of the problem.

> **Exercise 1.** Find a place in the NP-completeness proof of MQ that uses the quadratic badness function, and show that the underlying fact does not hold for linear functions. Does it hold for other functions than quadratic?

Similarly, ML only has a dynamic programming solution thanks to the linear badness function.

> **Exercise 2.** Explain how the linearity of the badness function is essential for the dynamic programming solution of ML.

> **Exercise 3.** The discussion of the ML problem only gave the cost computation. How can the actual page assignment be derived from the given construction? What is the time and space complexity of this part of the algorithm?

Contents